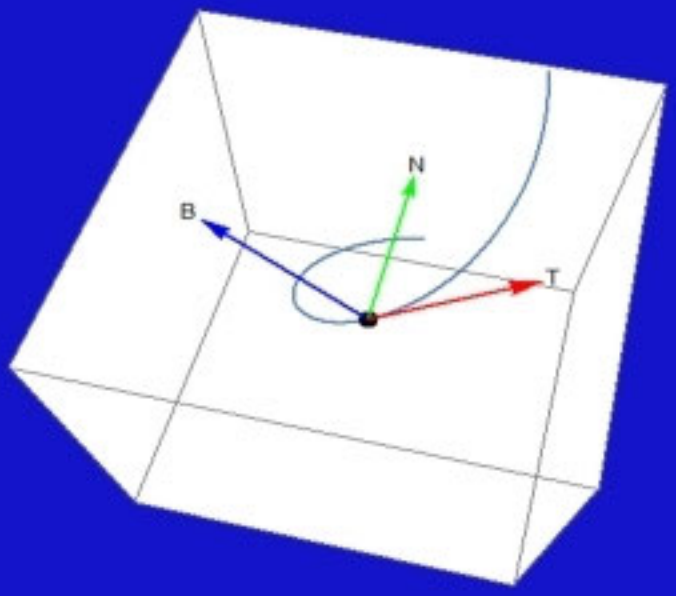
Steven Tan

VECTOR CALCULUS
Using Mathematica®

Second Edition



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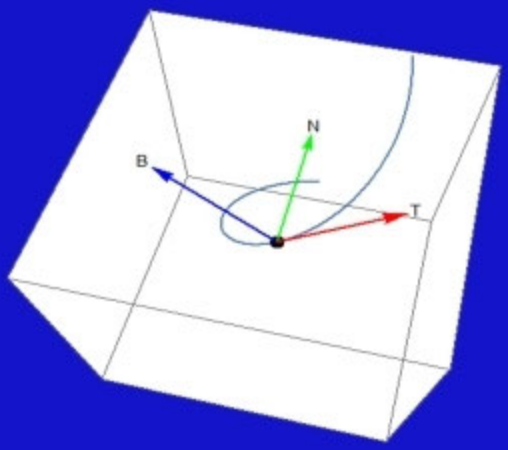

Vector Calculus Using Mathematica®

Second Edition

With 165 fully solved examples, 255 figures and 9 animations

Steven Tan

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Vector Calculus Using Mathematica®

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Dedication

To Svetlana

Preface to the Second Edition

The second edition is an addition of the first. Two new chapters on line integrals, Green's Theorem, Stokes's Theorem and Gauss's Theorem have been added.

The author would like to thank everyone who pointed out mistakes in the first edition. We hope we have found and corrected these, and have not introduced any more with the new material.

Preface to the First Edition

As far as the laws of mathematics refer to reality, they are not certain;

and as far as they are certain, they do not refer to reality.

(Illuminated Geometry, Sidelights on Relativity by A. Einstein)

This book is intended for an undergraduate course in vector calculus taken by students majoring in science, or engineering. There are many standard textbooks on vector calculus, however, this book takes rather different approach.

First of all, this book is different from most textbooks because it is the author's attempt to introduce vector calculus with the aid of Mathematica® computer algebra system to represent them and to calculate with them. It is written with and for Mathematica® version 11. However, most illustrations are backward compatible with earlier versions of Mathematica® or have equivalent representations.

Secondly, the unique features of this book, which set it apart from the existing textbooks, are the large number of illustrative examples. It is the author's opinion a novice in science or engineering needs to see a lot of examples in which mathematics is used to be able to "speak the language." Throughout the book these examples and all illustrations can be replicated and used to learn and discover vector calculus in a new and exciting way. Since we cover the problems from the core courses, the reader can practice with our solutions, and then modify our solutions to solve the particular problems assigned. This should help the reader move up the problem

solving skills and to use Mathematica® to visualize the results and to develop a deeper intuitive understanding. Usually, visualization provides much more insight than the formulas themselves.

This book covers vector calculus in one, two and three variables. The prerequisites are the standard courses in calculus. A knowledge of computer programming would be beneficial but not essential. From the point of view of mathematicians, this book is less rigor because the author has put less emphasis on proofs but more on examples and illustrations to clarify concepts.

It is important to note that this book should not be thought of as a substitute for the text in a chapter in a standard introductory textbook. Although there is an introduction to each chapter where the basics are presented, this introduction is brief. This book is therefore designed to be used in conjunction with any standard textbook. You could think of this book as supplementing a textbook by providing examples using Mathematica®. Or you could think of a textbook as supplementing this book by providing additional conceptual and theoretical background; and proofs of theorems.

We assume the reader has some familiarity with Mathematica®, so we could focus directly on vector calculus. This book does not discuss programming in Mathematica® nor does it teach all the principles and techniques of applications using Mathematica®. However, Appendix A provides a basic introduction to Mathematica®. New users will find that the materials in this appendix enable them to become familiar with Mathematica® within a few hours. The reader can learn the essentials of Mathematica® through examples described in the book. Mathematica® commands and techniques are introduced as the need arises.

Although extreme care was taken to correct all the misprints, it is very unlikely that I have been able to catch all of them. I shall be most grateful to those readers kind enough to bring to my attention any remaining mistakes, typographical or otherwise. Please feel free to contact me at:

GedankenBooks@gmail.com

Steven Tan

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Chapter 1. Vectors

Initialization

This section is reserved for Mathematica notebook initialization.

At the start of each chapter, we shall clear all values and definitions of any previously defined symbols using Clear command. Here is also the place where we set some default options. Most Mathematica modules are defined in the texts when the needs arise.

```
Clear["Global`*"]

SetOptions[ParametricPlot,PlotStyle->AbsoluteThickness[2]];

SetOptions[ParametricPlot3D,PlotStyle->AbsoluteThickness[2]];

SetOptions[Graphics,BaseStyle->AbsoluteThickness[2]];

SetOptions[Graphics3D,BaseStyle->AbsoluteThickness[2]];
```

Using SetOptions command we set specified default options for symbols that we are going to use thereafter. Unless overwrite it explicitly the options for these commands are set according to the values in SetOptions. For example, the ParametricPlot command's PlotStyle option is set to AbsoluteThickness[2] by default.

1.1 Algebraic Properties of Vectors

We begin with some reviews of the notion of a vector.

Definition 1.1

A vector in $\mathbb{R}^2$ is an ordered pair of real numbers. A vector in $\mathbb{R}^3$ is an ordered triple of real numbers. Similarly, a vector in $\mathbb{R}^n$ is an ordered n-tuple of real numbers.

In Mathematica, vectors are represented as lists of numbers or variables encased in braces.

```
vector2D={v1,v2}

vector3D={v1,v2,v3}
```

Show that addition of vectors is commutative, that is, A + B = B + A.

```
a={a1,a2,a3}

b={b1,b2,b3}
```

a+b==b+a

Show that the addition of vectors is associative, that is, A + (B + C) = (A + B) + C.

c={c1,c2,c3}

a+(b+c)==(a+b)+c

If A, B, and C are vectors, and m and n are scalars, then

1. A + B = B + A Commutative law for addition

2. A + (B + C) = (A + B) + C Associative law for addition

3. m(nA) = (m n)A = n(mA) Associative law for scalar multiplication

4. (m + n)A = mA + nA Distributive law for scalars

5. m(A + B) = mA + mB Distributive law for vectors

1.2 Geometric Properties of Vectors

A vector can be visualized in $\mathbb{R}^2$ or $\mathbb{R}^3$ as an arrow that begins at the origin and ends at a point. Such a depiction is often called the position vector of the point (a_1, a_2) or (a_1, a_2, a_3) in $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively.

(* A vector in $\mathbb{R}^2$ *)

```
Graphics[{Arrowheads[0.1],Arrow[{{0,0},{2,1}}]},Axes->True]
```

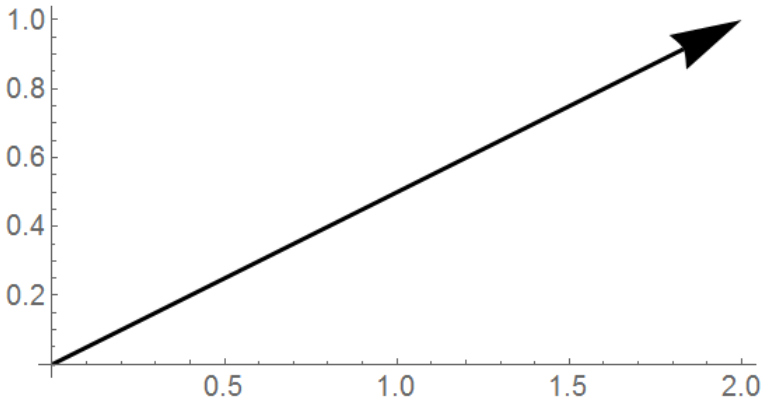


Figure 1.1. The vector a=(2,1) in $\mathbb{R}^2$.

(* A vector in $\mathbb{R}^3$ *)

```
Graphics3D[{Arrowheads[0.1],Arrow[{{0,0,0},{3,3,3}}]},Axes->True]
```

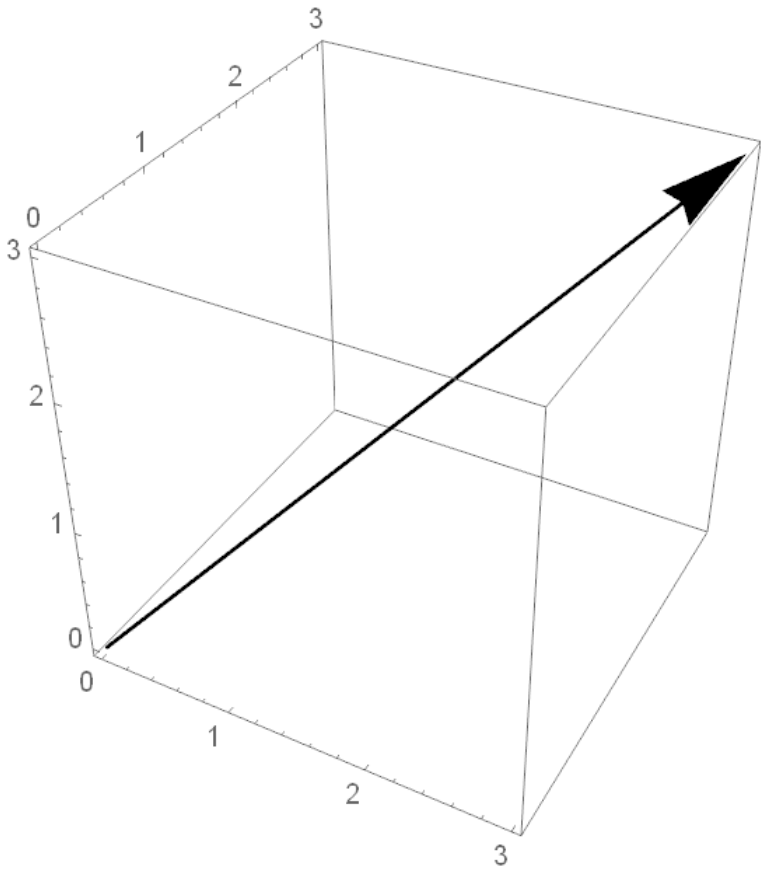


Figure 1.2. The vector a=(3,3,3) in $\mathbb{R}^3$.

Two vectors are equivalent as long as they are parallel and have the same lengths. Thus, we may represent the vector $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at the origin or with its tail at any other point.

```
origin={0,0};

a={2,1};

b={1,2};

Graphics[{Blue,Arrowheads[.08],Arrow[{origin,a}],Arrow[{origin,b}],Arr
ow[{origin,a+b}],{Dashed,Arrow[{a,a+b}]},
```

{Dashed,Arrow[{b,a+b}]},Text[Style["a",Bold,FontSize->14],
{1,0.3}],Text[Style["b",Bold,FontSize->14],{0.5,1.3}],Text[Style["a +
b",Bold,FontSize->14],{1.6,1.3}]},Axes->True]

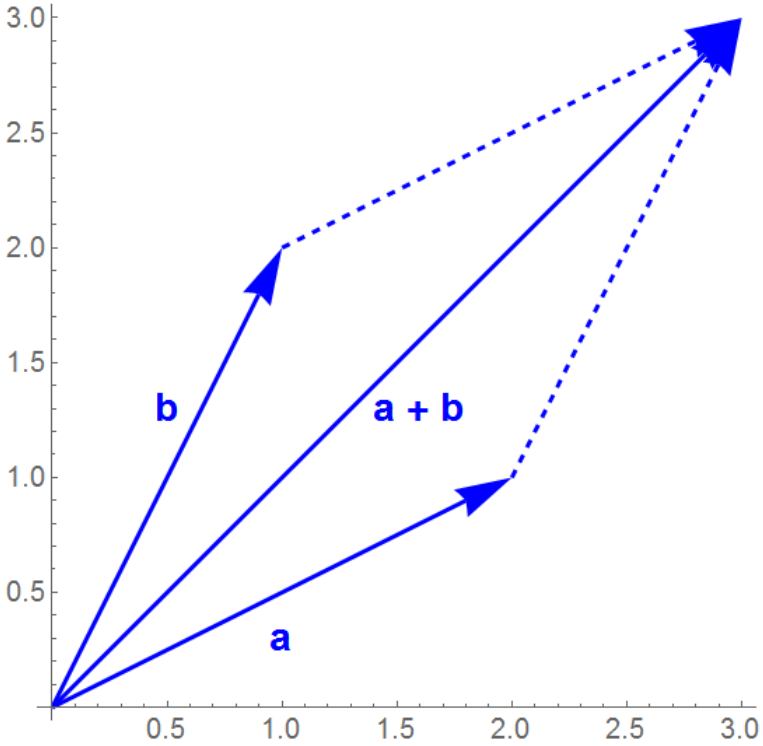


Figure 1.3. The vector a+b is represented by the arrow that begins at the common initial point of a and b and runs along the diagonal of the parallelogram determined by a and b.

1.3 Parametric Equations of Curves

A curve in $\mathbb{R}^2$ is the set of points (x,y) whose coordinates are given by a function of a parameter t, such as

$$\begin{cases} x \equiv f(t), \\ y \equiv g(t). \end{cases} \tag{1.1}$$

Similarly, a curve in $\mathbb{R}^3$ is the set of points (x,y,z) whose coordinates are given by a function of a parameter t

$$\begin{cases} x \equiv f(t), \\ y \equiv g(t), \\ z \equiv h(t). \end{cases} \tag{1.2}$$

Example 1.1. The vector parametric equations for a line in $\mathbb{R}^2$ through the point (b_1, b_2) and parallel to (a_1, a_2) are given by

$$\begin{cases} x = a_1 t + b_1, \\ y = a_2 t + b_2, \end{cases}$$

where $a_1, a_2, b_1, b_2, t \in \mathbb{R}$.

Example 1.2. Similarly, the vector parametric equations for a line in $\mathbb{R}^n$ through the point $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ and parallel to $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ are

$$\begin{cases} x_1 = a_1 t + b_1, \\ x_2 = a_2 t + b_2, \\ \quad\vdots \\ x_n = a_n t + b_n, \end{cases}$$

where

$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, t \in \mathbb{R}$.

In vector notation, this can be written as

$\mathbf{r}(t) = \mathbf{b} + t\,\mathbf{a}$,

where $\mathbf{r} = (x_1, x_2, \ldots, x_n)$.

Definition 1.2

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two nonzero, nonparallel vectors in $\mathbb{R}^3$. The vector parametric equations for a line in $\mathbb{R}^3$ through the point b and parallel to a is defined as

(1.3) $\mathbf{r}(t) = \mathbf{b} + t\,\mathbf{a}$,

where $t \in \mathbb{R}$ and $\mathbf{r} \in \mathbb{R}^3$.

Example 1.3. Write a set of parametric equations for the line through the points (5,-3,4) and (0,1,9). Draw the line.

Solution:

```
Clear[a,b]

a={0,1,9}-{5,-3,4}

b={5,-3,4}


(* The equation of the line through b and parallel to a *)

line=a t+b
```

Thus, the set of parametric equation for the line is

$$\begin{cases} x(t) = -5\,t + 5, \\ y(t) = 4\,t - 3, \\ z(t) = 5\,t + 4. \end{cases}$$

In vector notation, it is written as

r(t)=(-5t+5,4t-3,5t+4).

Note that we can use either the point (5,-3,4) or (0,1,9) as the through-point for b.

```
Module[{plot},plot[1]=ParametricPlot3D[line,{t,-1.5,1.5},PlotStyle-
>Hue[0.7],Boxed->False,AxesOrigin->{0,0,0},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{{Dashed,AbsoluteThickness[2],Arrowheads[0.04],Ar
row[{{0,0,0},b}]},
{Red,AbsoluteThickness[2],Arrowheads[0.04],Arrow[{b,b+0.5a}]}}];

plot[3]=Graphics3D[{{Text[Style["b",Bold],{3.5,-3,3.3}]},
{Text[Style["a",Bold],{5,-3,5.5}]},{Text[Style["(5, -3, 4)",Bold],
{7.1,-3,4.5}]},{Text[Style["(0, 1, 9)",Bold],{4,-3,10}]}}];
```

```
plot[4]=Graphics3D[{{Red,PointSize[0.025],Point[b]},
{PointSize[0.025],Point[{0,1,9}]}}];

Show[Array[plot,4],ViewPoint->{5,-10,2}]

]
```

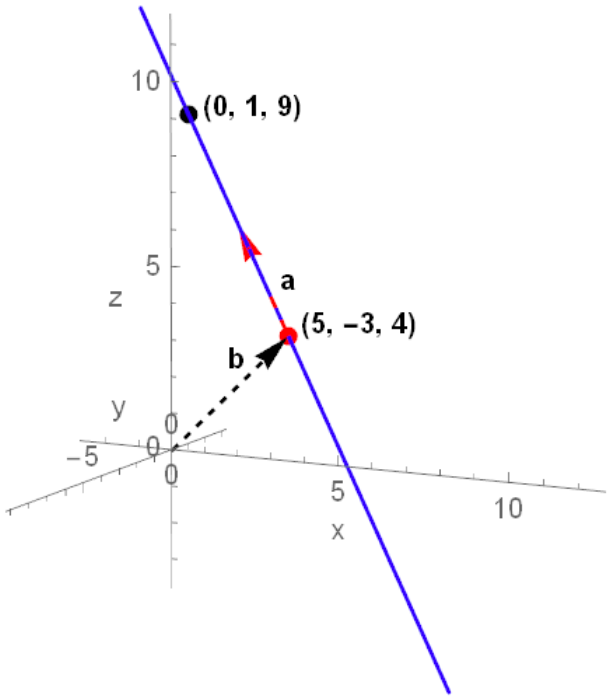


Figure 1.4. The line in space parametrized by the equation r(t)=(-5t+5,4t-3,5t+4).

1.3.1 The right circular helix of unit radius

The parametric equations for a right circular helix of radius 1 are defined as

$$\begin{cases} x = \cos t, \\ y = \sin t, \\ z = t, \end{cases} \qquad t \in \mathbb{R}.$$

```
ParametricPlot3D[{Cos[t],Sin[t],t},{t,-5,5},PlotStyle-
>Hue[0.7],AspectRatio->2]
```

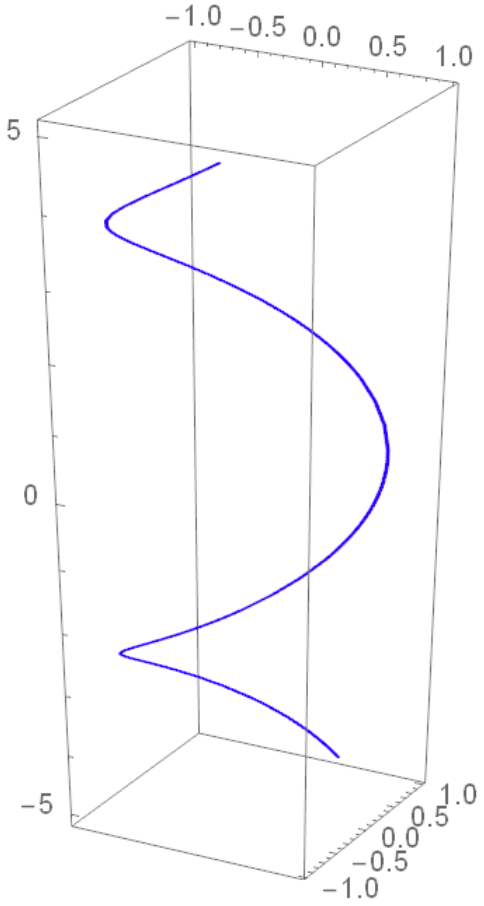


Figure 1.5. The right circular helix of radius 1.

1.3.2 A three-dimensional astroid

The parametric equations for a three-dimensional astroid are given by

$$\begin{cases} x = \cos^3 t, \\ y = \sin^3 t, \\ z = \cos 2t, \end{cases} \qquad 0 \le t \le 2\pi.$$

ParametricPlot3D[{Cos[t]^3,Sin[t]^3,Cos[2t]}//Evaluate,{t,0,2π},PlotStyle->Hue[0.7]]

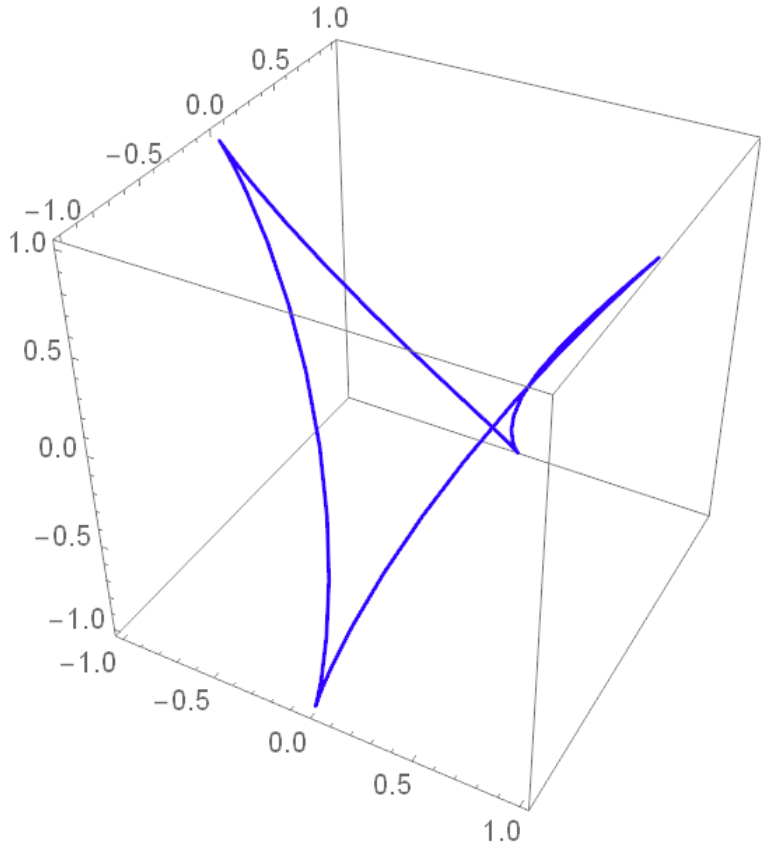


Figure 1.6. A three-dimensional astroid.

1.3.3 A cycloid

If a wheel rolls along a straight line at constant speed without slipping, a point on the circumference traces a curve called a cycloid (see Tan, S. Handbook of Famous Plane Curves Using Mathematica, Chapter 2.13 Cycloid).

The parametric equations for a cycloid are given by

$$\begin{cases} x = a\,(t - \sin t), \\ y = a\,(1 - \cos t), \end{cases} \qquad t \in \mathbb{R}.$$

ParametricPlot[{t-Sin[t],1-Cos[t]}//Evaluate,{t,0,4π},PlotStyle->Hue[0.7]]

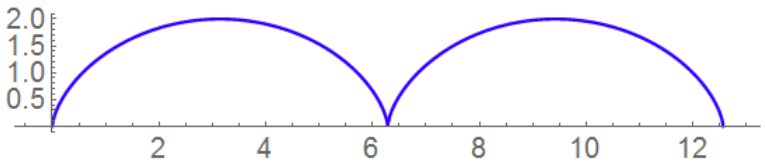


Figure 1.7. The cycloid for a=1.

Module[{x,y,cycl},

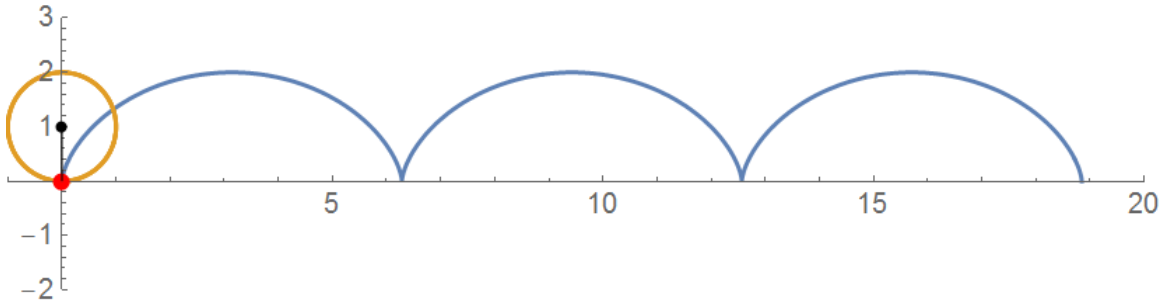
x[t_]:=t-Sin[t];

y[t_]:=1-Cos[t];

cycl[u_]:=ParametricPlot[{{x[t],y[t]},{u+Cos[t],1+Sin[t]}},{t,0,6 π},Epilog-> {{PointSize[0.01],Black,Point[{u,1}],PointSize[0.015],Red,Point[{x[u],y[u]}]},Line[{{u,1},{x[u],y[u]}}]},PlotRange->{{-1,20},{-2,3}},ImageSize->550];

Animate[cycl[u],{u,0,6π},AnimationRunning->False]

]



Animation 1.1. The locus of a point on the rim of a circle of radius 1 rolling along a straight line generates the cycloid (t-sin t,1-cos t).

1.3.4 The involute of a circle

The involute of a circle is formed by unwinding a taut string which has been wrapped around a circle (see Tan, S. Handbook of Famous Plane Curves Using Mathematica, Chapter 2.31 Involute of a Circle).

The parametric equations are

$$\begin{cases} x = a(\cos t + t \sin t), \\ y = a(\sin t - t \cos t). \end{cases} \qquad t \in \mathbb{R}.$$

ParametricPlot[{{Cos[t],Sin[t]},{Cos[t]+t Sin[t],Sin[t]-t Cos[t]}}//Evaluate, {t,0,2π},PlotStyle->Table[Hue[0.7i],{i,0,1}]]

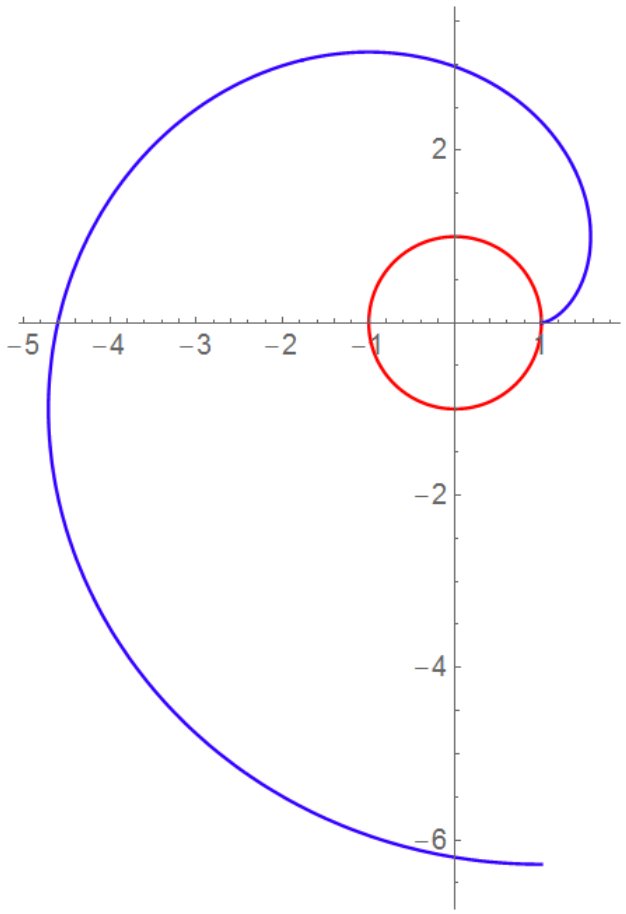


Figure 1.8. The involute of a circle.

1.4 The Dot Product

Definition 1.3

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two vectors in $\mathbb{R}^n$. The dot product of a and b is defined as

$$\mathbf{a} \cdot \mathbf{b} \equiv a_1 b_1 + a_2 b_2 + \ldots a_n b_n.$$

In $\mathbb{R}^3$, the analogous definition is

(1.4)

$$\mathbf{a} \cdot \mathbf{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3,$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

In Mathematica, there are two ways to compute the dot product, that is,

1. Using the period key between two vectors

```
a.b
```

2. Using Dot command

```
Dot[a,b]
```

Properties of dot products.

If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^n$ and $k \in \mathbf{R}$, then

1. a · a>=0

   a · a=0 if and only if a=0.

2. a · b=b · a

3. a · (b+c)= a · b + a · c

4. k (a · b)=(k a) · b=a · (k b)

Definition 1.4

The length of a vector $\mathbf{a} = (a_1, a_2, a_3)$, denoted $\| \mathbf{a} \|$ is defined as

(1.5)

$$\| \mathbf{a} \| \equiv \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} .$$

In Mathematica, we define the length of a vector using the dot product:

```
lengthn[v_]:=N[Sqrt[v.v]]

lengthn[a]

lengthn[b]
```

lengthn[c]

Throughout this book, we will be concerned only with vectors in $\mathbb{R}^2$ and vectors in $\mathbb{R}^3$.

Theorem 1.5

If a and b are any two vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, then the dot product of a and b is defined as

(1.6) $$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta \, .$$

where θ is the angle between a and b.

The angle between two vectors can be obtained from equation (1.6)

(1.7) $$\theta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} \, ,$$

where $0 \le \theta \le \pi$.

So, we can compute the angle between two vectors a and b in Mathematica using

θ=ArcCos[a.b/(lengthn[a] lengthn[b])]

Notice that the angle is in radians.

To convert this value to degrees, type

$\theta*180/\pi$

The notion of the angle between two vectors gives rise to the idea of the projection of b onto a, written as

(1.8) $\text{proj}_{\mathbf{a}}\,\mathbf{b}$

The magnitude of $\text{proj}_{\mathbf{a}}\,\mathbf{b}$ is defined as

$\|\mathbf{b}\|\cos\theta$,

where θ is the angle between a and b.

The direction of $\text{proj}_{\mathbf{a}}\,\mathbf{b}$ is either the same as that of a or opposite to a depending on the value of cos θ.

Thus, the magnitude is

(1.9)

$$\|\text{proj}_{\mathbf{a}}\,\mathbf{b}\| \equiv \|\mathbf{b}\|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$$

.

Hence, the projection of b onto a is

(1.10)

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\,\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}$$

.

1.5 The Cross Product of Vectors in $\mathbb{R}^3$

Definition 1.6

Let a and b be two vectors in $\mathbb{R}^3$. The cross product of a and b is denoted by $\mathbf{a} \times \mathbf{b}$. The length of $\mathbf{a} \times \mathbf{b}$ is defined as

(1.11)

$$\| \mathbf{a} \times \mathbf{b} \| = \| \mathbf{a} \| \, \| \mathbf{b} \| \sin \theta$$

,

where θ is the angle between a and b.

Note that the cross product of vectors only work on three-dimensional vectors.

There are two ways to compute the cross product of vectors in Mathematica:

1. Using the Cross command

2. Using the × sign

(Notes: to generate the × sign, press the Esc key, then type the word "cross" and followed by another Esc key.)

a={1,2,-1};

b={3,2,4};

Cross[a,b]

a × b

Properties of the cross products.

If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^3$ and $k \in \mathbf{R}$, then

1. a × b= - b × a (anti-commutativity)

2. a × (b+c)=a × b+a × c (distributivity)

3. (a+b) × c=a × c+b × c (distributivity)

4. k (a × b)=(k a) × b= a × (k b)

a={a1,a2,a3};

b={b1,b2,b3};

a × b

From the result above we see that

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - b_2 a_3,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

Written in vector form

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - b_2 a_3)\,\mathbf{i} + (a_3 b_1 - a_1 b_3)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k}$$

The cross products can also be written in determinant form as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.12}$$

Using Mathematica, it is straightforward to prove the properties of the cross products.

1.5.1 The Area of a Parallelogram

The area of a parallelogram spanned by two vectors a and b is given by

$$\| \mathbf{a} \times \mathbf{b} \| \tag{1.13}$$

```
a={-1,-2,0};

b={-3,1,0};

(* The area of parallelogram spanned by a and b *)

Sqrt[(Cross[a,b]).( Cross[a,b])]

lengthn[Cross[a,b]]
```

Alternatively, we can use the Norm command to compute the norm of a × b, that is $\|\mathbf{a} \times \mathbf{b}\|$.

```
Norm[Cross[a,b]]
```

1.5.2 The Volume of a Parallelepiped

The combination of dot product and cross product can be used to find the volume of a parallelepiped.

Let a parallelepiped be spanned by the vectors a, b, and c. The volume of the parallelepiped is defined as

(1.14)

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})| = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$$

The identity is known as triple scalar product.

```
a={3,-1,0};

b={-2,0,1};

c={1,-2,4};
```

```
(* The volume of the parallelepiped *)
```

Abs[a.Cross[b,c]]

Abs[b.Cross[c,a]]

Abs[c.Cross[a,b]]

Note that we have used the Abs command to compute the absolute value of the volume.

a={a1,a2,a3};

b={b1,b2,b3};

c={c1,c2,c3};

We can use Mathematica to verify the following identity

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

c.(Cross[a,b])==Det[{c,a,b}]//Simplify

Example 1.4. What is the volume of the parallelepiped with vertices (3,0,-1), (4,2,-1), (-1,1,0), (3,1,5), (0,3,0), (4,3,5), (-1,2,6), and (0,4,6)?

Solution:

Write the vertices as

a=(3,0,-1), b=(4,2,-1), c=(-1,1,0),

d=(3,1,5), e=(0,3,0), f=(4,3,5),

g=(-1,2,6), h=(0,4,6).

```
a={3,0,-1};

b={4,2,-1};

c={-1,1,0};

d={3,1,5};

e={0,3,0};

f={4,3,5};

g={-1,2,6};

h={0,4,6};
```

```
Graphics3D[{PointSize[0.03],Point[{a,b,c,d,e,f,g,h}],Text[Style["a",Blue,B
old,16],a+0.2],Text[Style["b",Blue,Bold,16],b+0.2],Text[Style["c",Blue,Bol
d,16],c+0.2],Text[Style["d",Blue,Bold,16],d+0.2],Text[Style["e",Blue,Bold,
16],e+0.2],Text[Style["f",Blue,Bold,16],f+0.2],Text[Style["g",Blue,Bold,16
],g+0.2],Text[Style["h",Blue,Bold,16],h+0.2],{Line[{{a,b},{a,c},{a,d}}]},
{Dashed,Line[{{e,c},{e,b},{h,g},{h,f},{d,g},{d,f},{c,g},{e,h},
{b,f}}]}},Axes->True,ImageSize->300]
```

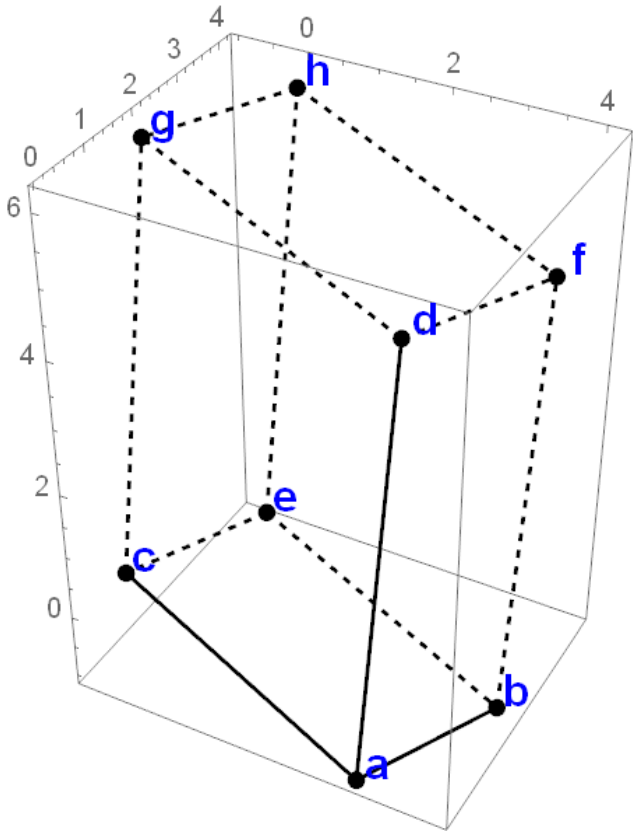


Figure 1.9. The parallelepiped with vertices (3,0,-1), (4,2,-1), (-1,1,0), (3,1,5), (0,3,0), (4,3,5), (-1,2,6), and (0,4,6).

We can find the volume of the parallelepiped by using equation (1.14):

$$\left|\left(\vec{ab} \times \vec{ac}\right) \cdot \vec{ad}\right| = 53$$

Abs[Cross[b-a, c-a].(d-a)]

1.6 Equations for Planes

1.6.1 Coordinate Equation of Planes

Definition 1.7

Let $P_0(x_0, y_0, z_0) \in \mathbb{R}^3$ be a point that lies in a plane Π and n=A i+B j+C k a normal vector to Π. Let P(x,y,z) be any point in the plane. Then Π is defined as the vector equation

(1.15) $$\mathbf{n} \cdot \overrightarrow{P_0 P} = 0 ,$$

where

$$\overrightarrow{P_0 P} = (x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k}$$

.

Thus, the equation of the plane Π may be written as

$$A\,(x - x_0) + B\,(y - y_0) + C\,(z - z_0) = 0$$

.

Equivalently,

(1.16) $$A\,x + B\,y + C\,z = D ,$$

where $D = A x_0 + B y_0 + C z_0$.

If C=0, then the normal vector n=A i+B j is parallel to the xy-plane. Hence the plane is a vertical plane. The equation of the plane becomes

$A(x - x_0) + B(y - y_0) = 0$.

Example 1.5. Find a set of parametric equations for the line of intersection of the planes x+2y-3z=5 and 5x+5y-z=1.

Solution:

We need to find:

(i) a point on the line of intersection

Such point must satisfy the system of simultaneous equations given by the two planes:

$$\begin{cases} x + 2y - 3z = 5 \\ 5x + 5y - z = 1 \end{cases}$$

Choose z=0, thus

x=-23/5, y=24/5, z=0.

Solve[{x+2y==5,5x+5y==1},{x,y}]

(ii) a vector parallel to the line of intersection

Since the line of intersection is perpendicular to the two normal vectors to the planes, a vector parallel to the line of intersection must also be perpendicular to both normal vectors to the planes.

The normal vectors to the planes can be read from the equations of the planes. These are

$\mathbf{n}_1 = (1, 2, -3)$,

and $\mathbf{n}_2 = (5, 5, -1)$, respectively.

Thus, a vector parallel to the line of intersection can be obtained from the cross product of $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\mathbf{n}_1 \times \mathbf{n}_2 = (1, 2, -3) \times (5, 5, -1)$$
$$= (13, -14, -5).$$

.

Cross[{1,2,-3},{5,5,-1}]

By equation (1.3), the vector parametric equation for the line of intersection is of the form

r(t)=b+t a.

r(t)=(-23/5,24/5,0)+t (13,-14,-5),

or

$$\begin{cases} x = 13\,t - \frac{23}{5}, \\ y = -14\,t + \frac{24}{5}, \\ z = -5\,t. \end{cases}$$

plot1=ContourPlot3D[{x+2y-3z==5,5x+5y-z==1},{x,-3,3},{y,-3,3},
{z,-5,3},ContourStyle->Opacity[0.4],Mesh->None]

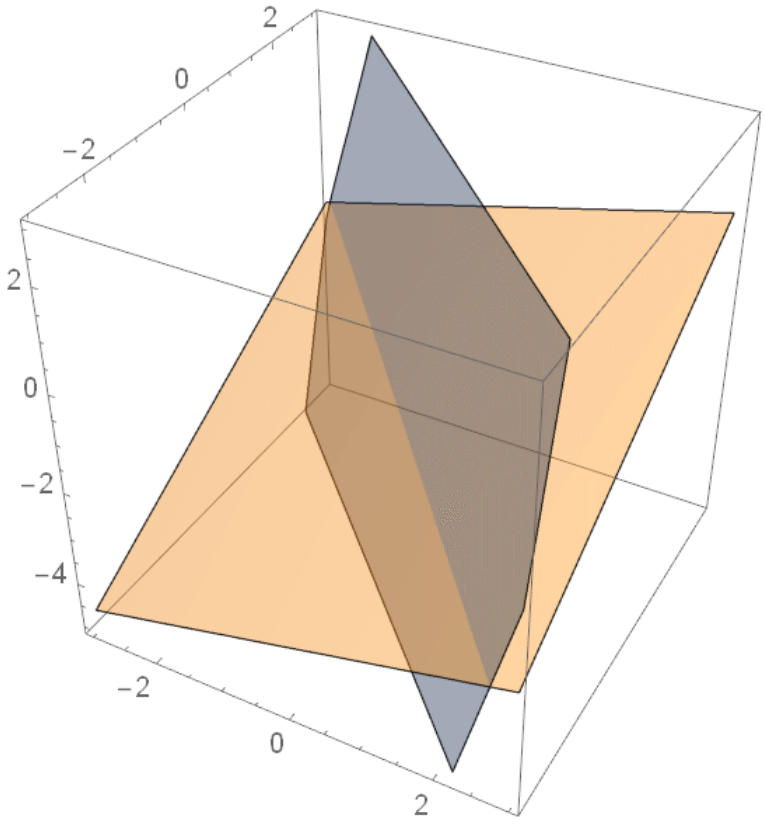


Figure 1.10. The planes x+2y-3z=5 and 5x+5y-z=1.

plot2=ParametricPlot3D[{13t-23/5,-14t+24/5,-5t},{t,-10,10},PlotStyle->
{Dashed,Hue[0]}]

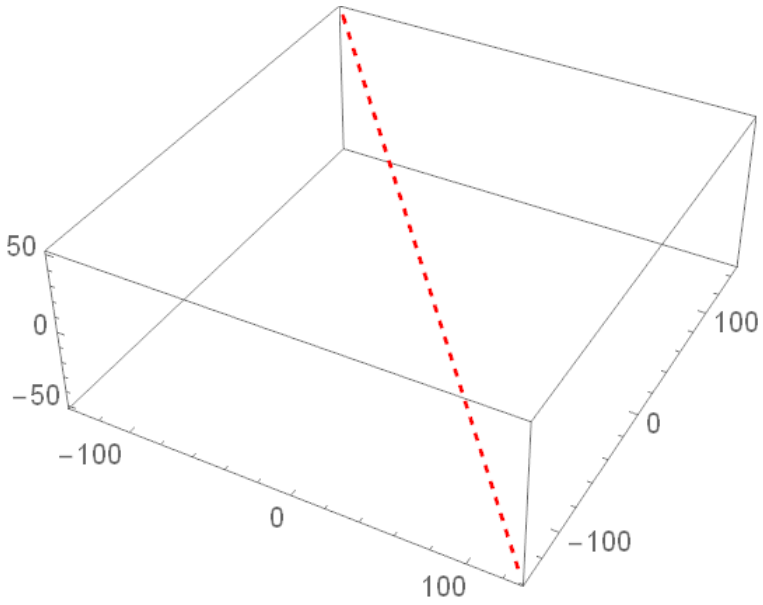


Figure 1.11. The intersection line of the planes x+2y-3z=5 and 5x+5y-z=1

Show[plot1,plot2]

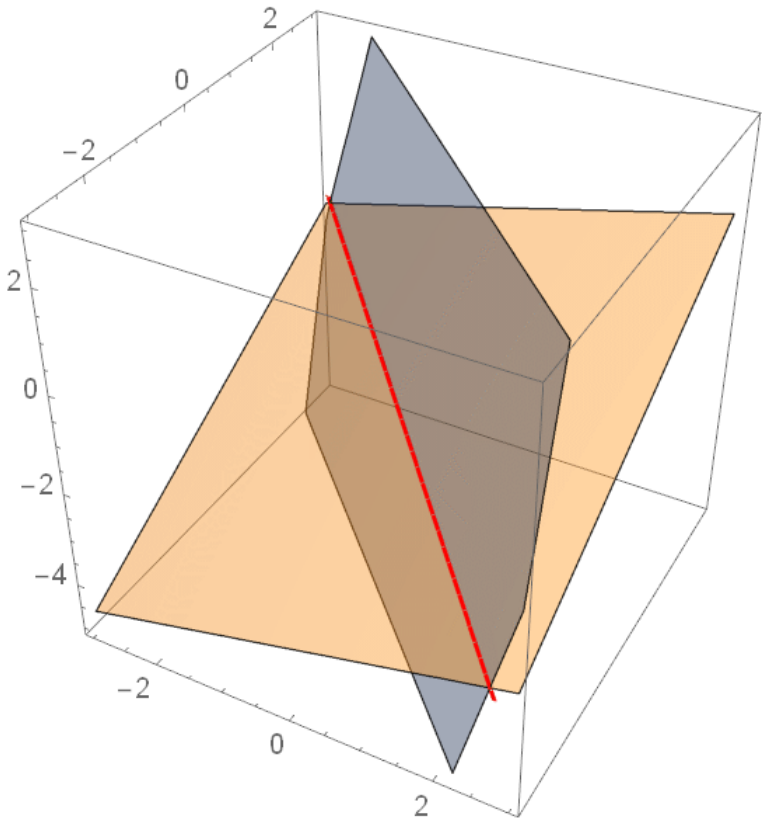


Figure 1.12. The planes x+2y-3z=5 and 5x+5y-z=1 and their intersection line.

1.6.2 Parametric Equations of Planes

Definition 1.8

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ be two nonzero, nonparallel vectors in $\mathbb{R}^3$. Let $P_0(c_1, c_2, c_3)$ be a point in the plane Π. A vector parametric equation for Π that is parallel to a and b is defined as

(1.17) $$\mathbf{r}(s, t) = s\,\mathbf{a} + t\,\mathbf{b} + \mathbf{c} \,,$$

where $s, t \in \mathbb{R}$, $\mathbf{r} \in \mathbb{R}^3$, and c is the position vector of P_0 .

Example 1.6. Find the parametric equation of a plane that contains the lines l_1 and l_2 , where

$$l_1 : x = 2t + 5,\ y = -3t - 6,\ z = 4t + 10$$

and

$$l_2 : x = 5t - 1,\ y = 10t + 3,\ z = 7t - 2$$

Solution:

Rewrite the equation of the lines as

$$l_1 : \begin{cases} x = 2t + 5, \\ y = -3t - 6, \\ z = 4t + 10 \end{cases} \qquad l_1 : \mathbf{r}_1(t) = (5, -6, 10) + t\,(2, -3, 4)$$

$$l_2 : \begin{cases} x = 5t - 1, \\ y = 10t + 3, \\ z = 7t - 2 \end{cases} \qquad l_1 : \mathbf{r}_2(t) = (-1, 3, -2) + t\,(5, 10, 7)$$

```
plot1=ParametricPlot3D[{{2t+5,-3t-6,4t+10},{5t-1,10t+3,7t-2}},
{t,-5,5},PlotStyle->{{Dashed,AbsoluteThickness[2]}},ViewPoint->
{1,5,10}]
```

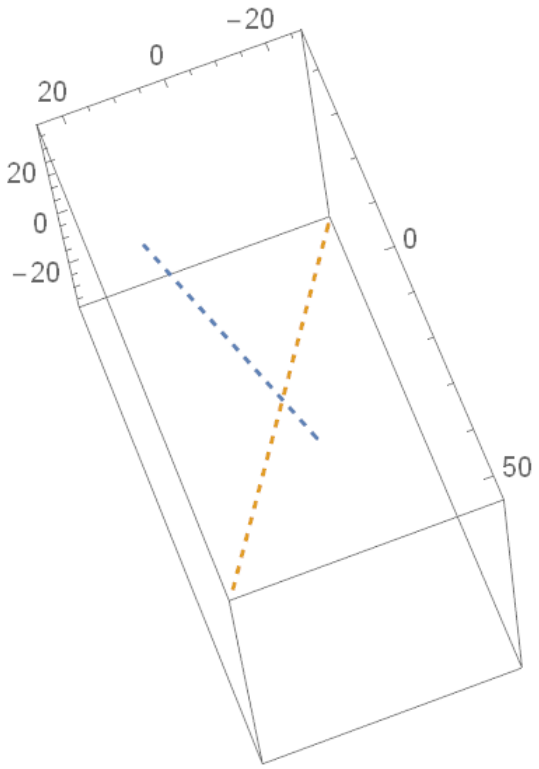


Figure 1.13. The lines l_1 and l_2 are

$$\mathbf{r}_1(t) = (5, -6, 10) + t\,(2, -3, 4)$$

and

$$\mathbf{r}_2(t) = (-1, 3, -2) + t\,(5, 10, 7)$$

, respectively.

The vector parallel to l_1 can be read from the equation, that is,

(2,-3,4).

Likewise, the vector parallel to l_2 is

(5,10,7).

We choose the point (5,-6,10) from l_1 as the point P_0 where the plane passes through.

By Definition 1.8, the vector parametric equation for the plane is defined as

r(s,t)=s a+t b+c.

So, the parametric equation of the plane is

r(s,t)=s (2,-3,4) +t (5,10,7)+(5,-6,10),

or in component form

$$\begin{cases} x = 2s + 5t + 5 \\ y = -3s + 10t - 6 \\ z = 4s + 7t + 10 \end{cases}$$

```
plot2=ParametricPlot3D[{2s+5t+5,-3s+10t-6,4s+7t+10},{s,-5,5},
{t,-5,5},PlotStyle->Opacity[0.3],Mesh->None];


Show[plot2,plot1,ViewPoint->{-20,50,10}]
```

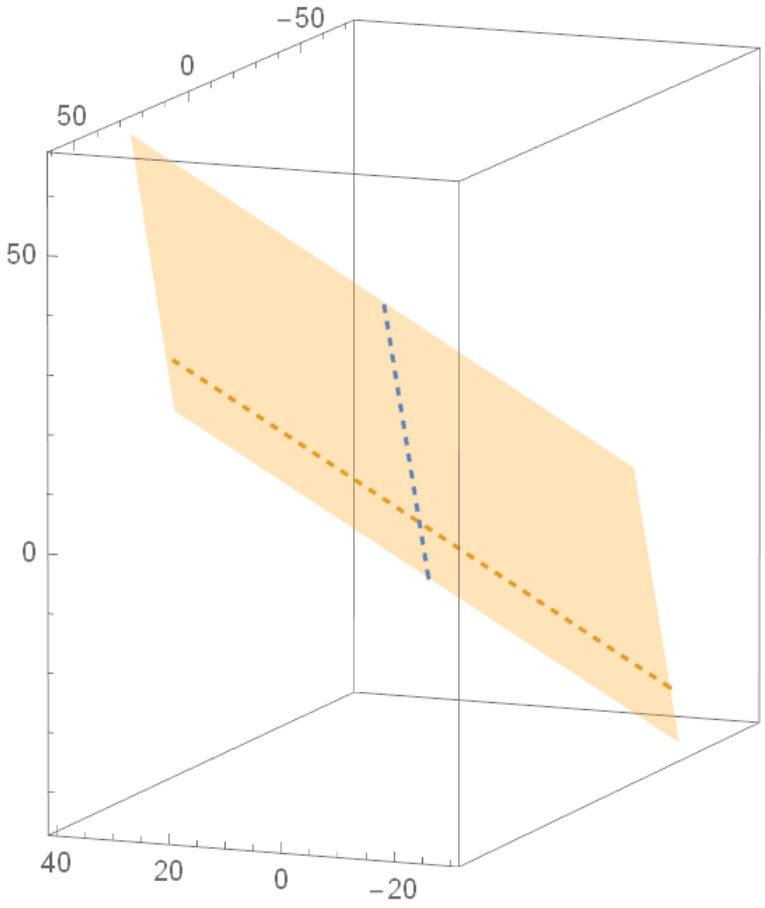


Figure 1.14. The plane that contains the lines

$$\mathbf{r}_1(t) = (5, -6, 10) + t\,(2, -3, 4)$$

and

$$\mathbf{r}_2(t) = (-1, 3, -2) + t\,(5, 10, 7)$$

.

1.7 Distance Examples

1.7.1 Distance between a point and a line

Example 1.7. Find the distance from the point (-11,10,20) to the line whose parametric equations are x=5-t, y=3, z=7t+8.

Solution:

Rewrite the parametric equations of the line in vector form:

$$\begin{cases} x = 5 - t, \\ y = 3, \\ z = 8 + 7t. \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix} t$$

.

By Definition 1.2, the line passes through the point B=(5,3,8), and a vector parallel to the line is a=(-1,0,7).

Let the point be P=(-11,10,20). So, the vector $\overrightarrow{BP}$ is

$$\overrightarrow{BP} = (-11, 10, 20) - (5, 3, 8) = (-16, 7, 12)$$

.

Let $\overrightarrow{BQ}$ be the projection of the vector $\overrightarrow{BP}$ onto the line (see Figure 1.15). By equation (1.10):

$$\overrightarrow{BQ} \equiv \text{proj}_{\mathbf{a}} \overrightarrow{BP} = \frac{\mathbf{a} \cdot \overrightarrow{BP}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} = \frac{(-1, 0, 7) \cdot (-16, 7, 12)}{(-1, 0, 7) \cdot (-1, 0, 7)} (-1, 0, 7)$$
$$= (-2, 0, 14).$$

.

{-1,0,7}.{-16,7,12}/{-1,0,7}.{-1,0,7} {-1,0,7}

So, the distance from the point P to the line is

$$\| \overrightarrow{QP} \| = \| \overrightarrow{BP} - \overrightarrow{BQ} \|$$
$$= \| (-16, 7, 12) - (-2, 0, 14) \| = \| (-14, 7, -2) \|$$
$$= \sqrt{249}$$
$$\simeq 15.7797.$$

Sqrt[{-14,7,-2}.{-14,7,-2}]

%//N

Module[{a,B,P,Q,r},

a={-1,0,7};

B={5,3,8};

P={-11,10,20};

Q=(a.(P-B))/a.a a;

r=(P-B)-Q;

```
line=ParametricPlot3D[{5-t,3,7t+8},{t,-1,5},PlotStyle->
{Green,AbsoluteThickness[3]},AxesLabel->{"x","y","z"}];

points=Graphics3D[{PointSize[0.05],Point[P],{Red,Point[B]}}];

arrows=Graphics3D[{AbsoluteThickness[2],Blue,Arrow[{B,P}],Red,Arro
w[{B,(B+a)}],Blue,Arrow[{B,(B+Q)}],{Dashed,Arrow[{B+Q,
(B+Q+r)}]}}];

texts=Graphics3D[{Text[Style["P",Bold],P*1.08],Text[Style["B",Bold],B*0
.8],Text[Style["Q",Bold],(B+Q)*1.08],Text[Style["a",Bold],(B+a)+
{0,-3,0}],Text[Style[ToString[lengthn[r]],Bold],0.55*(2B+2Q+r)]}];

Print[Style["The distance from P to the line: "
<>ToString[lengthn[r]],FontSize->14,Bold]];

Show[line,points,arrows,texts,PlotRange->All,Boxed->False,Axes-
>False,ViewPoint->{5,10,2}]

]
```

The distance from P to the line: 15.7797

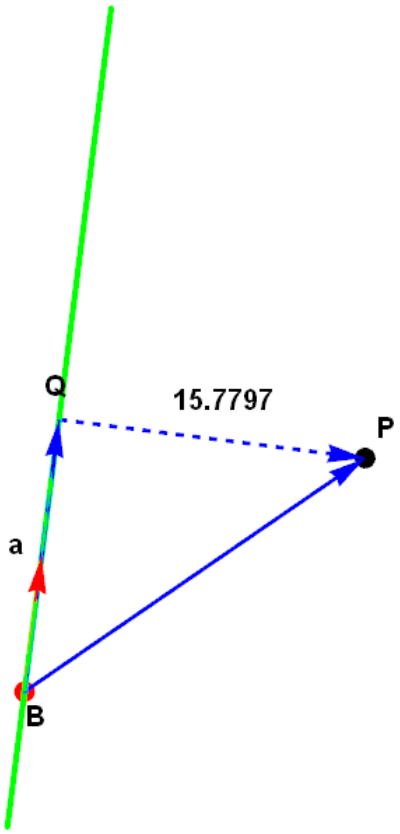


Figure 1.15. The distance from the point P=(-11,10,20) to the line x=5-t, y=3, z=7t+8

1.7.2 Distance between a point and a plane

Example 1.8. Find the distance from the point P=(2,8,5) to the plane x-2y-2z=1.

Solution:

By Definition 1.7 the normal vector to the plane is n=(1,-2,-2).

Let us choose a point R in the plane. Write R=(1,0,0). So,

$$\overrightarrow{PR} = (1, 0, 0) - (2, 8, 5) = (-1, -8, -5)$$

.

Let $\overrightarrow{PQ}$ be the projection of the vector $\overrightarrow{PR}$ onto the vector n (see Figure 1.16), thus by equation (1.10):

$$\overrightarrow{PQ} \equiv \text{proj}_{\mathbf{n}} \overrightarrow{PR} = \frac{\mathbf{n} \cdot \overrightarrow{PR}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = \frac{(1,-2,-2) \cdot (-1, -8, -5)}{(1,-2,-2) \cdot (1,-2,-2)} (1, -2, -2)$$

$$= \left(\frac{25}{9}, -\frac{50}{9}, -\frac{50}{9}\right)$$

{1,-2,-2}.{-1,-8,-5}/{1,-2,-2}.{1,-2,-2} {1,-2,-2}

So, the distance from the point P to the plane is

$$\| \overrightarrow{PQ} \| = \left\| \left(\frac{25}{9}, -\frac{50}{9}, -\frac{50}{9}\right) \right\| = \frac{25}{3} \simeq 8.33333$$

.

Sqrt[{25/9,-(50/9),-(50/9)}.{25/9,-(50/9),-(50/9)}]

lengthn[{25/9,-(50/9),-(50/9)}]

Module[{n,R,P,Q,r},

n={1,-2,-2};

```
R={1,0,0};

P={2,8,5};

Q=(n.(R-P))/n.n n;

plane=ContourPlot3D[x-2y-2z==1,{x,-6,8},{y,-6,8},{z,-8,8},Mesh-
>None,ContourStyle->{Green,Opacity[0.4]},AxesLabel->{"x","y","z"}];

points=Graphics3D[{PointSize[0.03],Point[P],{Red,Point[R]}}];

arrows=Graphics3D[{AbsoluteThickness[2],Blue,Arrow[{P,R}],Red,Arrow
[{R,(R+n)}],Blue,Dashed,Arrow[{P,(P+Q)}]}];

texts=Graphics3D[{Text[Style["P",Bold],P*1.08],Text[Style["R",Bold],R+
{0.6,0.5,0.1}],Text[Style["Q",Bold],(P+Q)*1.08],Text[Style["n",Bold],
(R+n)+{0,-0.5,0}],Text[Style[ToString[lengthn[Q]],Bold],0.5*(2P+Q)]}];

Print[Style["The distance from P to the plane: "
<>ToString[lengthn[Q]],FontSize->14,Bold]];

Show[plane,points,arrows,texts,PlotRange->All,ViewPoint->{8,-6,15}]

]
```

The distance from P to the plane: 8.33333

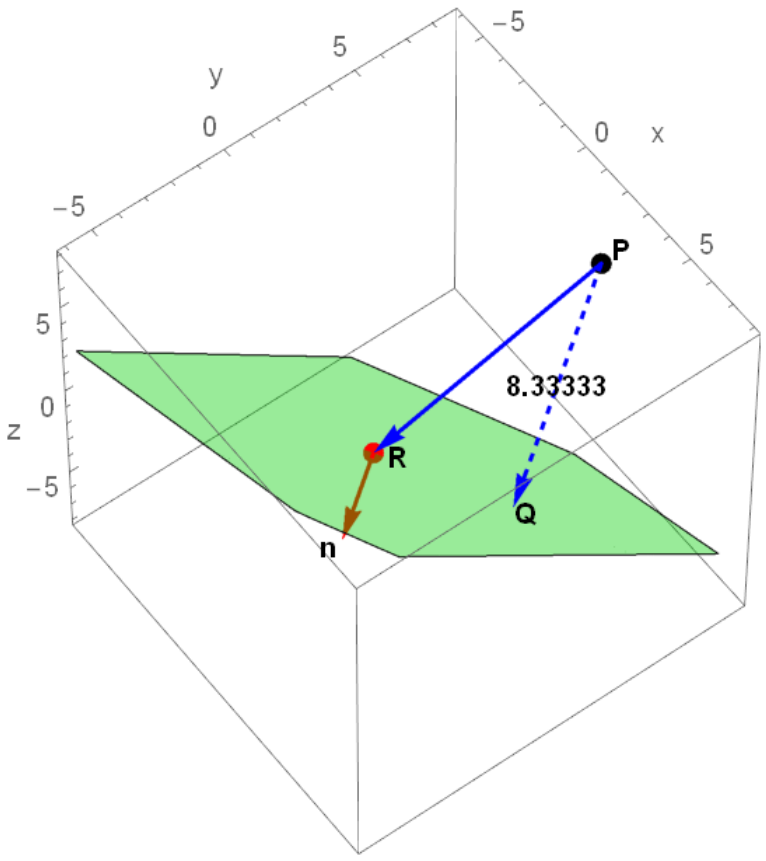


Figure 1.16. The distance from the point P=(2,8,5) to the plane x-2y-2z=1.

1.7.3 Distance between parallel planes

Example 1.9. Find the distance between the two planes given by the equations x-3y+2z=1 and x-3y+2z=8.

Solution:

From the equation of the two planes we see that the normal vector to the planes are the same, that is,

n=(1,-3,2).

Hence, the two planes are parallel.

Choose the points P_1 and P_2 from the two planes x-3y+2z=1 and x-3y+2z=8, respectively by setting x=y=0

$P_1 = (0, 0, 1/2)$ and $P_2 = (0, 0, 4)$.

So,

$\overrightarrow{P_1 P_2} = (0, 0, 7/2)$.

The projection of $\overrightarrow{P_1 P_2}$ onto the vector n by equation (1.10) is

$$\text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2} = \left(\frac{\mathbf{n} \cdot \overrightarrow{P_1 P_2}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = \left(\frac{1}{2}, -\frac{3}{2}, 1\right)$$

.

(* Compute the projection of $\overrightarrow{P_1 P_2}$ onto the vector n *)

{1,-3,2}.{0,0,7/2}/{1,-3,2}.{1,-3,2} {1,-3,2}

The desired distance is given by

$$\| \text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2} \| = \sqrt{7/2} \simeq 1.87083$$

.

(* The length of $\text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2}$ *)

```
Sqrt[{1/2,-(3/2),1}.{1/2,-(3/2),1}]

%//N


Module[{P1,P2,n,projP1P2,d},

P1={0,0,1/2};

P2={0,0,4};

n={1,-3,2};

projP1P2=(n.(P2-P1)/n.n)n;

d=Sqrt[projP1P2.projP1P2]//N;

points=Graphics3D[{PointSize[0.025],{Blue,Point[P1]},
{Red,Point[P2]}}];

arrows=Graphics3D[{Arrowheads[0.03],Arrow[{P1,P2}],
{Dashed,Arrow[{P1,P1+projP1P2}]}}];

texts=Graphics3D[{Text[Style["P1",Bold],P1+
{0.2,0.4,-0.4}],Text[Style["P2",Bold],P2+{0.2,0.4,0.4}]}];

planes=ContourPlot3D[{x-3y+2z==1,x-3y+2z==8},{x,-5,5},{y,-5,5},
{z,0,5},ContourStyle->Opacity[0.2],Mesh->None];

Print[Style["The distance between the two planes: "
<>ToString[d],FontSize->14,Bold]];

Show[points,arrows,texts,planes,Boxed->False,ViewPoint->{8,-3,-3}]

]
```

The distance between the two planes: 1.87083

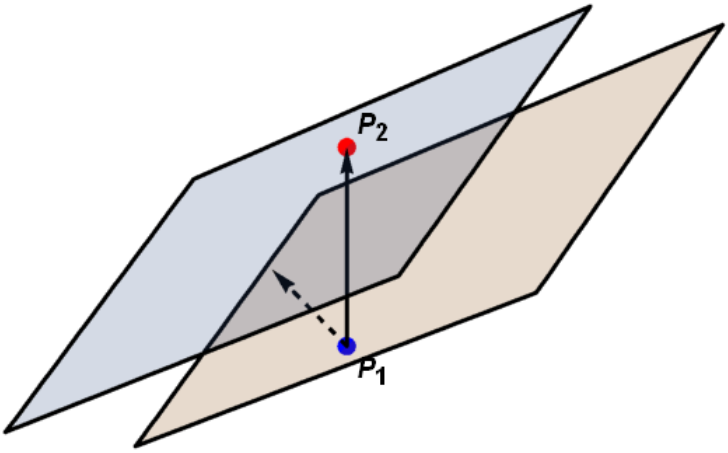


Figure 1.17. The distance between two parallel planes x-3y+2z=1 and x-3y+2z=8.

1.7.4 Distance between two skew lines

Two lines in $\mathbb{R}^3$ are called skew lines if they are neither intersecting nor parallel.

Example 1.10. Determine the distance between the two lines

$$l_1(t) = t\begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 15 \end{pmatrix}$$

and

$$l_2(t) = t\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}.$$

Solution:

lines=ParametricPlot3D[{{-1,3,15}+t{8,-1,0},{0,3,4}+t{0,3,1}},
{t,-5,5},AxesLabel->{"x","y","z"},ViewPoint->{30,-30,15}]

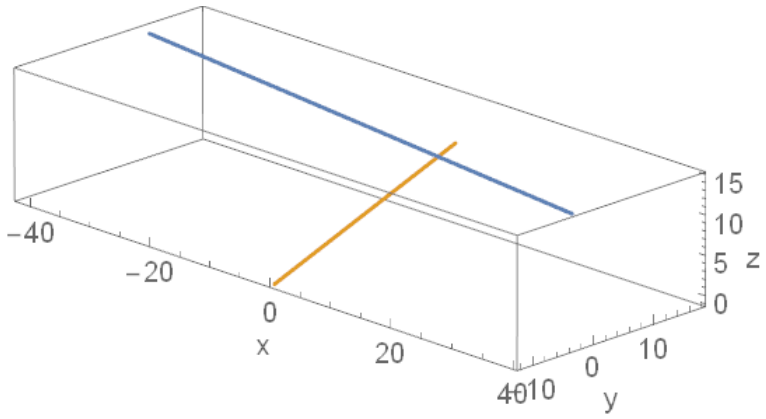


Figure 1.18. The two skew lines l_1 and l_2 .

Note that the two lines are neither intersecting nor parallel. Thus they are two skew lines.

From the vector parametric equations for the lines, we see that the points $B_1 = (-1, 3, 15)$ and $B_2 = (0, 3, 4)$ are on the first and second line, respectively. Hence,

$$\overrightarrow{B_1 B_2} = (0, 3, 4) - (-1, 3, 15) = (1, 0, -11)$$

.

For a vector n that is perpendicular to both lines, we may use $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2$, where $\mathbf{a}_1 = (8, -1, 0)$ is a vector parallel to the first line and $\mathbf{a}_2 = (0, 3, 1)$ is a vector parallel to the second line.

So,

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 = (-1, -8, 24)$$

.

Cross[{8,-1,0},{0,3,1}]

The projection of the vector $\overrightarrow{B_1 B_2}$ onto the normal vector n by equation (1.10) is

$$\text{proj}_{\mathbf{n}} \overrightarrow{B_1 B_2} = \left(\frac{\mathbf{n} \cdot \overrightarrow{B_1 B_2}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{265}{641}, \frac{2120}{641}, -\frac{6360}{641} \right)$$

.

(* Compute the projection of $\overrightarrow{B_1 B_2}$ onto the vector n *)

{-1,-8,24}.{1,0,-11}/{-1,-8,24}.{-1,-8,24} {-1,-8,24}

The desired distance is given by

$$\| \text{proj}_{\mathbf{n}} \overrightarrow{B_1 B_2} \| = \frac{265}{\sqrt{641}} \simeq 10.4669$$

.

(* The length of $\text{proj}_{\mathbf{n}} \overrightarrow{B_1 B_2}$ *)

```
Sqrt[{265/641,2120/641,-(6360/641)}.{265/641,2120/641,-(6360/641)}]

%//N


Module[{a1,B1,a2,B2,n,projB1B2,d},

a1={8,-1,0};

B1={-1,3,15};

a2={0,3,1};

B2={0,3,4};

n=Cross[a1,a2];

projB1B2=(n.(B2-B1)/n.n)n;

d=Sqrt[projB1B2.projB1B2]//N;

points=Graphics3D[{PointSize[0.025],{Blue,Point[B1]},
{Red,Point[B2]}}];

arrow=Graphics3D[{AbsoluteThickness[2],Arrowheads[0.02],Arrow[{B1,
B2}],{Dashed,Arrow[{B1,B1+projB1B2}]}}];

texts=Graphics3D[{Text[Style["B1",Bold],B1*1.2],Text[Style["B2",Bold],
B2*0.6]}];

Print[Style["The distance between the two skew lines: "
<>ToString[d],FontSize->14,Bold]];

Show[lines,points,arrow,texts,PlotRange->All,ViewPoint->
{30,-30,15},ImageSize->500]
```

]

The distance between the two skew lines: 10.4669

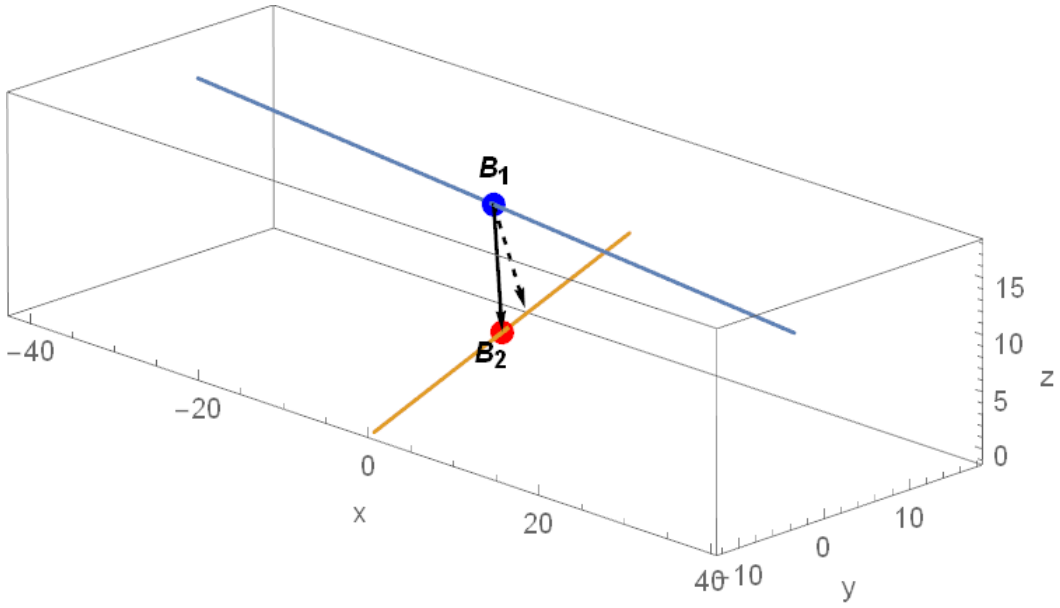


Figure 1.19. The distance between the two skew lines l_1 and l_2

1.8 Curvilinear Coordinates

A point in the plane and space can be labeled by Cartesian coordinates (x,y) and (x,y,z), respectively. Cartesian coordinates are also called rectangular coordinates. However, sometimes it is more convenient to use curvilinear coordinates such as polar coordinates (r,θ) in the plane; cylindrical (r,ϕ,z) and spherical coordinates (r,θ,ϕ) in space.

1.8.1 Polar Coordinates

In polar coordinates, the position vector of a point P is determined by r, the distance from the origin to the point P, and by θ, the angle that the vector makes with the x-axis.

```
Module[{o,p,er,eθ,arrow,point,frame,text},
o={0,0};
p={3,2};
er=(p-o)/lengthn[p-o];
eθ={{0,-1},{1,0}}.er;
point=Graphics[{Red,PointSize[0.04],Point[p]}];
arrow=Graphics[{Dashed,Arrow[{o,p}]}];
frame=Graphics[{Hue[0.7],Arrow[{p,p+er}],Arrow[{p,p+eθ}]}];
```

```
text=Graphics[{Text[Style["P",Italic,FontSize->Scaled[0.04]],p+
{0,-0.3}],Text[Style["r",Bold,FontSize->Scaled[0.04]],
{1.8,1.5}],Text[Style["θ",FontSize->Scaled[0.04]],
{0.6,0.2}],Text[Style["er",FontSize-
>Scaled[0.03]],p+er*1.15],Text[Style["eθ",FontSize-
>Scaled[0.03]],p+eθ*1.15]}];

Show[point,arrow,frame,text,Axes->True,AxesLabel->{"x","y"}]

]
```

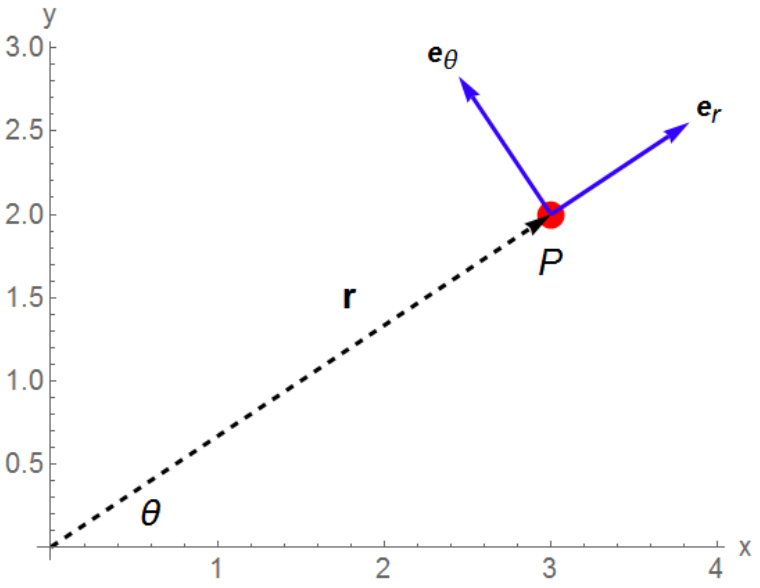


Figure 1.20. A point P as described by polar coordinates with basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

The relations between polar and Cartesian coordinates are given by

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases} \tag{1.18}$$

$$\text{(1.19)} \quad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \tan^{-1} \frac{y}{x}. \end{cases}$$

Just like a point P in the plane can be described by two orthogonal unit vectors i and j in Cartesian coordinates, the point P can also be described by two orthogonal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (see Figure 1.20). Observe that the directions of increasing r and θ are determined by $\mathbf{e}_r$ and $\mathbf{e}_\theta$, respectively. Thus, it is natural to define the unit vector $\mathbf{e}_r$ to be the direction of vector r in increasing r divided by its length:

$$\mathbf{e}_r \equiv \frac{\partial \mathbf{r}}{\partial r} \Big/ \left\| \frac{\partial \mathbf{r}}{\partial r} \right\| .$$

Similarly for the unit vector $\mathbf{e}_\theta$:

$$\mathbf{e}_\theta \equiv \frac{\partial \mathbf{r}}{\partial \theta} \Big/ \left\| \frac{\partial \mathbf{r}}{\partial \theta} \right\| .$$

With these definitions we can find the expression for the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in terms of i and j.

Since

r=x i+y j,

r=r cos θ i +r sin θ j.

Hence,

$$\text{(1.20)} \quad \begin{aligned} \mathbf{e}_r &= \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}, \\ \mathbf{e}_\theta &= -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j}. \end{aligned}$$

Thus, a position vector r in polar coordinates becomes

(1.21) $\mathbf{r} = r\,\mathbf{e}_r$.

Notice that the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are different from point to point because of the dependency of the angle θ which is usually a function of parameter t such as time in many physical applications.

When we differentiate the unit vectors with respect to θ we obtain

$$\frac{d\mathbf{e}_r}{d\theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \equiv \mathbf{e}_\theta,$$
$$\frac{d\mathbf{e}_\theta}{d\theta} = -\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} \equiv -\mathbf{e}_r.$$

Taking the derivatives of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ with respect to t gives

(1.22)

$$\dot{\mathbf{e}}_r \equiv \frac{d\mathbf{e}_r}{dt} = \frac{d\mathbf{e}_r}{d\theta}\,\frac{d\theta}{dt} = \dot{\theta}\,\mathbf{e}_\theta,$$
$$\dot{\mathbf{e}}_\theta \equiv \frac{d\mathbf{e}_\theta}{dt} = \frac{d\mathbf{e}_\theta}{d\theta}\,\frac{d\theta}{dt} = -\dot{\theta}\,\mathbf{e}_r,$$

,

where dot denotes differentiation with respect to t.

1.8.2 Cylindrical Coordinates

Polar coordinates in the plane can be extended to three dimensions in a straightforward manner. The polar coordinates are still used in place of the x- and y-coordinates and in addition a z-coordinate, which is treated in a Cartesian like manner. A point P in space is determined by the r and θ coordinates of its projection in the xy-plane and the z-coordinates. Thus, r is

the distance from P to the z-axis and θ is the angle between the projection in the xy-plane and the positive x-axis (see Figure 1.21).

```
Module[{o,p,θ,er,eθ,ez,arrow,point,line,frame,text},

o={0,0,0};

p={2,2,1};

θ=ArcTan[p[[2]]/p[[1]]];

er={Cos[θ],Sin[θ],0};

eθ={-Sin[θ],Cos[θ],0};

ez={0,0,1};

point=Graphics3D[{Red,PointSize[0.04],Point[p]}];

arrow=Graphics3D[{AbsoluteThickness[1],Dashed,Arrow[{{0,0,0},
{p[[1]],p[[2]],0}}],Arrow[{{0,0,p[[3]]},{p[[1]],p[[2]],p[[3]]}}]}];

line=Graphics3D[{AbsoluteThickness[1],Dashed,Line[{{p[[1]],p[[2]],0},
{p[[1]],p[[2]],p[[3]]}}]}];

frame=Graphics3D[{Hue[0.7],Arrow[{p,p+er}],Arrow[{p,p+eθ}],Arrow[{p,
p+ez}]}];

text=Graphics3D[{Text[Style["P",Italic,FontSize->Scaled[0.04]],p+
{0.3,-0.2,0}],Text[Style["r",Bold,FontSize->Scaled[0.04]],
{1.8,1.4,0}],Text[Style["θ",FontSize->Scaled[0.04]],
{0.8,0.3,0}],Text[Style["er",FontSize-
>Scaled[0.03]],p+er*1.2],Text[Style["eθ",FontSize-
>Scaled[0.03]],p+eθ*1.2],Text[Style["ez",FontSize-
>Scaled[0.03]],p+ez*1.2]}];
```

Show[point,arrow,line,frame,text,Axes->True,AxesStyle->Directive[AbsoluteThickness[1]],AxesOrigin->{0,0,0},Boxed->False,Ticks->None,AxesLabel->{"x","y","z"},LabelStyle->Directive[Bold,Blue]]

]

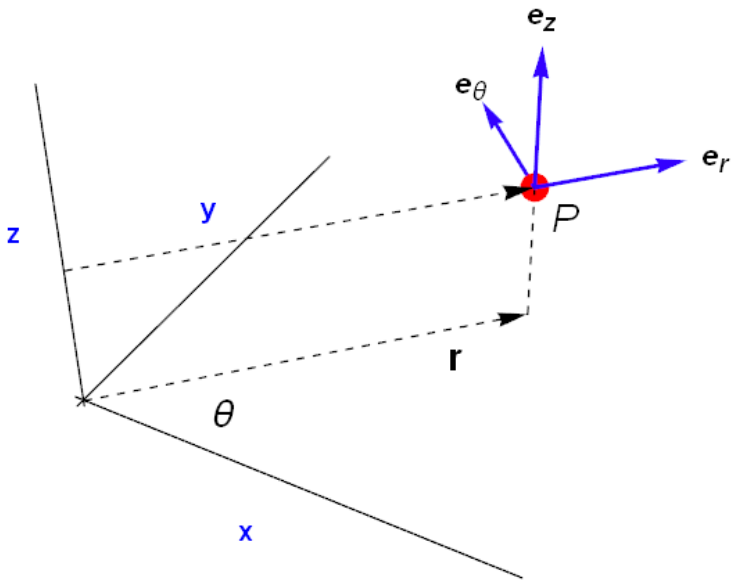


Figure 1.21. A point P as described in cylindrical coordinates with basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$.

The relations between cylindrical and Cartesian coordinates are given by

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z. \end{cases} \tag{1.23}$$

$$(1.24)\quad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \tan^{-1}\left(\frac{y}{x}\right), \\ z = z. \end{cases}$$

We can extend our results from polar coordinates to cylindrical coordinates. The three orthogonal unit vectors are $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$, expressed in terms of i, j, k, the basis vectors of Cartesian coordinates:

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \\ \mathbf{e}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \\ \mathbf{e}_z &= \mathbf{k}. \end{aligned} \quad (1.25)$$

and their derivatives:

$$\begin{aligned} \frac{d\mathbf{e}_r}{d\theta} &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \equiv \mathbf{e}_\theta, \\ \frac{d\mathbf{e}_\theta}{d\theta} &= -\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} \equiv -\mathbf{e}_r, \\ \frac{d\mathbf{e}_z}{d\theta} &= 0. \end{aligned}$$

Taking the derivatives of $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ with respect to t, we obtain

(1.26)

$$\begin{aligned} \dot{\mathbf{e}}_r &\equiv \frac{d\mathbf{e}_r}{dt} = \frac{d\mathbf{e}_r}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\,\mathbf{e}_\theta, \\ \dot{\mathbf{e}}_\theta &\equiv \frac{d\mathbf{e}_\theta}{dt} = \frac{d\mathbf{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\,\mathbf{e}_r, \\ \dot{\mathbf{e}}_z &\equiv \frac{d\mathbf{e}_z}{dt} = 0, \end{aligned}$$

where dot denotes differentiation with respect to t.

Since

r=x i+y j+z k,

the position vector r in cylindrical coordinates is represented by

(1.27) $\mathbf{r} = r\,\mathbf{e}_r + z\,\mathbf{e}_z$.

1.8.3 Spherical Coordinates

In spherical coordinates, a point P is determined by r, the distance from P to the origin; θ, the angle between the positive z-axis and the position vector of P; and ϕ, the angle between the projection of the position vector in the xy-plane and the positive x-axis (see Figure 1.22).

The range of values for r, θ and ϕ are

r>=0, 0<=θ<=π, 0<=ϕ<=2π, respectively.

```
Module[{o,p,θ,ϕ,er,eθ,eϕ,arrow,point,line,frame,text},
o={0,0,0};
p={2,2,1};
θ=ArcTan[Sqrt[p[[1]]^2+p[[2]]^2]/p[[3]]];
ϕ=ArcTan[p[[2]]/p[[1]]];
er={Sin[θ] Cos[ϕ],Sin[θ]Sin[ϕ],Cos[θ]};
eθ={Cos[θ] Cos[ϕ],Cos[θ]Sin[ϕ],-Sin[θ]};
eϕ={-Sin[ϕ],Cos[ϕ],0};
```

```
point=Graphics3D[{Red,PointSize[0.04],Point[p]}];

arrow=Graphics3D[{Dashed,Arrow[{o,p}]}];

line=Graphics3D[{AbsoluteThickness[1],Dashed,Line[{{0,0,0},
{p[[1]],p[[2]],0}}],Line[{{p[[1]],p[[2]],0},{p[[1]],p[[2]],p[[3]]}}]}];

frame=Graphics3D[{Hue[0.7],Arrow[{p,p+er}],Arrow[{p,p+eθ}],Arrow[{p,
p+eϕ}]}];

text=Graphics3D[{Text[Style["P",Italic,FontSize->Scaled[0.04]],p+
{0.3,-0.1,0}],Text[Style["r",Bold,FontSize->Scaled[0.04]],
{1.8,1,0.9}],Text[Style["θ",FontSize->Scaled[0.04]],
{0.2,0.2,0.3}],Text[Style["ϕ",FontSize->Scaled[0.04]],
{0.8,0.3,0}],Text[Style["er",FontSize-
>Scaled[0.03]],p+er*1.2],Text[Style["eθ",FontSize-
>Scaled[0.03]],p+eθ*1.2],Text[Style["eϕ",FontSize-
>Scaled[0.03]],p+eϕ*1.2]}];

Show[point,arrow,line,frame,text,Axes->True,AxesStyle-
>Directive[AbsoluteThickness[1]],AxesOrigin->{0,0,0},Boxed-
>False,Ticks->None,AxesLabel->{"x","y","z"},LabelStyle-
>Directive[Bold,Blue]]

]
```

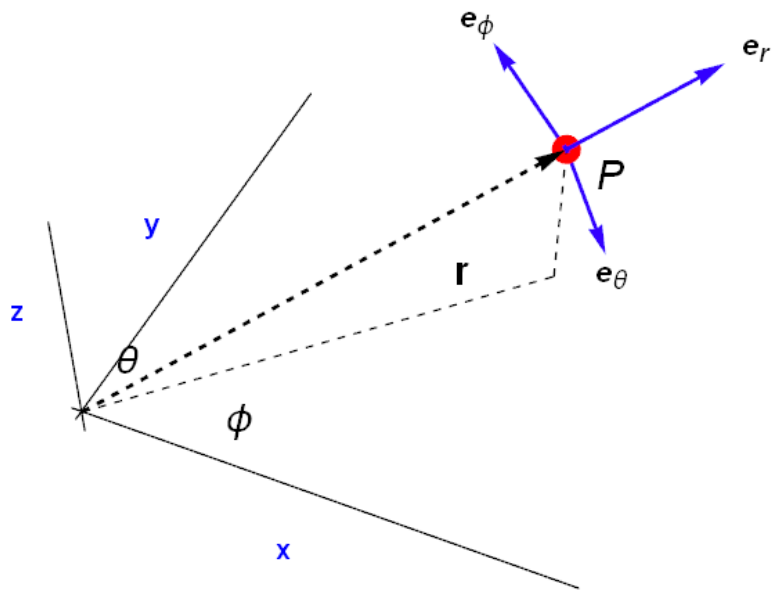


Figure 1.22. A point P as described in spherical coordinates with basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$.

The relations between spherical and Cartesian coordinates are given by

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta. \end{cases} \tag{1.28}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2}, \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right), \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right). \end{cases} \tag{1.29}$$

The three orthogonal unit vectors are defined as

$$\mathbf{e}_r \equiv \frac{\partial \mathbf{r}}{\partial r} \Big/ \left\| \frac{\partial \mathbf{r}}{\partial r} \right\|,$$

$$\mathbf{e}_\theta \equiv \frac{\partial \mathbf{r}}{\partial \theta} \Big/ \left\| \frac{\partial \mathbf{r}}{\partial \theta} \right\|,$$

$$\mathbf{e}_\phi \equiv \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left\| \frac{\partial \mathbf{r}}{\partial \phi} \right\|.$$

By using the above definitions the basis vectors can be obtained:

(1.31)

$$\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},$$

$$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},$$

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.$$

Since

r=x i + y j + z k,

the position vector r is represented by

r=r sin θ cos ϕ i + r sin θ sin ϕ j + r cos θ k.

Thus,

(1.30) $\mathbf{r} = r\,\mathbf{e}_r$.

Just as unit vectors in polar and cylindrical coordinates, $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are different from point to point because of the dependency of the angle θ and ϕ. In general, θ and ϕ are functions of t such as time in many physical applications.

The derivatives can be computed by hands though it is somewhat more involved. So, we use Mathematica to do some calculations.

First, we define the three orthogonal unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$:

```
Clear[θ,ϕ,er,eθ,eϕ]

er[t_]:={Sin[θ[t]] Cos[ϕ[t]],Sin[θ[t]] Sin[ϕ[t]],Cos[θ[t]]}

eθ[t_]:={Cos[θ[t]] Cos[ϕ[t]],Cos[θ[t]] Sin[ϕ[t]],-Sin[θ[t]]}

eϕ[t_]:={-Sin[ϕ[t]],Cos[ϕ[t]],0}
```

Next, we compute $\dot{\mathbf{e}}_r$:

```
er'[t]
```

(1.32)

$$\dot{\mathbf{e}}_r = \left(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi},\ \cos\theta\sin\phi\,\dot{\theta} + \cos\phi\sin\theta\,\dot{\phi},\ -\sin\theta\,\dot{\theta}\right)$$

.

Written in vector form:

$$\dot{\mathbf{e}}_r = \dot{\theta}\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix} + \sin\theta\,\dot{\phi}\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

.

But from equation (1.31) $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are

$\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}$ and $\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$, respectively.

Therefore,

(1.33) $\dot{\mathbf{e}}_r = \dot{\theta}\,\mathbf{e}_\theta + \sin\theta\,\dot{\phi}\,\mathbf{e}_\phi$.

Similarly, we compute $\dot{\mathbf{e}}_\theta$:

eθ'[t]

$\dot{\mathbf{e}}_\theta = \left(-\sin\theta\cos\phi\,\dot{\theta} - \cos\theta\sin\phi\,\dot{\phi},\ -\sin\theta\sin\phi\,\dot{\theta} + \cos\theta\cos\phi\,\dot{\phi},\ -\cos\theta\,\dot{\theta}\right)$

.

In vector form:

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} + \cos\theta\,\dot{\phi}\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

.

Therefore,

(1.34) $\dot{\mathbf{e}}_\theta = -\dot{\theta}\,\mathbf{e}_r + \cos\theta\,\dot{\phi}\,\mathbf{e}_\phi$.

Similarly for $\dot{\mathbf{e}}_\phi$

eϕ'[t]

$$\dot{\mathbf{e}}_\phi = \left(-\cos\phi\,\dot{\phi},\ -\sin\phi\,\dot{\phi},\ 0\right)$$

.

In vector form:

$$\dot{\mathbf{e}}_\phi = -\sin\theta\,\dot{\phi}\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} - \cos\theta\,\dot{\phi}\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix}$$

.

Therefore,

(1.35)

$$\dot{\mathbf{e}}_\phi = -\sin\theta\,\dot{\phi}\,\mathbf{e}_r - \cos\theta\,\dot{\phi}\ \mathbf{e}_\theta$$

.

Chapter 2. Curves

In this chapter, we study the differential geometry of curves in the plane and in space.

A curve in $\mathbb{R}^2$ is assigned a set of two coordinates associated with two unit vectors in the plane:

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\};$$

whereas a curve in $\mathbb{R}^3$, a set of three coordinates associated with three unit vectors in space:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

.

There are invariant properties that characterize curves. These include: arc length, curvature, and torsion. The discussion in this chapter leads to the famous Frenet formulas for curves.

Initialization

```
Clear["Global`*"]
SetOptions[ParametricPlot3D,PlotStyle->AbsoluteThickness[2]];
SetOptions[ParametricPlot,PlotStyle->AbsoluteThickness[2]];
```

2.1 Parametric Curves

Suppose a particle is moving in space such that its position vector at any time t is defined by the coordinates x(t),y(t), and z(t). We denote the position vector of the curve by

r(t)=(x(t),y(t),z(t)).

Assume that the component functions x(t), y(t) and z(t) are differentiable, that is, they can be differentiated any number of times. The curve is said to be smooth.

Definition 2.1

A parametrized curve in $\mathbb{R}^n$ is a smooth function $\mathbf{r} : \mathcal{I} \to \mathbb{R}^n$, where $\mathcal{I} \subset \mathbb{R}$ is an interval.

An interval can be of the form [a,b],[a,b),(a,b],[a,∞),(a,∞),(∞,b],(-∞,b),or (-∞,∞), where $a, b \in \mathbb{R}$. In this book, we are primarily concerned with n=2 for curves in the plane and n=3 for curves in space.

Definition 2.2

The velocity vector v of a particle moving in a curve $\mathbf{r} : \mathcal{I} \to \mathbb{R}^n$ at time $t \in \mathcal{I}$ is defined as

$$\mathbf{v}(t) \equiv \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

.

The speed v(t) of the particle is defined as the magnitude of the velocity vector, that is

$v(t) = \| \mathbf{r}'(t) \|$.

A curve is regular if its velocity vector is a continuous function and its speed is nonzero.

Furthermore, a(t)=r"(t) is called the acceleration vector of the particle moving in the curve, provided these derivatives exist.

Here are some example of curves:

Example 2.1. A circle of radius 1

The parametric equation of a circle of radius 1 is given by

$\mathbf{r} : \mathcal{I} \to \mathbb{R}^2, \quad \mathbf{r}(t) = (\cos t, \ \sin t), \quad t \in \mathbb{R}$

.

A circle is an example of plane curve.

ParametricPlot[{Cos[t],Sin[t]},{t,0,2π},ImageSize->200]

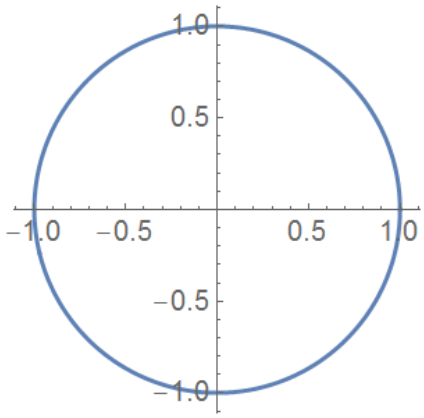


Figure 2.1. A circle of radius 1.

Example 2.2. A parabola

The parametric equation of a parabola is given by

$$\mathbf{r} : \mathcal{I} \to \mathbb{R}^2, \quad \mathbf{r}(t) = (t, t^2), \quad t \in \mathbb{R}$$

.

This is another example of a plane curve that models a particle traveling along the parabola y=x^2. We define the parameter t equals the x-coordinate.

ParametricPlot[{t,t^2},{t,-1,1},AspectRatio->1,ImageSize->200]

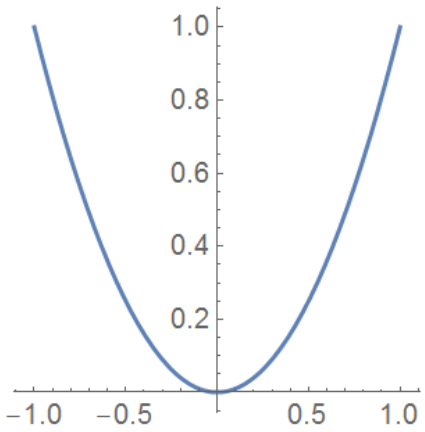


Figure 2.2. The parabola y=x^2.

Example 2.3. A right circular helix of radius 1

The parametric equation of a right circular helix of radius 1 is given by

$$\mathbf{r} : \mathcal{I} \to \mathbb{R}^3, \qquad \mathbf{r}(t) = (\cos t,\ \sin t,\ t), \qquad t \in \mathbb{R}$$

.

ParametricPlot3D[{Cos[t],Sin[t],t},{t,0,4π},PlotStyle->AbsoluteThickness[4],AspectRatio->1,ImageSize->300]

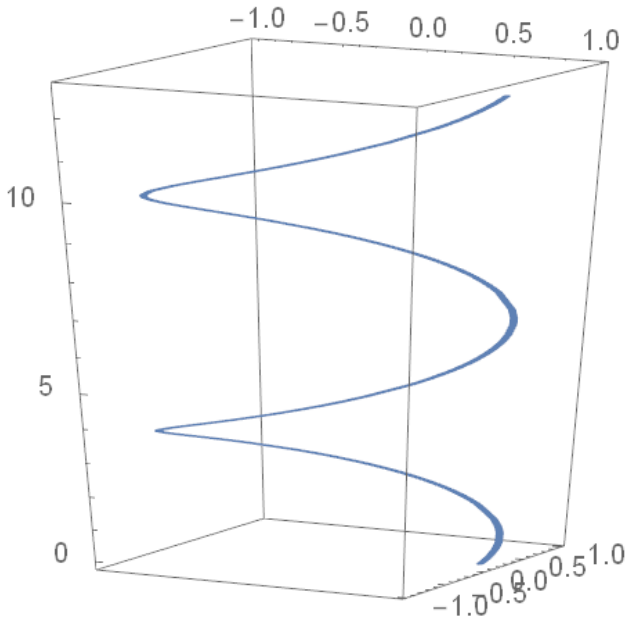


Figure 2.3. A right circular helix of radius 1.

Example 2.4. A toroidal spiral

The toroidal spiral is a space curve $\mathbf{r} : \mathcal{I} \to \mathbb{R}^3$ parametrized by

$$\mathbf{r}(t) = ((4 + \sin n\,t)\cos t,\ (4 + \sin n\,t)\sin t,\ \cos n\,t)$$

, $t \in [0,\ 2\,\pi]$.

Consider n=2.

ParametricPlot3D[{(4+Sin[2t])Cos[t],(4+Sin[2t])Sin[t],Cos[2t]},
{t,0,2π},AspectRatio->1]

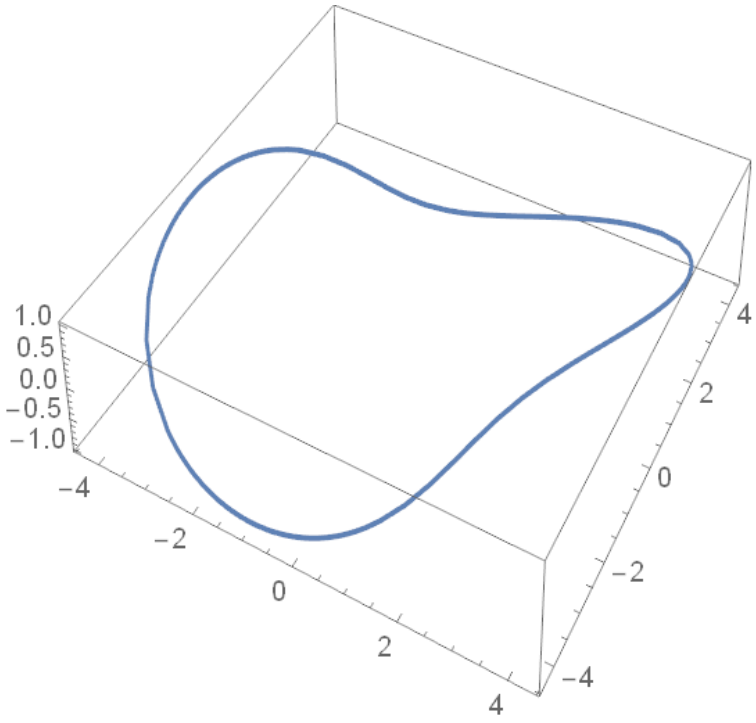


Figure 2.4. A toroidal spiral.

Example 2.5. A twisted cubic

$$\mathbf{r} : \mathcal{I} \to \mathbb{R}^3, \qquad \mathbf{r}(t) = \left(t, t^2, t^3\right), \qquad t \in [-1, 1]$$

.

ParametricPlot3D[{t,t^2,t^3},{t,-1,1},AspectRatio->1]

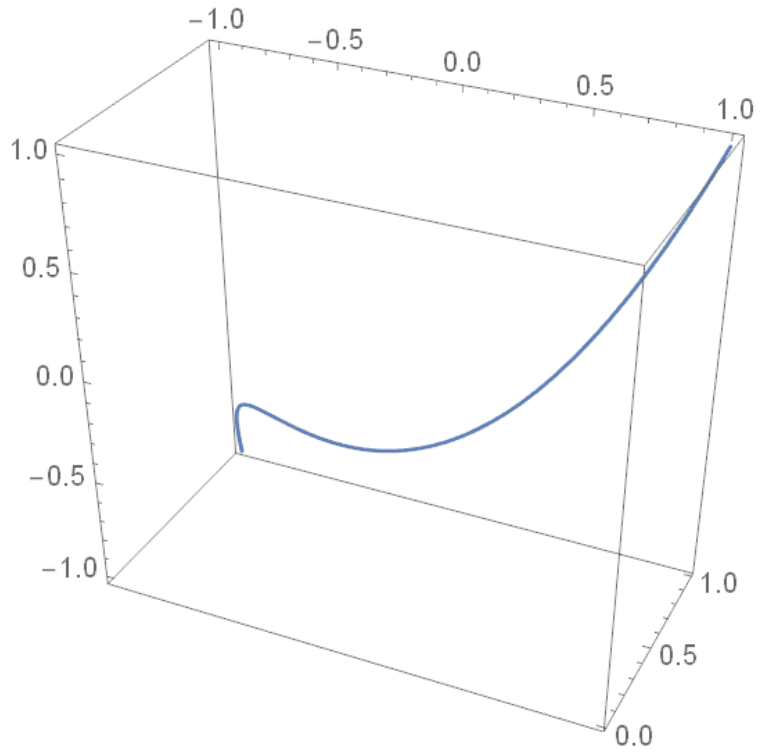


Figure 2.5. A twisted cubic.

We define some functions in Mathematica that we shall use repeatedly. First of all, the length of a vector:

```
vectorLength[r_]:=Sqrt[Simplify[r.r]]
```

The unit vectors:

```
unitVector[r_]:=Simplify[r/vectorLength[r]]
```

Next, we define the velocity vector, speed, and acceleration vector as in the following:

```
velocity[r_][t_]:=Simplify[D[r[t],t]]

speed[r_][t_]:=Simplify[vectorLength[velocity[r][t]]]

acceleration[r_][t_]:=Simplify[D[velocity[r][t],t]]
```

As an example, we find the velocity vector, speed, and acceleration vector of the twisted cubic.

```
twistedCubic[t_]={t,t^2,t^3}

velocity[twistedCubic][t]

speed[twistedCubic][t]

acceleration[twistedCubic][t]
```

Next example, we find the velocity vector, speed, and the acceleration vector of the toroidal spiral.

```
toroidalSpiral[n_][t_]={(4+Sin[n t])Cos[t],(4+Sin[n t])Sin[t],Cos[n t]}

velocity[toroidalSpiral[n]][t]
```

speed[toroidalSpiral[n]][t]

acceleration[toroidalSpiral[n]][t]

2.1.1 Reparametrization

A curve can be reparametrized. For example, consider the curve:

$\tilde{\mathbf{r}}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$.

Suppose there exists a differentiable function h such that $\tilde{\mathbf{r}} = \mathbf{r} \circ h$, where

$\tilde{x}(t) \equiv x(h(t)),$
$\tilde{y}(t) \equiv y(h(t)),$
$\tilde{z}(t) \equiv z(h(t)).$

Then,

$\tilde{x}'(t) = x'(h(t))\, h'(t),$
$\tilde{y}'(t) = y'(h(t))\, h'(t),$
$\tilde{z}'(t) = z'(h(t))\, h'(t),$

where prime denotes differentiation with respect to t.

The curve $\tilde{\mathbf{r}}(t)$ is said to be a reparametrization of r(t). If h'(t)>0, it is called a positive reparametrization of r. Similarly, if h'(t)<0, it is called a negative reparametrization of r.

Example 2.6. Consider the following two regular space curves:

$\mathbf{r}(t) = (\cos t, \sin t, t), \qquad t \in [-2\pi, 2\pi]$

,

$$\tilde{\mathbf{r}}(t) = (\cos 2t, \sin 2t, 2t), \qquad t \in [-\pi, \pi]$$

.

ParametricPlot3D[{Cos[t],Sin[t],t},{t,-2π,2π},PlotStyle->AbsoluteThickness[4],AspectRatio->1,ImageSize->300]

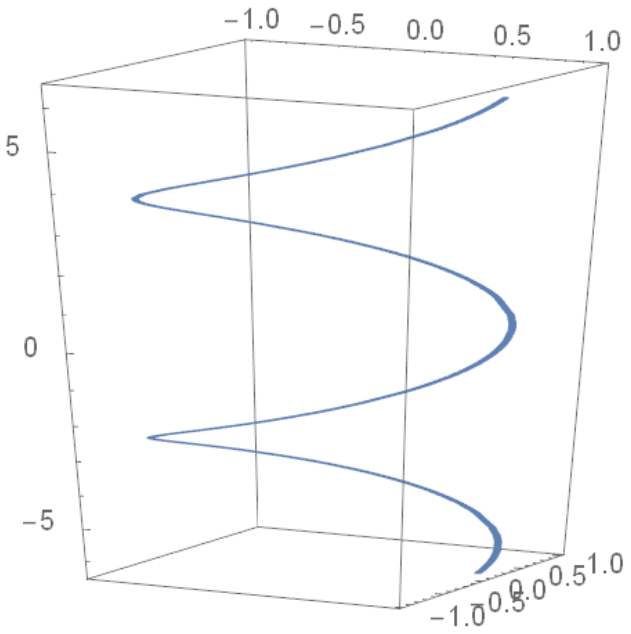


Figure 2.6. The space curve r(t)=(cos t,sin t,t).

ParametricPlot3D[{Cos[2t],Sin[2t],2t},{t,-π,π},PlotStyle->{AbsoluteThickness[4],Hue[0]},AspectRatio->1,ImageSize->300]

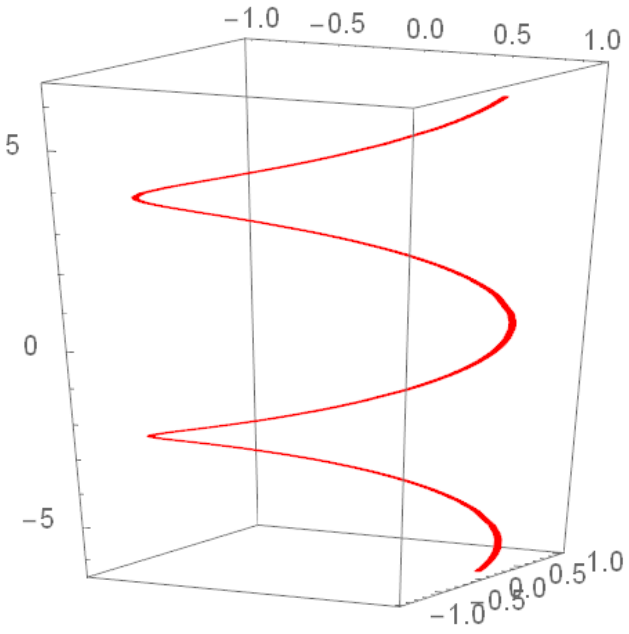


Figure 2.7. The space curve

$$\tilde{\mathbf{r}}(t) = (\cos 2t, \sin 2t, 2t)$$

.

Obviously, the two curves trace out the same path.

Notice that

$$\tilde{\mathbf{r}}(t) = (\mathbf{r} \circ h)(t) = \mathbf{r}(h(t)) \text{ ,}$$

where h(t)=2t.

Since h'(t)=2>0, it is a positive reparametrization.

We say that $\tilde{\mathbf{r}}$ is a reparametrization of r and refer to $\tilde{\mathbf{r}}$ and r as two parametrization of the same path.

2.1.1.1 Reparametrization by arc length

The arc length function s(t) on the curve r(t), starting at a is defined as

(2.1) $$s(t) \equiv \int_a^t \| \mathbf{r}'(u) \| \, du \,,$$

where ||r'(u)|| is the speed of the curve r(t).

By the Fundamental Theorem of Calculus, we get

(2.2) $$\frac{ds(t)}{dt} = s'(t) = \| \mathbf{r}'(t) \| \,.$$

The Inverse Function Theorem implies that the function s(t) has an inverse t(s), and that

(2.3) $$\frac{dt}{ds} \Big|_{s(t)} = \frac{1}{\frac{ds}{dt} \Big|_{t(s)}} \,.$$

If we define $\tilde{\mathbf{r}}$ as a function of arc length s such that

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) \,,$$

then

$$\tilde{\mathbf{r}}' = \frac{dt}{ds} \mathbf{r}'(t(s)) \,.$$

Hence,

$$\| \tilde{\mathbf{r}}' \| = \| \frac{dt}{ds} \mathbf{r}'(t(s)) \| = \left(\frac{dt}{ds} \Big|_{s(t)} \right) \| \mathbf{r}'(t(s)) \| = \left(\frac{dt}{ds} \Big|_{s(t)} \right) \left(\frac{ds}{dt} \Big|_{t(s)} \right) = 1$$

.

Thus, curve which is parametrized by arc length has unit-speed.

We give an example in the following on how to find the reparametrization of a curve with arc length.

Example 2.7. Re-parametrize the curve below with arc length:

$$\mathbf{r} : \mathcal{I} \to \mathbb{R}^3, \quad \mathbf{r}(t) = (\cosh t, \sinh t, t), \quad t > 0$$

.

and show that it has unit-speed.

Solution:

First we compute the arc length function s(t) by equation (2.1):

$$s(t) \equiv \int_0^t \| \mathbf{r}'(u) \| du$$,

and define the starting point at t=0.

Clear[r]

r[t_]:={Cosh[t],Sinh[t],t}

ParametricPlot3D[r[t],{t,0,2}]

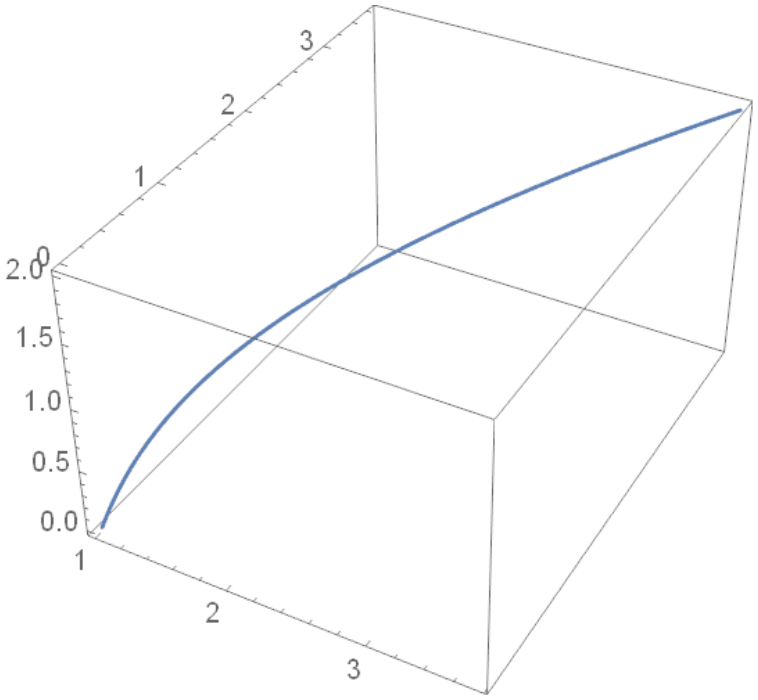


Figure 2.8. The curve parametrized by r(t)=(cosh t,sinh t,t)

(* Evaluate the arc length function s(t) *)

Assuming[t>0,Integrate[speed[r][u]//Evaluate,{u,0,t}]]

Note that we have used Assuming command to include the assumption t>0 to the Integrate command.

So, we obtain

$$s(t) = \sqrt{2}\,\sinh t.$$

$$t(s) = \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right).$$

Thus, we can re-parametrize the curve by arc length s as

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) = \left(\cosh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right), \sinh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right), \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right)$$

.

rtilde[s_]=
{Cosh[ArcSinh[s/Sqrt[2]]],Sinh[ArcSinh[s/Sqrt[2]]],ArcSinh[s/Sqrt[2]]}

Evaluating the magnitude of the velocity vector we obtain

$$\| \tilde{\mathbf{r}}' \| = 1$$.

That is, the reparametrized curve has unit-speed.

speed[rtilde][s]

After showing the property that curve parametrized by arc length has unit-speed, next we show that reparametrization preserves the length of curves.

Let $\tilde{\mathbf{r}}(t)$ be a reparametrization of the curve r(t) given by

(2.4) $$\tilde{\mathbf{r}}(t) = (\mathbf{r} \circ h)(t) = \mathbf{r}(h(t))$$,

where h is a differentiable function and h'>0 for positive parametrization.

Then

$$\tilde{\mathbf{r}}'(t) = \mathbf{r}'(h(t))\, h'(t),$$

$$\| \tilde{\mathbf{r}}'(t) \| = \| \mathbf{r}'(h(t)) \|\, h'(t)$$

.

We compute the length of r(t) within the domain of its definition (a,b):

$$\int_a^b \| \mathbf{r}'(u) \| \, du$$.

Using the change of variable, u=h(t), whose domain of definition (c,d), we get

du=h'(t) dt.

So, the integral becomes

$$\int_a^b \| \mathbf{r}'(u) \| \, du = \int_c^d \| \mathbf{r}'(h(t)) \| \, h'(t) \, dt$$

.

But

$$\| \tilde{\mathbf{r}}'(t) \| = \| \mathbf{r}'(h(t)) \| \; h'(t)$$

, thus

(2.5)

$$\int_a^b \| \mathbf{r}'(u) \| \, du = \int_c^d \| \tilde{\mathbf{r}}'(t) \| \, dt$$

.

This implies that the positive reparametrization preserves the arc length.

Similarly, it can be shown for negative reparametrization where h'<0, the arc length is preserved. Hence, the arc length is a geometric invariant.

2.1.2 The Frenet Formulas

In this section we shall consider unit-speed curves, that is, curves parametrized by arc length s.

Let $\mathbf{r} : \mathcal{I} \to \mathbb{R}^3$ be a unit-speed curve, where $\mathcal{I} \subset \mathbb{R}$ is an interval. So,

$$\| \mathbf{r}'(s) \| = 1 \qquad \text{for each } s \text{ in } \mathcal{I}$$

.

Define T(s)=r'(s).

T is called the unit tangent vector field of r.

Since T has constant length, its derivative T'(s)=r''(s) measures the way the curve is turning in $\mathbb{R}^3$. T'(s) is called the curvature vector field of r.

A real-valued function κ>=0 such that

(2.6)

$$\kappa(s) \equiv \| \mathbf{T}'(s) \| \qquad \text{for all } s \in \mathcal{I}$$

is called the curvature function of r.

In other words, the curvature κ of a path in $\mathbb{R}^3$ is the rate of change of the direction of T per unit change in distance along the path. It measures the failure of the curve to be linear.

Since ||T||=T · T=1. Differentiating ||T|| gives

2 T · T'=0.

So, T' is always orthogonal to T, that is, normal to the curve r.

Define an unit normal vector field N(s) as

(2.7) $$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}.$$

Since T' is always orthogonal to T, we have

(2.8) $$\mathbf{T}(s) \cdot \mathbf{N}(s) = 0.$$

N(s) is often called the principal normal vector field of r.

By definition of the curvature function of r (equation 2.6) we obtain

(2.9) $$\mathbf{N}(s) = \frac{1}{\kappa}\mathbf{T}'(s).$$

Define a vector field B(s) such that

(2.10) $$\mathbf{B}(s) \equiv \mathbf{T}(s) \times \mathbf{N}(s).$$

B(s) is called the binormal vector field of r.

The three vectors T(s), N(s), B(s) constitute a Frenet frame on r.

In summary,

(2.11)

$$\mathbf{T}(s) = \mathbf{r}'(s), \qquad \mathbf{N}(s) = \frac{1}{\kappa}\mathbf{T}'(s), \qquad \mathbf{B}(s) \equiv \mathbf{T}(s) \times \mathbf{N}(s)$$

.

(2.12)

$$\mathbf{T} \cdot \mathbf{T} = 1, \qquad \mathbf{N} \cdot \mathbf{N} = 1, \qquad \mathbf{B} \cdot \mathbf{B} = 1.$$

$$\mathbf{T} \cdot \mathbf{N} = 0, \qquad \mathbf{T} \cdot \mathbf{B} = 0, \qquad \mathbf{N} \cdot \mathbf{B} = 0.$$

Next, we establish the derivatives of T', N', B' in terms of T, N, B.

We have T=r',

so T'=r''=κ N.

Now, differentiate T · B=0 gives

T'· B + T · B'=0,

T'· B=-T · B'.

Substituting T'=κ N into the above equation gives

κ N · B=-T · B'.

But N · B=0, thus

$\mathbf{T} \cdot \mathbf{B}' = 0$.

Since $\mathbf{T} \cdot \mathbf{N} = 0$, it suggests that $\mathbf{B}'$ is a scalar multiple of $\mathbf{N}$.

Define

(2.13) $$\mathbf{B}' = -\tau \mathbf{N},$$

where τ is defined as the torsion function of the curve r.

The torsion of a curve in space measures the failure of the curve to lie in a plane. Hence, the torsion of a plane curve is zero.

Theorem 2.3 (Frenet Formulas for unit-speed curves)

If $\mathbf{r} : \mathcal{I} \to \mathbb{R}^3$ is a unit-speed curve with positive curvature κ and torsion τ, then

(2.14) $$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N}, \\ \mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \mathbf{B}' &= -\tau \mathbf{N}. \end{aligned}$$

Proof.

The first and third formulas are the definition of curvature and torsion, equation (2.9) and (2.13), respectively, so we just need to verify the second formula.

We use orthonormal expansion to express N' in terms of the frame field (T, N, B):

N'=(N'·T)T + (N'·N)N + (N'·B)B,

where N'·T, N'·N, and N'·B are coefficients of the expansion.

Differentiating N ·T=0, we get

N'·T + N·T'=0

N'·T =- N·T'=- N · κ N=-κ.

Since N is a unit vector field, N'·N=0.

Differentiating N ·B=0, we get

N'·B + N·B'=0

N'·B =- N·B'=- N · (-τ N)=τ.

Hence, the Frenet formulas.

Example 2.8. Show that the following curve has unit-speed. Compute the Frenet frame T, N, B.

$$\mathbf{r}(s) = \left(\frac{1}{3}(1+s)^{3/2},\ \frac{1}{3}(1-s)^{3/2},\ \frac{1}{\sqrt{2}}s\right)$$

.

Solution:

```
Clear[r]

r[s_]={1/3 (1+s)^(3/2),1/3 (1-s)^(3/2),1/Sqrt[2] s}


(* Compute the speed of the curve *)

speed[r][s]


ParametricPlot3D[r[s],{s,0,10},ViewPoint->{1,-1,4}]
```

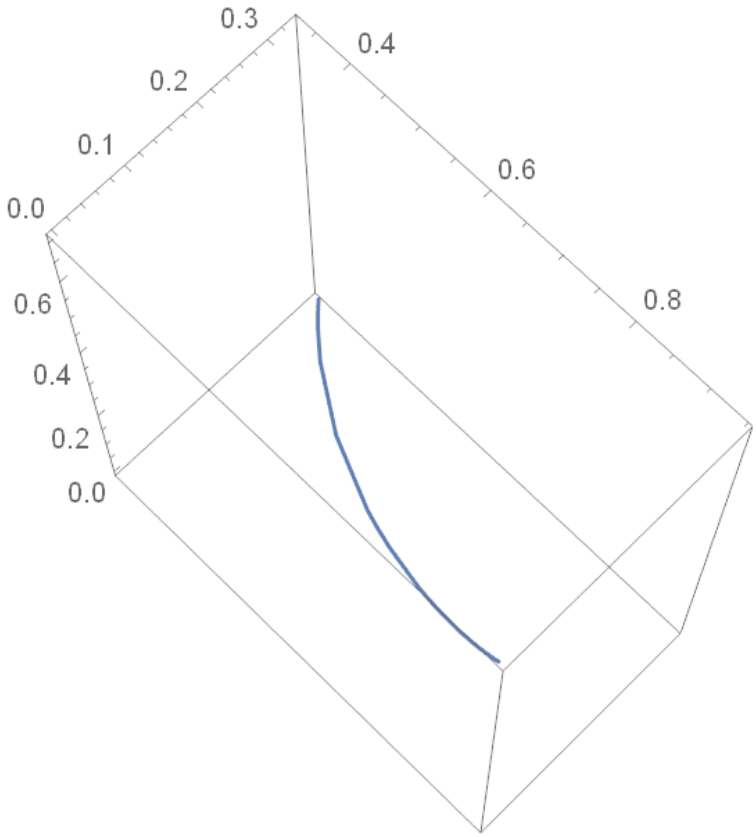


Figure 2.9. The space curve

$$\mathbf{r}(s) = \left(\frac{1}{3}(1+s)^{3/2},\ \frac{1}{3}(1-s)^{3/2},\ \frac{1}{\sqrt{2}}s\right)$$

.

(* Compute the unit tangent vector T *)

uT=D[r[s],s]//Simplify

(* Compute the unit normal vector N by equation (2.7) *)

uN=unitVector[D[uT,s]]//Simplify//PowerExpand

(* Compute the unit binormal vector B by equation (2.10) *)

uB=Cross[uT,uN]//Simplify

uT.uT//Simplify

uN.uN//Simplify

uB.uB//Simplify

uT.uN//Simplify

uT.uB//Simplify

uN.uB//Simplify

The Frenet formulas, equation (2.14) in Theorem 2.3 are valid only for unit-speed curves, that is, curves parametrized by arc length. In practice,

however, given an arbitrary-speed curve it may be difficult to re-parametrize it into a unit-speed curve. Thus, we need to have Frenet formulas for arbitrary-speed curves. This leads to the following lemma:

Lemma 2.4 (Frenet Formulas for arbitrary-speed curves)

If $\mathbf{r} : \mathcal{I} \to \mathbb{R}^3$ is a regular curve with positive curvature κ and torsion τ, then

(2.15)

$$\begin{aligned} \mathbf{T}' &= \kappa v \mathbf{N}, \\ \mathbf{N}' &= -\kappa v \mathbf{T} + \tau v \mathbf{B}, \\ \mathbf{B}' &= -\tau v \mathbf{N}. \end{aligned}$$

Proof.

Let $\tilde{\mathbf{r}}(s(t))$ be a unit-speed reparametrization of r.

Write $\mathbf{r}(t) = \tilde{\mathbf{r}}(s(t))$,

where s(t) is the arc length function.

Let $(\tilde{\mathbf{T}}, \tilde{\mathbf{N}}, \tilde{\mathbf{B}})$ be the Frenet frame of $\tilde{\mathbf{r}}$ and denote the curvature and torsion of $\tilde{\mathbf{r}}$ by $\tilde{\kappa}$ and $\tilde{\tau}$, respectively.

By definition:

(2.16)

$$\mathbf{T}(t) = \tilde{\mathbf{T}}(s(t)), \quad \mathbf{N}(t) = \tilde{\mathbf{N}}(s(t)), \quad \mathbf{B}(t) = \tilde{\mathbf{B}}(s(t))$$

.

(2.17)

$$\kappa(t) = \tilde{\kappa}(s(t)), \quad \tau(t) = \tilde{\tau}(s(t))$$

.

Using chain rule, the derivative of T with respect to t gives

$$\frac{d\mathbf{T}(t)}{dt} = \frac{d\,\tilde{\mathbf{T}}(s)}{ds}\,\frac{ds}{dt} = V\,\frac{d\,\tilde{\mathbf{T}}(s)}{ds}\,,$$

where $V \equiv \frac{ds}{dt}$.

But

$$\frac{d\,\tilde{\mathbf{T}}(s)}{ds} = \tilde{\mathbf{T}}'(s) = \tilde{\kappa}(s)\,\tilde{\mathbf{N}}(s) = \kappa\,\mathbf{N}$$

by equation and (2.14), (2.16) and (2.17). So,

$$\frac{d\mathbf{T}}{dt} = V\,\kappa\,\mathbf{N}\,.$$

Similarly, for N:

$$\frac{d\mathbf{N}(t)}{dt} = \frac{d\,\tilde{\mathbf{N}}(s)}{ds}\,\frac{ds}{dt} = V\,\frac{d\,\tilde{\mathbf{N}}(s)}{ds}\,.$$

$$\frac{d\,\tilde{\mathbf{N}}(s)}{ds} = \tilde{\mathbf{N}}' = -\kappa\,\tilde{\mathbf{T}} + \tau\,\tilde{\mathbf{B}} = -\kappa\,\mathbf{T} + \tau\,\mathbf{B}$$

.

So,

$$\frac{d\mathbf{N}}{dt} = -\kappa\,V\,\mathbf{T} + \tau\,V\,\mathbf{B}\,.$$

For B:

$$\frac{d\mathbf{B}(t)}{dt} = \frac{d\,\tilde{\mathbf{B}}(s)}{ds}\,\frac{ds}{dt} = v\,\frac{d\,\tilde{\mathbf{B}}(s)}{ds}\,.$$

$$\frac{d\,\tilde{\mathbf{B}}(s)}{ds} = \tilde{\mathbf{B}}' = -\tau\,\tilde{\mathbf{N}} = -\tau\,\mathbf{N}\,.$$

So,

$$\frac{d\mathbf{B}}{dt} = -\tau\,v\,\mathbf{N}\,.$$

Note that for curves parametrized by arc length, v=1, so we get the Frenet formulas in Theorem 2.3.

The Frenet equations can be written compactly using matrix notation:

(2.18)

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = |\mathbf{v}| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

.

Now, we can find the expression for velocity vector and acceleration vector in terms of the Frenet frame.

Let r(t) be a regular curve with arbitrary-speed function v, then the velocity vector is just

(2.19) $$\mathbf{v} = v\,\mathbf{T}\,,$$

where T is the unit tangent vector to the curve.

Taking the derivative of v with respect to t gives the acceleration vector:

$$\mathbf{a} \equiv \mathbf{v}' = \frac{dv}{dt}\,\mathbf{T} + v\,\frac{d\mathbf{T}}{dt}\,.$$

By Lemma 2.4:

$$\mathbf{a} = \frac{dv}{dt}\,\mathbf{T} + \kappa\, v^2\,\mathbf{N}\,. \tag{2.20}$$

From this result, we notice that the acceleration vector lies in the plane spanned by T and N. This plane is called the osculating plane of the curve.

By equation (2.10) the unit binormal vector B is perpendicular to both T and N:

$$\mathbf{B} \equiv \mathbf{T} \times \mathbf{N}\,. \tag{2.21}$$

Theorem 2.5

If $\mathbf{r} : \mathcal{I} \to \mathbb{R}^3$ is a regular curve with nonzero curvature κ, then

(2.22)

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}, \qquad \tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2}$$

.

(2.23)

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{\|\mathbf{v} \times \mathbf{a}\|}, \qquad \mathbf{N} = \mathbf{B} \times \mathbf{T}$$

.

Proof.

From equation (2.19), (2.20) and (2.21):

(2.24)

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= (v\,\mathbf{T}) \times \left(\frac{dv}{dt}\,\mathbf{T} + \kappa\, v^2\,\mathbf{N}\right) \\ &= v\,\frac{dv}{dt}\,\mathbf{T} \times \mathbf{T} + \kappa\, v^3\,\mathbf{T} \times \mathbf{N} \\ &= \kappa\, v^3\,\mathbf{B}.\end{aligned}$$

(2.25)

$$\|\mathbf{v} \times \mathbf{a}\| = \|\kappa\, v^3\|\,\|\mathbf{B}\| = \kappa\, v^3$$

.

Since κ>=0, v>0 and ||B||=1

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$.

This is the first equation of (2.22).

To find τ we take the derivative of a with respect to t, thus

$$\mathbf{a}' = \left(\frac{dv}{dt}\,\mathbf{T} + \kappa\, v^2\,\mathbf{N}\right)' = \frac{d^2 v}{dt^2}\,\mathbf{T} + \frac{dv}{dt}\,\mathbf{T}' + \kappa'\, v^2\,\mathbf{N} + 2\,\kappa\, v\,\frac{dv}{dt}\,\mathbf{N} + \kappa\, v^2\,\mathbf{N}'$$

.

Using equation (2.15):

$$\mathbf{a}' = \frac{d^2 v}{dt^2}\mathbf{T} + \kappa v \frac{dv}{dt}\mathbf{N} + \kappa' v^2 \mathbf{N} + 2\kappa v \frac{dv}{dt}\mathbf{N} - \kappa^2 v^3 \mathbf{T} + \kappa v^3 \tau \mathbf{B}$$

.

Taking the dot product of $(\mathbf{v} \times \mathbf{a})$ and a' gives

$$(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}' = \left(\kappa v^3 \mathbf{B}\right) \cdot \left(\frac{d^2 v}{dt^2}\mathbf{T} + \kappa v \frac{dv}{dt}\mathbf{N} + \kappa' v^2 \mathbf{N} + 2\kappa v \frac{dv}{dt}\mathbf{N} - \kappa^2 v^3 \mathbf{T} + \kappa v^3 \tau \mathbf{B}\right)$$
$$= \kappa^2 v^6 \tau,$$

where we have used the orthonormality of T, N, and B.

But from equation (2.25) we have $\kappa^2 v^6 = \| \mathbf{v} \times \mathbf{a} \|^2$, thus

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2}$$.

So, equation (2.22) is proved.

The first equation of (2.23) $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is equivalent to equation (2.19).

From equation (2.24) and (2.25), we get the second equation of (2.23):

$$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{\|\mathbf{v} \times \mathbf{a}\|}$$.

By definition of vector cross product, taking the cross product of equation (2.21) with T leads to the third equation of (2.23):

$$\begin{aligned}\mathbf{B}\times\mathbf{T} &= (\mathbf{T}\times\mathbf{N})\times\mathbf{T} = -\mathbf{T}\times(\mathbf{T}\times\mathbf{N}) \\ &= -[(\mathbf{T}\cdot\mathbf{N})\mathbf{T} - (\mathbf{T}\cdot\mathbf{T})\mathbf{N}] \\ &= \mathbf{N}.\end{aligned}$$

Next, the curvature and torsion of a curve is defined accordingly using Mathematica.

```
(* Define the curvature of a curve by equation (2.22) *)

κ[r_][t_]:=

Module[{tt,a,b},

a=vectorLength[Cross[r'[tt], r''[tt]]];

b=vectorLength[r'[tt]]^3;

Simplify[a/b/.tt->t]

]


(* Define the torsion of a curve by equation (2.22) *)

τ[r_][t_]:=

Module[{tt,a,b},

a=(Cross[r'[tt], r''[tt]]).r'''[tt];

b=Cross[r'[tt], r''[tt]];

Simplify[a/b.b/.tt->t]
```

]

Next, we define some modules in Mathematica for the Frenet frame based on equation (2.23):

```
(* Define the unit tangent vector field: T *)
UT[r_][t_]:=
Module[{α,tt},
α=r'[tt];
Simplify[α/vectorLength[α]/.tt->t]
]

(* Define the unit binormal vector field: B *)
UB[r_][t_]:=
Module[{α,tt},
α=Cross[r'[tt], r''[tt]];
Simplify[α/vectorLength[α]/.tt->t]
]

(* Define the unit normal vector field: N *)
UN[r_][t_]:=Simplify[Cross[UB[r][t], UT[r][t]]]
```

```
(* Define the Frenet frame on curve r at t=t0 *)

frenetFrame[r_,t0_]:=Graphics3D[{AbsoluteThickness[1.5],
{PointSize[0.03],Point[r[t]]},{Red,Arrow[{r[t],r[t]+UT[r][t]}]},
{Green,Arrow[{r[t],r[t]+UN[r][t]}]}, {Blue,Arrow[{r[t],r[t]+UB[r][t]}]},
Text["T", r[t]+1.05*UT[r][t]],Text["N", r[t]+1.05*UN[r][t]],Text["B",
r[t]+1.05*UB[r][t]]}/.t->t0]
```

Example 2.9. A curve r(t) parametrized by

r(t)=(t cos t,t sin t,t)

lies on a double cone and passes through the vertex at t=0.

(a) Find the velocity vector, speed and acceleration vector of the curve.

(b) Find the arc length of the curve starting from the origin to t=2.

(c) Find the curvature and the torsion.

(d) Find the Frenet frame.

(e) Plot the Frenet frame at t=0.

Solution:

```
Clear[rhelix]

rhelix[t_]={t Cos[t],t Sin[t],t}
```

curvePlot=ParametricPlot3D[rhelix[t],{t,-4π,4π},AxesLabel->{"x","y","z"}]

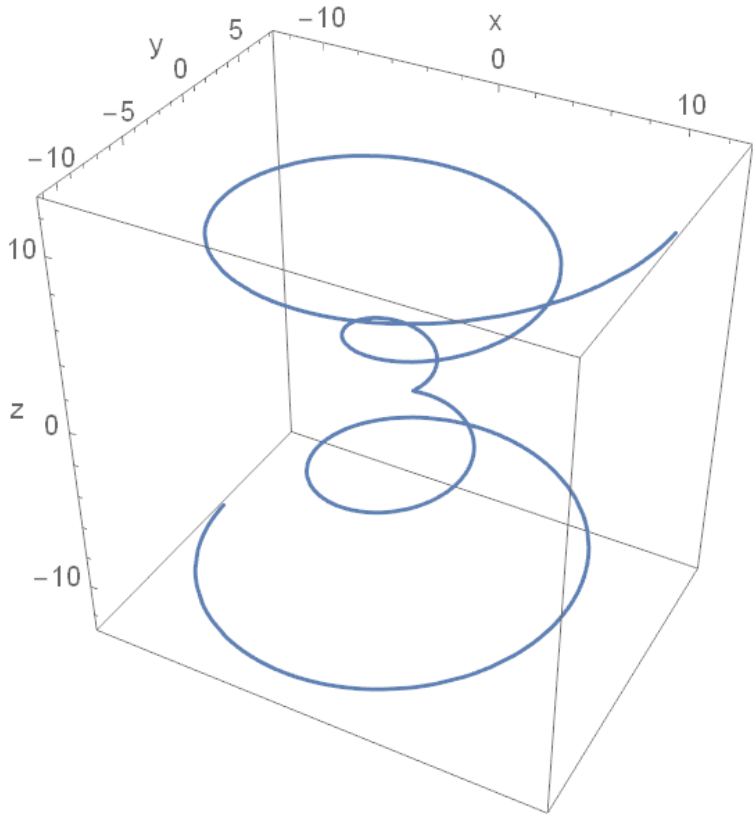


Figure 2.10. The curve r(t)=(t cos t,t sin t,t).

(a)

(* Compute the velocity vector *)

velocity[rhelix][t]

(* Compute the speed *)

speed[rhelix][t]

(* Compute the acceleration vector *)

acceleration[rhelix][t]

(b)

The arc length of the curve starting from the origin to t=2 can be evaluated using equation (2.1)

$$s \equiv \int_0^2 \| (\cos t - t \sin t,\ t \cos t + \sin t,\ 1) \| \, dt$$
$$= \sqrt{6} + \sinh^{-1}\left(\sqrt{2}\right)$$
$$\simeq 3.59571.$$

(* The arc length of the curve from t=0 to t=2 *)

Integrate[speed[rhelix][u]//Evaluate,{u,0,2}]

%//N

(c)

(* The curvature *)

κ[rhelix][t]

(* The torsion *)

τ[rhelix][t]

(d)

```
(* The Frenet frame: T, B, N *)
UT[rhelix][t]
UB[rhelix][t]
UN[rhelix][t]

UT[rhelix][t] . UT[rhelix][t]//Simplify
UB[rhelix][t] . UB[rhelix][t]//Simplify
UN[rhelix][t] . UN[rhelix][t]//Simplify

UT[rhelix][t] . UB[rhelix][t]//Simplify
UT[rhelix][t] . UN[rhelix][t]//Simplify
UB[rhelix][t] . UN[rhelix][t]//Simplify
```

(e)

```
curvePlot=ParametricPlot3D[rhelix[t],{t,-(π/2),π/3},PlotRange->{{-1,1},
{-1,1},{-1,1}},AxesLabel->{"x","y","z"}];
Show[curvePlot,frenetFrame[rhelix,0],ViewPoint->{1/4,-1,2}]
```

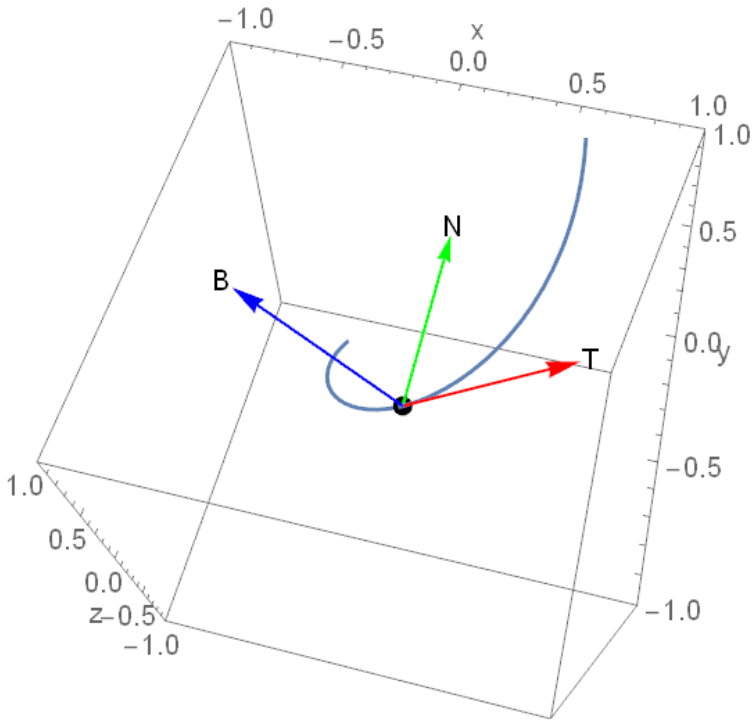
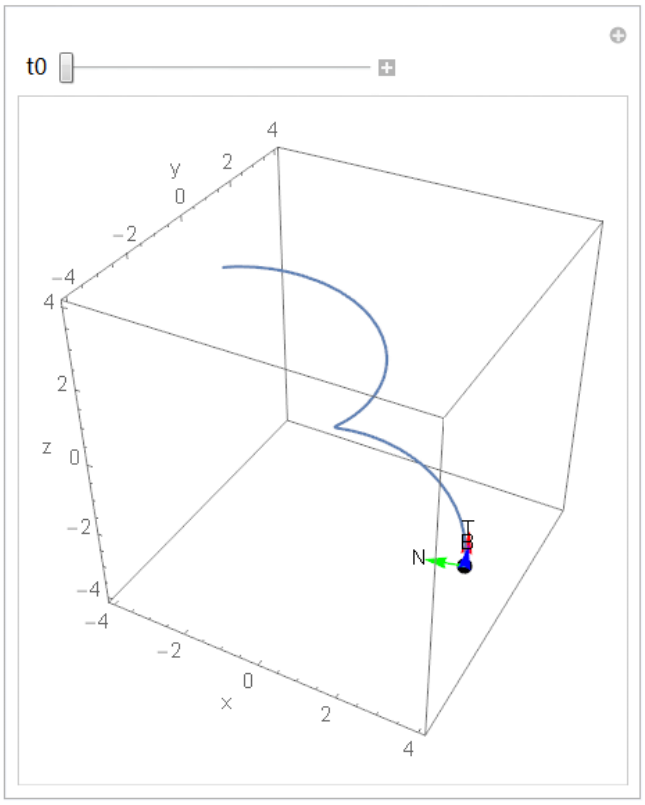


Figure 2.11. The Frenet frame for the curve r(t)=(t cos t,t sin t,t) at t=0 (the origin).

The following animation shows the Frenet frame as it traces out the curve:

```
curvePlot=ParametricPlot3D[rhelix[t],{t,-π,π},AxesLabel->{"x","y","z"}];

Module[{α,β},

α=curvePlot;

β=rhelix;

Manipulate[Show[α,frenetFrame[β,t0],PlotRange->{{-4,4},{-4,4},{-4,4}}],
{{t0,-π},-π,π}]

]
```



Animation 2.12. The Frenet frame as it traces out the curve r(t)=(t cos t,t sin t,t).

Example 2.10. Consider the toroidal spiral defined in Example 2.4. Let n=2,

$$\mathbf{r}(t) = ((4 + \sin 2t)\cos t,\ (4 + \sin 2t)\sin t,\ \cos 2t)$$

(a) Find the velocity vector, speed and acceleration vector of the curve.

(b) Find the arc length of the curve starting from the origin to t=2.

(c) Find the curvature and the torsion.

(d) Find the Frenet frame.

(e) Plot the Frenet frame at t=π/2.

Solution:

(a)

(* Compute the velocity vector *)

velocity[toroidalSpiral[2]][t]

(* Compute the speed *)

speed[toroidalSpiral[2]][t]

(* Compute the acceleration vector *)

acceleration[toroidalSpiral[2]][t]

(b) Using equation (2.1) we get:

s≃9.70111.

(* The arc length of the curve from t=0 to t=2 *)

NIntegrate[speed[toroidalSpiral[2]][u]//Evaluate,{u,0,2}]

Note that we have used NIntegrate command to compute the arc length numerically because analytical integration will take long time to complete.

(c)

(* The curvature *)

κ[toroidalSpiral[2]][t]

(* The torsion *)

τ[toroidalSpiral[2]][t]

(d)

(* The Frenet frame: T, B, N *)

UT[toroidalSpiral[2]][t]

UB[toroidalSpiral[2]][t]

UN[toroidalSpiral[2]][t]

UT[toroidalSpiral[2]][t] . UT[toroidalSpiral[2]][t]//Simplify

UB[toroidalSpiral[2]][t] . UB[toroidalSpiral[2]][t]//Simplify

UN[toroidalSpiral[2]][t] . UN[toroidalSpiral[2]][t]//Simplify

UT[toroidalSpiral[2]][t] . UB[toroidalSpiral[2]][t]//Simplify

UT[toroidalSpiral[2]][t] . UN[toroidalSpiral[2]][t]//Simplify

UB[toroidalSpiral[2]][t] . UN[toroidalSpiral[2]][t]//Simplify

(e)

curvePlot=ParametricPlot3D[{(4+Sin[2t])Cos[t],(4+Sin[2t])Sin[t],Cos[2t]},
{t,0,2π},AxesLabel->{"x","y","z"}]

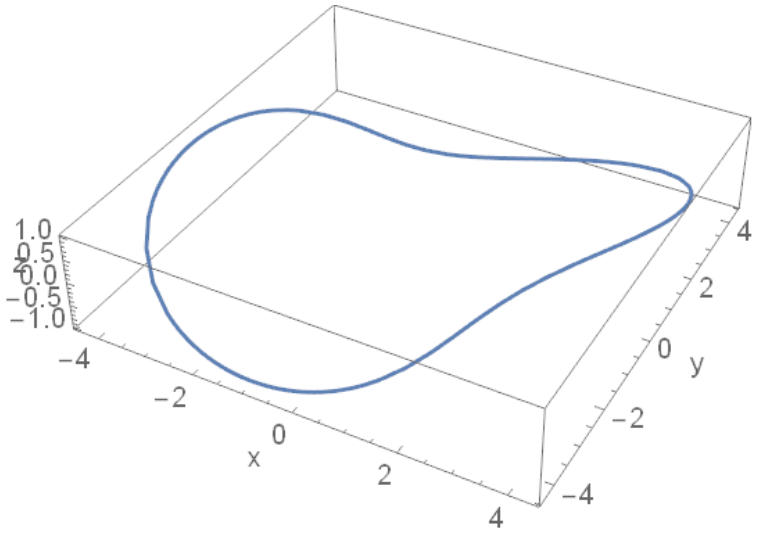


Figure 2.13. The toroidal spiral r(t)=((4+sin 2 t)cos t,(4+sin 2 t)sin t,cos 2 t).

Show[curvePlot,frenetFrame[toroidalSpiral[2],π/2],PlotRange->{{-5,5},
{-5,5},{-2,2}}]

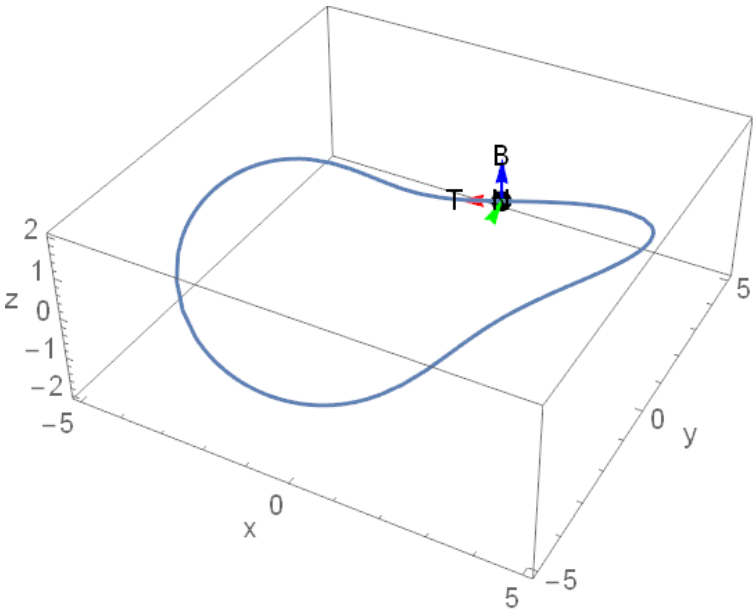
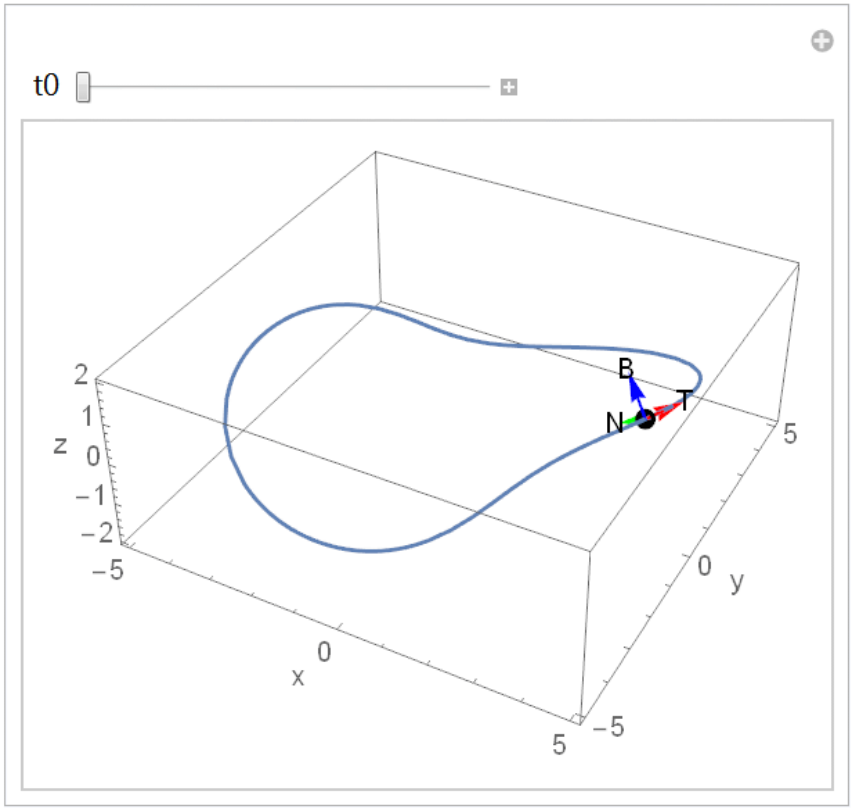


Figure 2.14. The toroidal spiral r(t)=((4+sin 2 t)cos t,(4+sin 2 t)sin t,cos 2 t) with the Frenet frame at t=π/2.

```
Module[{α,β},

α=curvePlot;

β=toroidalSpiral[2];

Manipulate[Show[α,frenetFrame[β,t0],PlotRange->{{-5,5},{-5,5},{-2,2}}],
{{t0,0},0,2π,2π/30}]

]
```



Animation 2.15. The Frenet frame as it traces out the curve r(t)=((4+sin 2 t)cos t,(4+sin 2 t)sin t,cos 2 t).

In the next example, we will illustrate vectors fields related to a plane curve. In this case, the binormal vector field is always perpendicular to the plane.

For this purpose, we define below a Mathematica module called planeFields to show velocity, speed, acceleration, unit tangent vector, and unit normal vectors fields for any curve in the plane.

```
planeFields[r_,t0_]:=Module[{uT,uN},

uT[r][t_]:=Simplify[unitVector[velocity[r][t]]];

uN[r][t_]:=Simplify[unitVector[D[uT[r]
[t],t]]];Graphics[{{Magenta,PointSize[0.03],Point[r[t]]},{Red,Arrow[{r[t],
```

```
r[t]+velocity[r][t]}]},{Green,Dashed,Thickness[0.01],Arrow[{r[t], r[t]+uT[r]
[t]}]},{Blue,Arrow[{r[t], r[t]+acceleration[r][t]}]},{Dashed,Arrow[{r[t],
r[t]+uN[r][t]}]}, Text["v",r[t]+1.05*velocity[r]
[t]],Text["a",r[t]+1.05*acceleration[r][t]],Text["T",r[t]+1.2*uT[r]
[t]],Text["N",r[t]+1.2*uN[r][t]]

    }/.t->t0]

]
```

Example 2.11. The cycloid is a plane curve defined parametrically by

r(t)=(t-sin t,1-cos t), $t \in \mathbb{R}$.

(a) Find the velocity vector, speed and acceleration vector of the curve.

(b) Find the length of the cycloid from t=0 to t=4π.

(c) Find the curvature and torsion.

(d) Plot the related vector fields at t=2π/3.

Solution:

```
cycloid[t_]:={t-Sin[t],1-Cos[t]}
```

```
cycloidPlot= ParametricPlot[Evaluate[cycloid[t]],{t,0,4π},PlotRange->
{{-1,5π},{-1,3}},ImageSize->Large]
```

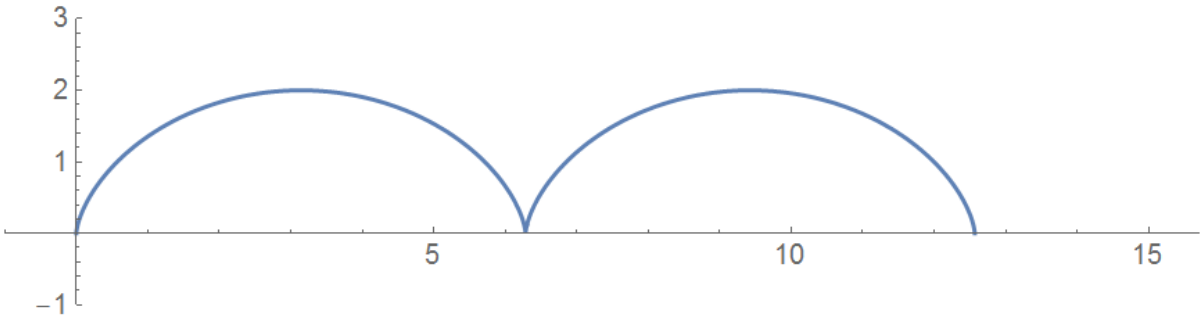


Figure 2.16. The cycloid r(t)=(t-sin t,1-cos t).

(a)

(* Compute the velocity vector *)

velocity[cycloid][t]

(* Compute the speed *)

speed[cycloid][t]

(* Compute the acceleration vector *)

acceleration[cycloid][t]

(b) Using equation (2.1):

(* The length of the cycloid from t=0 to t=4π *)

Integrate[speed[cycloid][u]//Evaluate,{u,0,4π}]

(c)

Since our curvature and torsion functions are defined for curve in space, we need to first convert the plane curve to space curve by adding a zero components for the z-axis. Thus,

cycloid3D[t_]=Append[cycloid[t],0]

(* The curvature of the cycloid *)

κ[cycloid3D][t]

(* The torsion of the cycloid *)

τ[cycloid3D][t]

Apparently, the torsion is zero because the curve lies in the plane.

(d)

Show[cycloidPlot,planeFields[cycloid,2π/3]]

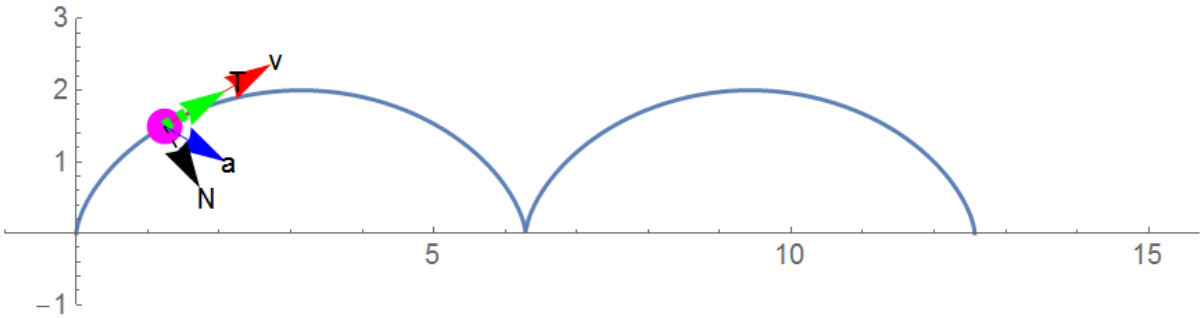
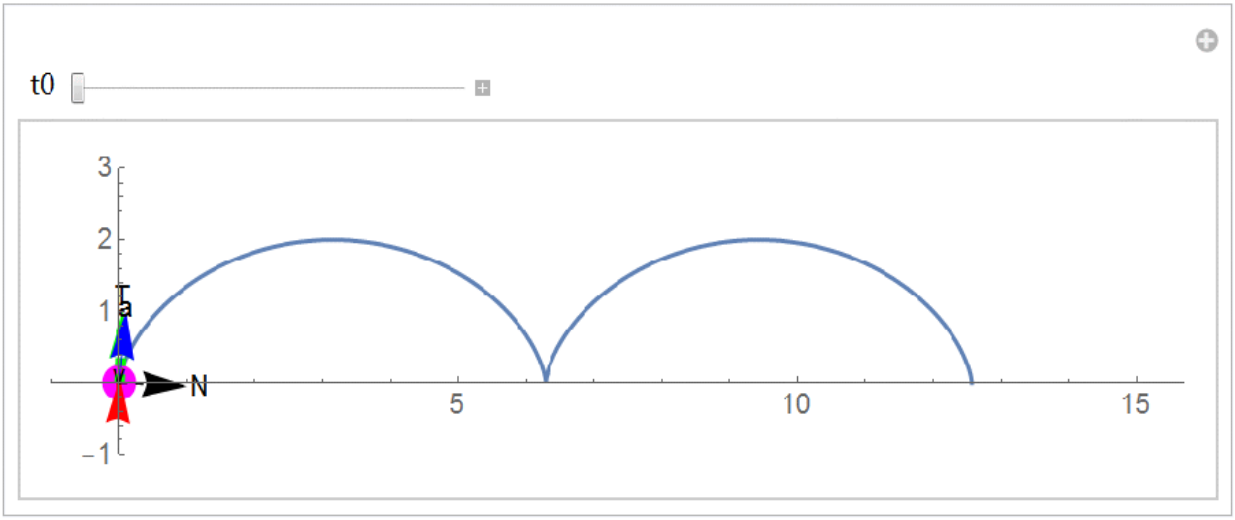


Figure 2.17. The cycloid r(t)=(t-sin t,1-cos t) with the velocity vector, acceleration vector, unit tangent vector and the unit normal vector at t=2π/3.

Here is an animation of the movement of the related vector fields as they trace out the path of the cycloid:

```
Module[{α,β},

α=cycloidPlot;

β=cycloid;

Manipulate[Show[α,planeFields[β,t0]],{{t0,0.1},0.1,4π,2π/20}]

]
```



Animation 2.18. The vector fields as they trace out the path of the cycloid r(t)=(t-sin t,1-cos t).

Example 2.12. Consider the cardioid $\mathbf{r}: \mathcal{I} \to \mathbb{R}^2$, where $\mathcal{I} \subset \mathbb{R}$ is defined by

r(t)=2 cos t (1+cos t) i + 2 sin t (1+cos t) j, 0<=t<=2π.

(a) Find the velocity vector, speed and acceleration vector of the curve.

(b) Find the length of the cardioid.

(c) Find the curvature and torsion.

(d) Plot the related vector fields at t=1.

Solution:

cardioid[t_]:={2 Cos[t] (1+Cos[t]),2 Sin[t] (1+Cos[t])}

cardioidPlot= ParametricPlot[Evaluate[cardioid[t]],{t,0,2π},PlotRange->
{{-4,7},{-4,4}}]

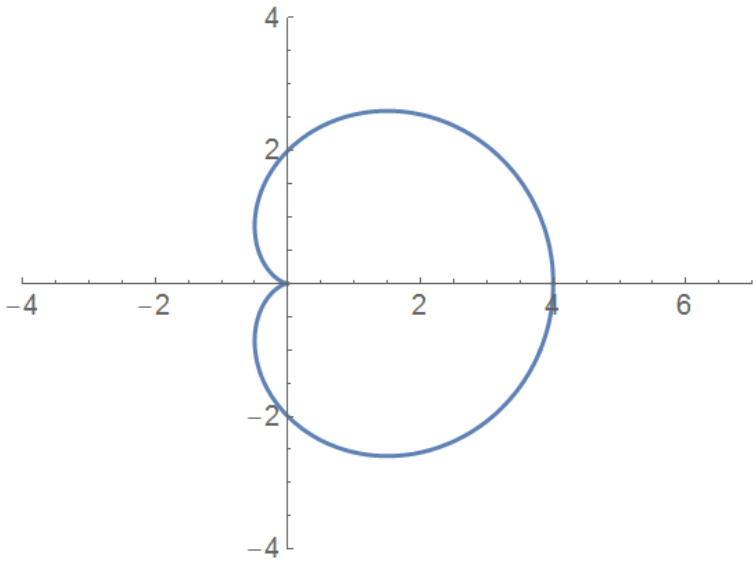


Figure 2.19. The cardioid r(t)=2 cos t (1+cos t) i + 2 sin t (1+cos t) j.

(a)

(* Compute the velocity vector *)

velocity[cardioid][t]

(* Compute the speed *)

speed[cardioid][t]

(* Compute the acceleration vector *)

acceleration[cardioid][t]

(b)

(* The length of the cardioid *)

Integrate[speed[cardioid][u]//Evaluate,{u,0,2π}]

(c)

Since our curvature and torsion functions are defined for curve in space, we need to first convert the plane curve to space curve by adding a zero components for the z-axis. Thus,

cardioid3D[t_]=Append[cardioid[t],0]

(* The curvature of the cardioid *)

κ[cardioid3D][t]

(* The torsion of the cardioid *)

τ[cardioid3D][t]

Apparently, the torsion is zero because the curve lies in the plane.

(d)

Show[cardioidPlot,planeFields[cardioid,1]]

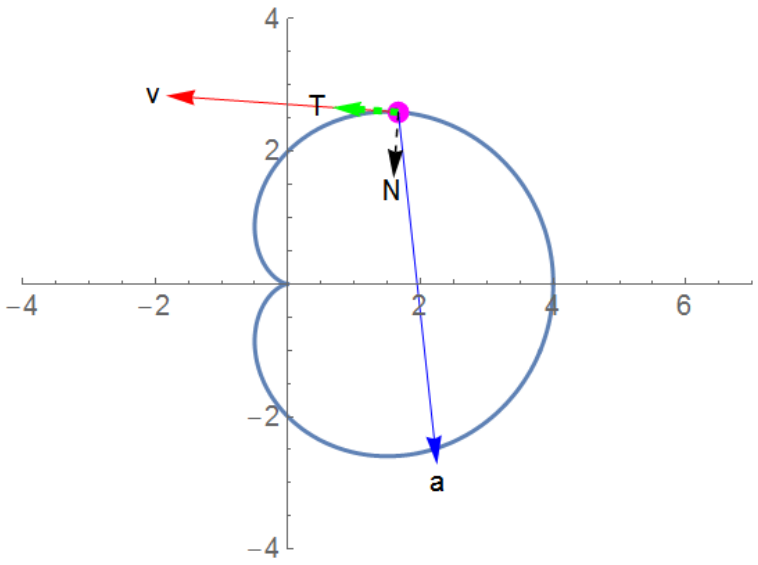


Figure 2.20. The cardioid r(t)=2 cos t (1+cos t) i + 2 sin t (1+cos t) j with the velocity vector, acceleration vector, unit tangent vector and the unit normal vector at t=1.

Below is an animation of the movement of the related vector fields as they trace out the path of the cardioid:

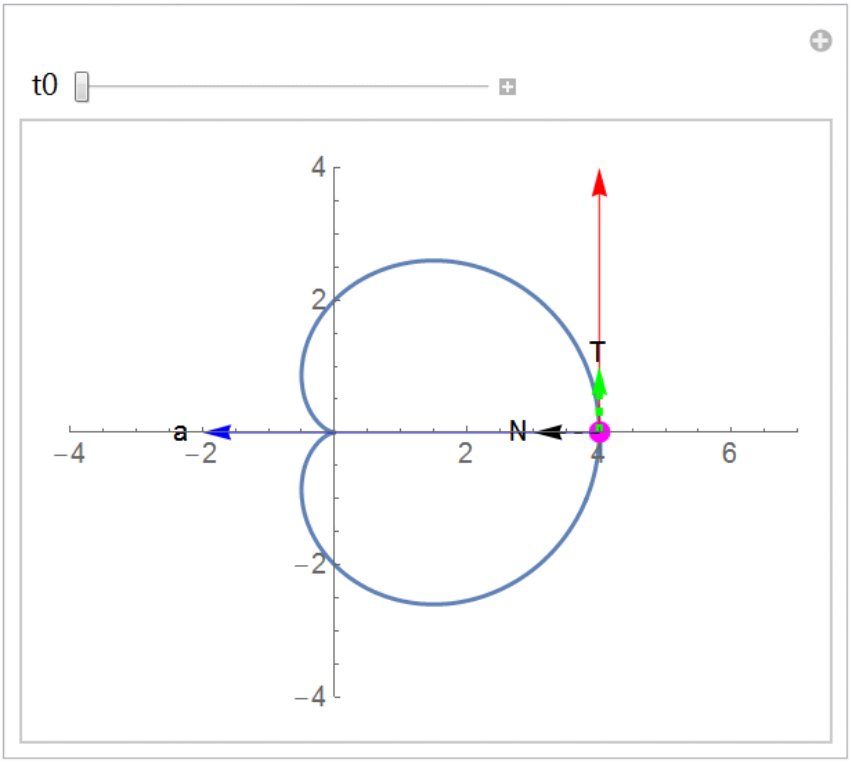
Module[{α,β},

α=cardioidPlot;

β=cardioid;

Manipulate[Show[α,planeFields[β,t0]],{{t0,0},0,2π,2π/25}]

]



Animation 2.21. The movement of the related vector fields as they trace out the path of the cardioid r(t)=2 cos t (1+cos t) i + 2 sin t (1+cos t) j.

As we have mentioned earlier, the velocity and acceleration vectors lie in the osculating plane spanned by T and N (equation 2.20). The vectors T and N contain the information about the direction in which the curve is heading and the direction in which the curve is bending, respectively. In single variable calculus we know that a line with unit tangent vector T is a best linear approximation to the curve. From the animation above, however, it suggests that the path of a curve can also be approximated by a circle in the osculating plane. So, this leads us to the following discussion about osculating circle.

2.2 Osculating Circles

In this section, we learn how to construct osculating circles to curves. Before doing so, we shall review again the curvature of curves. Note that the curvature formula we obtain in Theorem 2.5 involves cross product of velocity vector and acceleration vector which is only defined in three-dimensional space. Therefore, we shall need to use other method to compute the curvature and the unit normal vector of curves in the plane.

Recall Lemma 2.4 where we have

(2.26) $$\mathbf{T}' = \kappa v \mathbf{N} .$$

Thus,

$$\| \mathbf{T}' \| = | \kappa | \ | v | \ \| \mathbf{N} \| .$$

But ||N||=1, κ>=0 and speed is always positive, so we obtain

(2.27) $$\kappa = \frac{\| \mathbf{T}' \|}{v} .$$

For curves that are parametrized by arc length, this is just $\kappa(s) \equiv \| \mathbf{T}'(s) \|$ which we have encountered in Section 2.1.2.

The curvature for plane curves is defined below using equation (2.26):

κ2[r_][t_]:=

```
Module[{κ},

κ/.Solve[D[UT[r][t],t]==κ*speed[r][t]*UN[r][t],κ][[1]]//Simplify

]
```

A word of caution: since UT, UB and UN are vectors defined in three-dimensional space, we shall need to convert the plane curve to space curve by adding a vanishing z-component before submitting it to κ2.

In Section 2.1.2 we define a unit normal vector field N(s) (equation 2.7) by

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}.$$

For $s \equiv s(t)$,

$$\mathbf{T}'(s) = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{1}{v}\mathbf{T}'(t).$$

Taking the norm, we get

$$\|\mathbf{T}'(s)\| = \frac{1}{v}\|\mathbf{T}'(t)\|.$$

Thus,

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

The corresponding unit normal vectors are identical, therefore, N is independent of parametrization. By equation (2.21), the binormal vector B also turns out to be independent of parametrization.

Therefore, the Frenet frame {T, N, B} is independent of parametrization.

Theorem 2.6. Curvature of a curve in the plane.

Let r(t)=(x(t),y(t)) be a regular curve in $\mathbb{R}^2$. The curvature at the point (x(t),y(t)) is given by

(2.28) $$\kappa 2(t) = \frac{|x' y'' - x'' y'|}{\left(x'^2 + y'^2\right)^{3/2}},$$

where prime denotes differentiation with respect to t.

Proof.

First, we convert the plane curve to a space curve by adding a zero z-component and then use Theorem 2.5 (equation 2.22) to compute the curvature of the curve.

$$K = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}.$$

We use Mathematica to prove this theorem:

```
Clear[r,x,y]

r[t_]:={x[t],y[t],0}

vectorLength[Cross[r'[t], r''[t]]]/vectorLength[r'[t]]^3
```

Hence,

$$\kappa 2(t) = \frac{|x'y'' - x''y'|}{\left(x'^2 + y'^2\right)^{3/2}}$$.

Example 2.13. Show that the curvature of a circle with radius ρ is

κ2=1/ρ.

Solution:

The parametric equation of a circle with radius ρ is given by

r(t)=(ρ cos t,ρ sin t), $t \in [0, 2\pi]$.

First, we need to add a vanishing z-component to make the curve three-dimensional.

```
Clear[r,ρ,x,y]

r[t_]:={ρ Cos[t],ρ Sin[t],0}

(* Compute the curvature using equation (2.26) *)

κ2[r][t]//PowerExpand
```

Alternatively using Theorem 2.5, first equation of (2.22) to compute the curvature, we obtain the same result.

(* Compute the curvature using equation (2.22) *)

κ[r][t]//PowerExpand

Example 2.14. A plane curve called the spiral of Cornu is defined parametrically by

r(t)=(x(t),y(t)),

where $x(t) = \int_0^t \sin \frac{u^2}{2}\, du$ and $y(t) = \int_0^t \cos \frac{u^2}{2}\, du$.

Use Theorem 2.6 to show that the curvature is $|t|$.

Solution:

(* Define the spiral of Cornu *)

cornuSpiral[t_]:=

Module[{x,y},

x[t]=Integrate[Sin[u^2/2],{u,0,t}];

y[t]=Integrate[Cos[u^2/2],{u,0,t}];

{x[t],y[t]}

]

ParametricPlot[cornuSpiral[t]//Evaluate,{t,-3π,3π}]

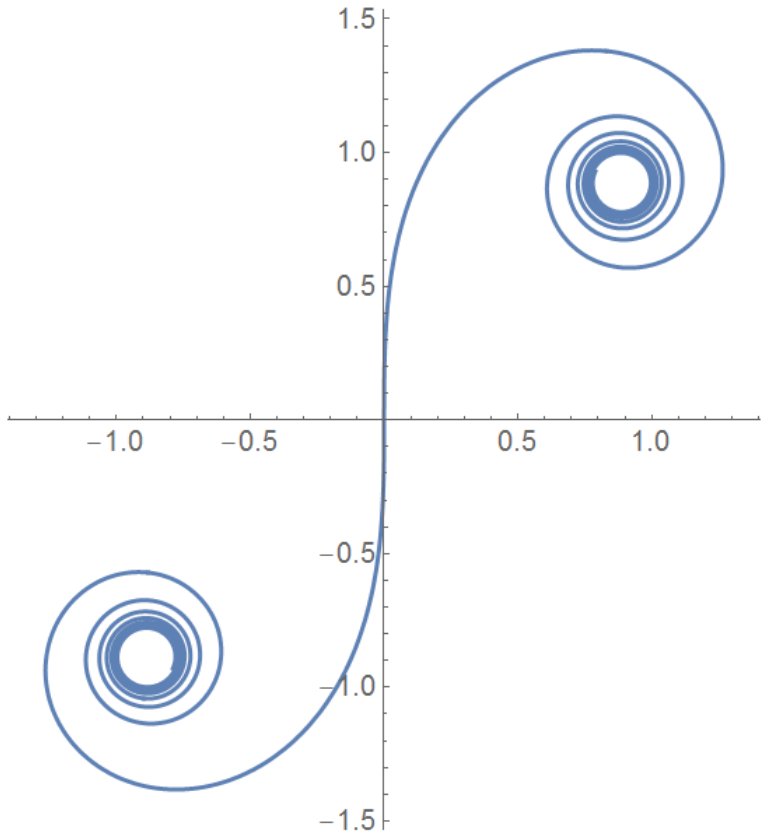


Figure 2.22. The spiral of Cornu.

(* Compute the curvature using Theorem 2.6, equation (2.28) *)

Module[{x,y},

x[t_]=cornuSpiral[t][[1]];

y[t_]=cornuSpiral[t][[2]];

1/(x'[t]^2+y'[t]^2)^(3/2) Abs[x'[t]*y''[t]-x''[t]*y'[t]]//Simplify

]

We can also compute the curvature of the Cornu spiral using κ2 module. However, the curve needs to be made three-dimensional by adding a vanishing z-component before submitting it to the module.

```
cornuSpiral3D[t_]:=Append[cornuSpiral[t],0]
```

```
κ2[cornuSpiral3D][t]//PowerExpand
```

Corollary 2.7. Let r be a curve parametrized by arc length in $\mathbb{R}^2$. The turning angle at a point P on r(s) is the angle between the unit tangent vector and the x-axis at P. Let θ(s) be the turning angle at r(s). The curvature at P is

(2.29) $$\kappa(s) = \left| \frac{d\theta}{ds} \right| .$$

Proof.

Since r is a plane curve parametrized by arc length, the unit tangent vector is

T(s)=v(s),

T'(s)=v'(s)=a(s).

Notice that T(s)=(cos θ(s),sin θ(s)), where θ(s) is the angle between the unit tangent vector and the x-axis at P (see Figure 2.23).

```
Module[{r,a,u,v,s,z},
```

```
r[s_]={s,s^2};

a=1;

u=r[a];

v=r'[a];

z[0]=ParametricPlot[Evaluate[r[t]],{t,0,2},PlotRange->{{0,2.5},
{0,4}},AspectRatio->1];

z[1]={Hue[0.7],AbsoluteThickness[3],Arrow[{u,u+v}]};

z[2]={Dashed,Hue[0.7],AbsoluteThickness[3],Line[{u,u+{1,0}}]};

z[3]={Hue[0.7],PointSize[0.03],Point[u]};

z[4]=Circle[u,0.2,{0,ArcTan[v[[2]]/v[[1]]]}];z[5]=Text["r(s)",u+
{-0.4,-0.3}];z[6]=Text["r'(s)",u+1.1v];z[7]=Text["θ(s)",u+{0.3,0.2}];

z[8]=Text[Style["P",Italic],{0.9,1.1}];

Show[z[0],Epilog->Array[z,8],AxesLabel->{"x","y"}]

]
```

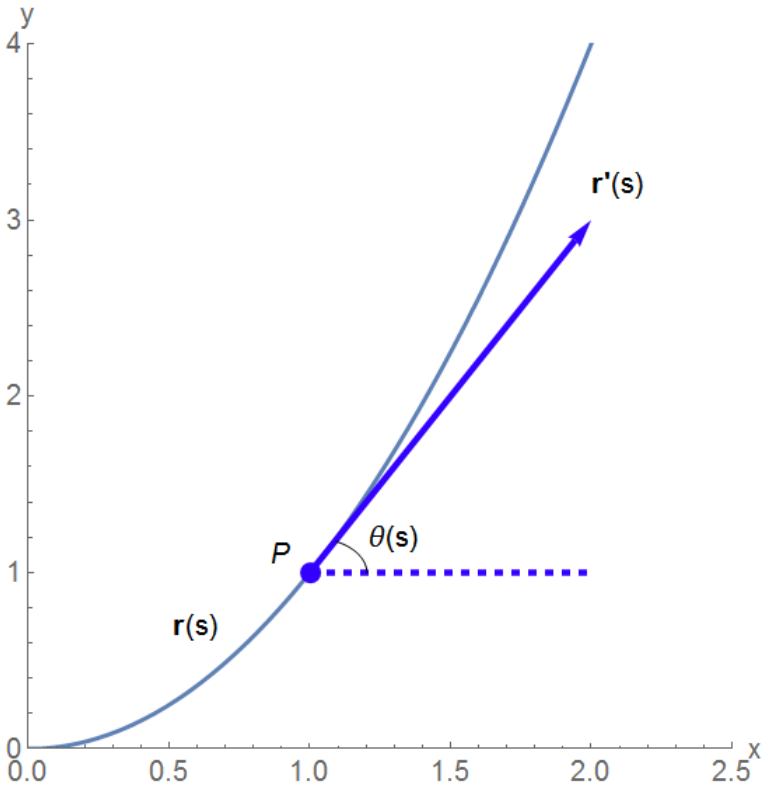


Figure 2.23. The unit tangent vector r'(s) is making an angle θ (called the turning angle) with the x-axis at P.

We compute the curvature by using Theorem 2.5 (equation 2.22)

$$K = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

.

We use Mathematica to prove this corollary:

Clear[T]

T[s_]:={Cos[θ[s]],Sin[θ[s]],0}

(* Compute the curvature using Theorem 2.5 *)

vectorLength[Cross[T[s], T'[s]]]/vectorLength[T[s]]^3

Hence,

$\kappa(s) = \left| \frac{d\theta}{ds} \right|$.

Definition 2.8.

Let $\mathbf{r} : (a, b) \in \mathcal{I} \to \mathbb{R}^2$ be a regular plane curve with nonzero curvature κ2(t) where a

(2.30) $\mathbf{r}(t) + \frac{1}{\kappa 2(t)} \mathbf{N}(t)$ **.**

The radius of this circle, ρ=1/κ2, is called the radius of curvature of the curve at the corresponding point.

Though it is defined for plane curves, the same definition is also applicable to space curves as is illustrated in the examples below.

The osculating circles lie in the plane spanned by T(t) and N(t) referred to as the osculating plane. The centers of all osculating circles to a curve form another curve called the evolute to the curve.

As we have seen in Example 2.12, a circle of radius 1/κ2 has curvature κ2, thus the curve and the osculating circle has the same curvature at point of contact.

Since the osculating plane is spanned by T(t) and N(t), the osculating circle of radius ρ can be parametrized by

$$\theta \mapsto \rho \cos\theta\,(-\mathbf{N}(t)) + \rho \sin\theta\,(\mathbf{T}(t)), \qquad \theta \in [0, 2\pi]$$

,

where N points toward the center of the circle.

Using Definition 2.8, the points on the circle is then

$$\mathbf{r}(t) + \frac{1}{\kappa 2(t)}\,\mathbf{N}(t) + \rho \cos\theta\,(-\mathbf{N}(t)) + \rho \sin\theta\,(\mathbf{T}(t))$$

,

$$\mathbf{r}(t) + \frac{1}{\kappa 2(t)}\,[(1 - \cos\theta)\,\mathbf{N}(t) + \sin\theta\,\mathbf{T}(t)]$$

.

Let us implement this in Mathematica by defining the module oscCircle to draw an osculating circle to the curve, and oscCenter to draw the center point of the osculating circle.

```
(* Define the osculating circle of the curve r at t0 *)

oscCircle[r_,t0_]:=

Module[{α},

α=r;

ParametricPlot3D[Evaluate[α[t]+1/κ2[α][t] ((1-Cos[θ])UN[α][t]+Sin[θ]
UT[α][t])/.t->t0],{θ,0,2π},PlotStyle->
{Red,Dashed,AbsoluteThickness[2]},PlotRange->All]

]
```

```
(* Define the center point of the osculating circle *)

oscCenter[r_,t0_]:=

Module[{α},

α:=r;

Graphics3D[{PointSize[0.03],Red,Point[α[t]+1/κ2[α][t] UN[α][t]]}/.t->t0]

]
```

Next, we define point function to plot a point in the curve:

```
(* Define a function to plot a point in the curve r at t0 *)

point[r_,t0_]:=Graphics3D[{PointSize[0.03],Blue,Point[r[t]]}/.t->t0]
```

Example 2.15. A parametric equation of the helix is defined by

r(t)=(cos t,sin t,t), **where** $t \in \mathbb{R}$ **.**

(a) Compute the curvature of the curve.

(b) Plot the curve and its osculating circle at t=0.

(c) Plot the evolute to the curve on the same graph.

Solution:

(a)

helix[t_]:={Cos[t],Sin[t],t}

(* The curvature of the helix *)

κ2[helix][t]

(b)

helixPlot=ParametricPlot3D[Evaluate[helix[t]],{t,-4π,4π},PlotRange-> {{-6,6},{-6,6},{-12,12}}]

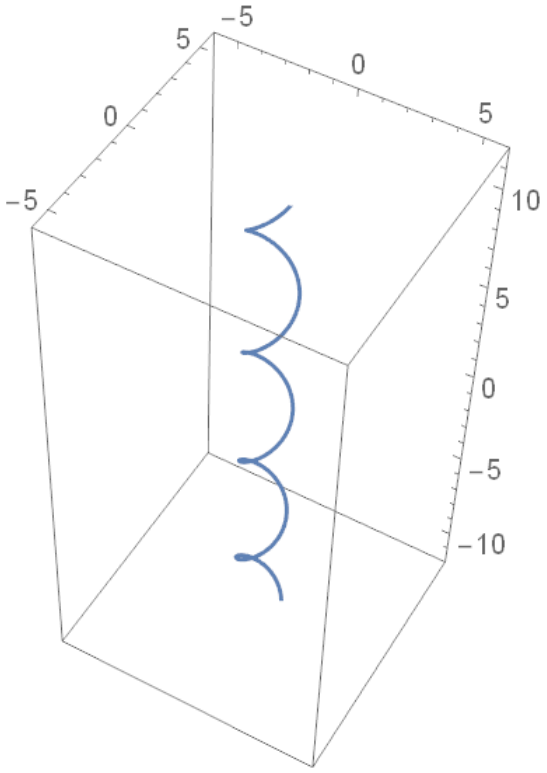


Figure 2.24. The helix r(t)=(cos t,sin t,t).

Show[helixPlot,point[helix,0],oscCircle[helix,0],oscCenter[helix,0],Plot Range->{{-6,6},{-6,6},{-12,12}}]

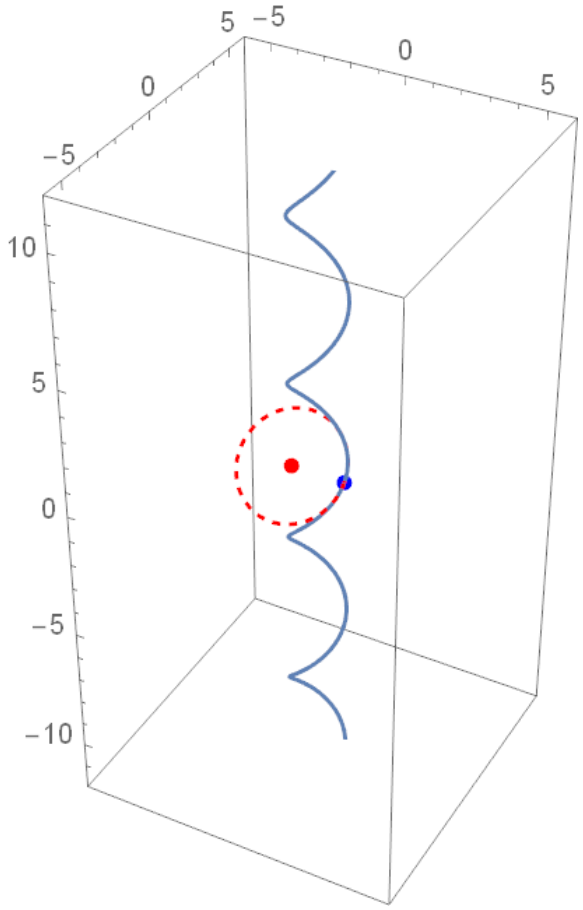
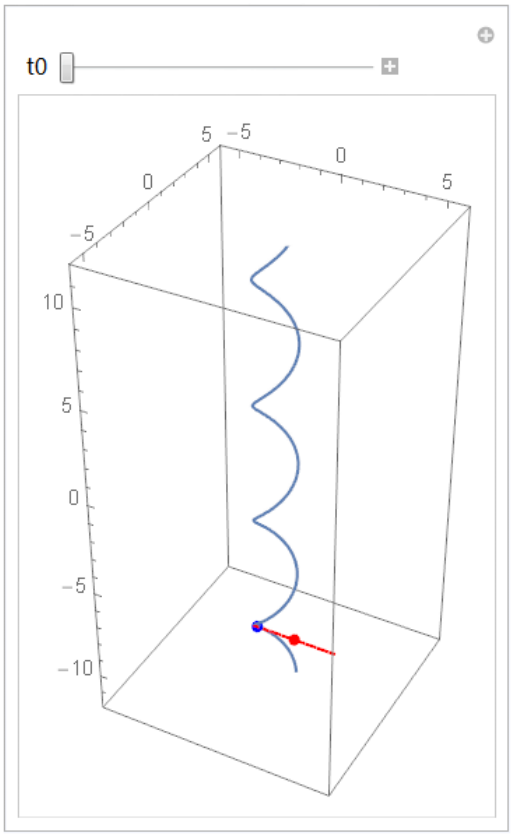


Figure 2.25. The helix r(t)=(cos t,sin t,t) and the osculating circle at t=0.

Module[{},Manipulate[Show[helixPlot,point[helix,t0],oscCircle[helix,t0],oscCenter[helix,t0],PlotRange->{{-6,6},{-6,6},{-12,12}}], {{t0,-3π},-3π,4π,2π/20}]

]



Animation 2.26. The movement of the osculating circle as it traces out the helix r(t)=(cos t,sin t,t).

(c)

```
helixPlot2=ParametricPlot3D[Evaluate[{helix[t],helix[t]+1/κ2[helix][t]
UN[helix][t]}],{t,-4π,4π},PlotRange->{{-6,6},{-6,6},{-12,12}},PlotStyle->
{Automatic,{Red,Dashed}}]
```

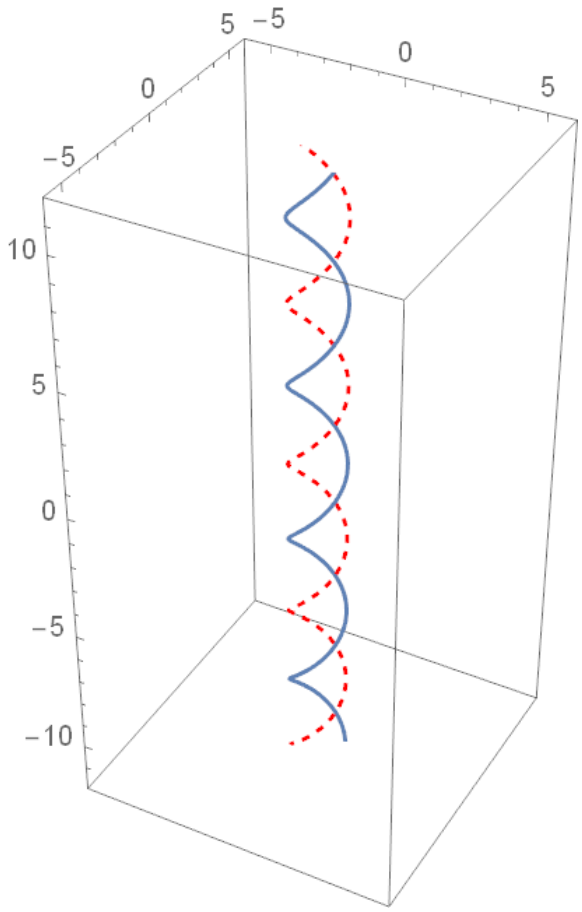
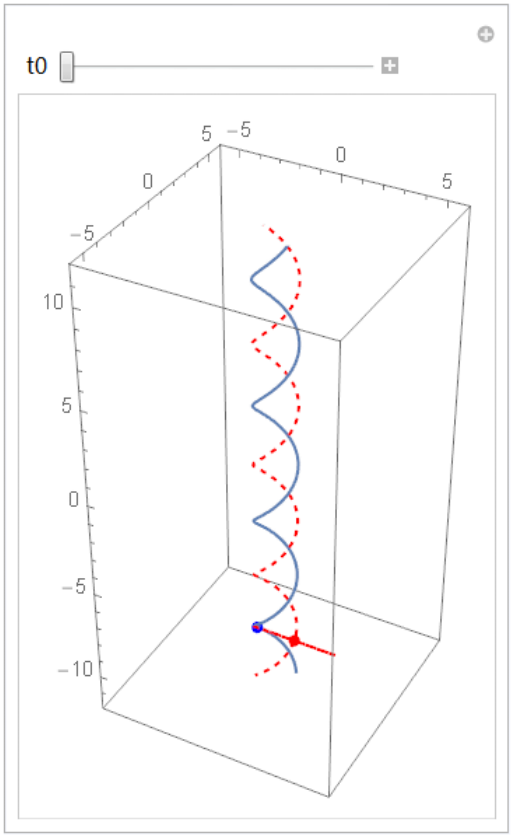


Figure 2.27. The centers of all osculating circles to the helix form another curve called the evolute to the helix (the red dashed curve).

Module[{},Manipulate[Show[helixPlot2,point[helix,t0],oscCircle[helix,t0],oscCenter[helix,t0],PlotRange->{{-6,6},{-6,6},{-12,12}}], {{t0,-3π},-3π,4π,2π/20}]

]



Animation 2.28. The helix r(t)=(cos t,sin t,t) and its evolute; with the osculating circles.

The above animation shows that the centers of all osculating circles trace out another curve called the evolute to the original curve. It demonstrates that the evolute of the helix is another helical curve.

Example 2.16. A curve is parametrized by

r(t)=(t cos t,t sin t,sin 2t), $-2\pi \le t \le 2\pi$.

(a) Compute the curvature of the curve.

(b) Plot the curve and its osculating circle at t=0.

(c) Plot the evolute to the curve on the same graph.

Solution:

(a)

```
Clear[r]

r[t_]:={t Cos[t],t Sin[t],Sin[2t]}


κ2[r][t]
```

(b)

```
curvePlot=ParametricPlot3D[Evaluate[r[t]],{t,-2π,2π},PlotRange->
{{-8,8},{-8,8},{-4,4}}]
```

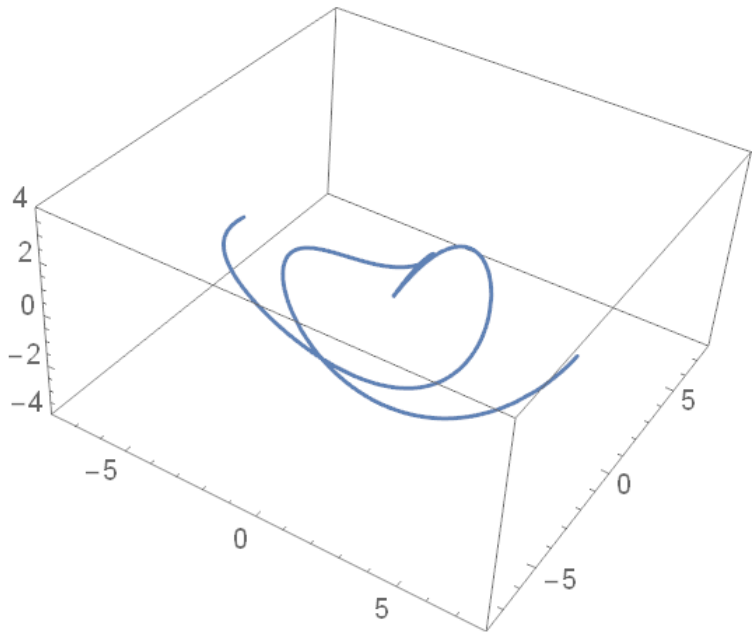


Figure 2.29. The curve r(t)=(t cos t,t sin t,sin 2t).

**Show[curvePlot,point[r,0],oscCircle[r,0],oscCenter[r,0],PlotRange->
{{-8,8},{-8,8},{-4,4}}]**

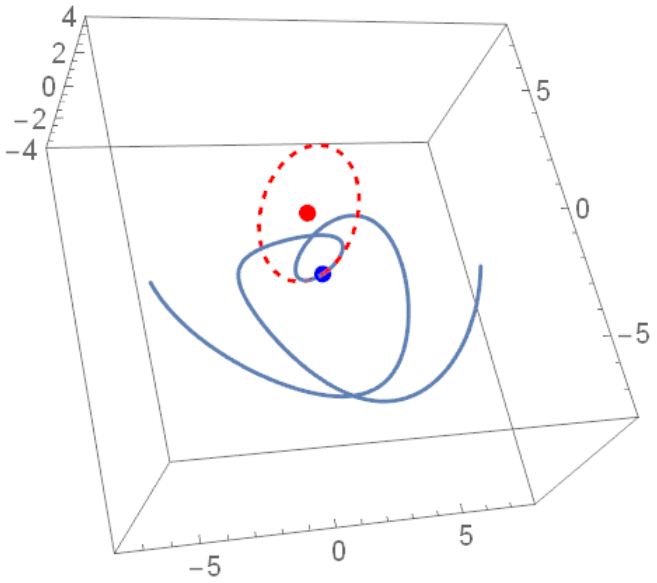


Figure 2.30. The curve r(t)=(t cos t,t sin t,sin 2t) and the osculating circle to the curve at t=0.

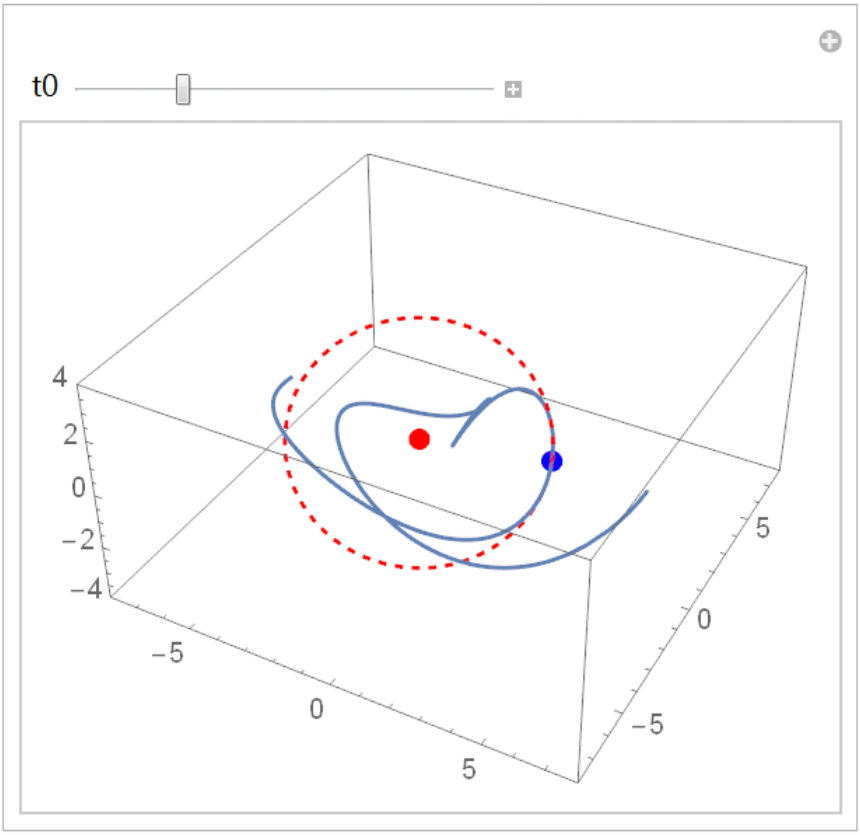
Module[{α,β},

α=curvePlot;

β=r;

Manipulate[Show[α,point[β,t0],oscCircle[β,t0],oscCenter[β,t0],PlotRange->{{-8,8},{-8,8},{-4,4}}],{{t0,-π},-2π,2π,2π/20}]

]



Animation 2.31. The movement of the osculating circle as it traces out the curve r(t)=(t cos t,t sin t,sin 2t)

(c)

curvePlot2=ParametricPlot3D[Evaluate[{r[t],r[t]+1/κ2[r][t] UN[r][t]}], {t,0,2π},PlotRange->{{-7,7},{-7,7},{-4,4}},PlotStyle-> {Automatic,Magenta},ImageSize->500]

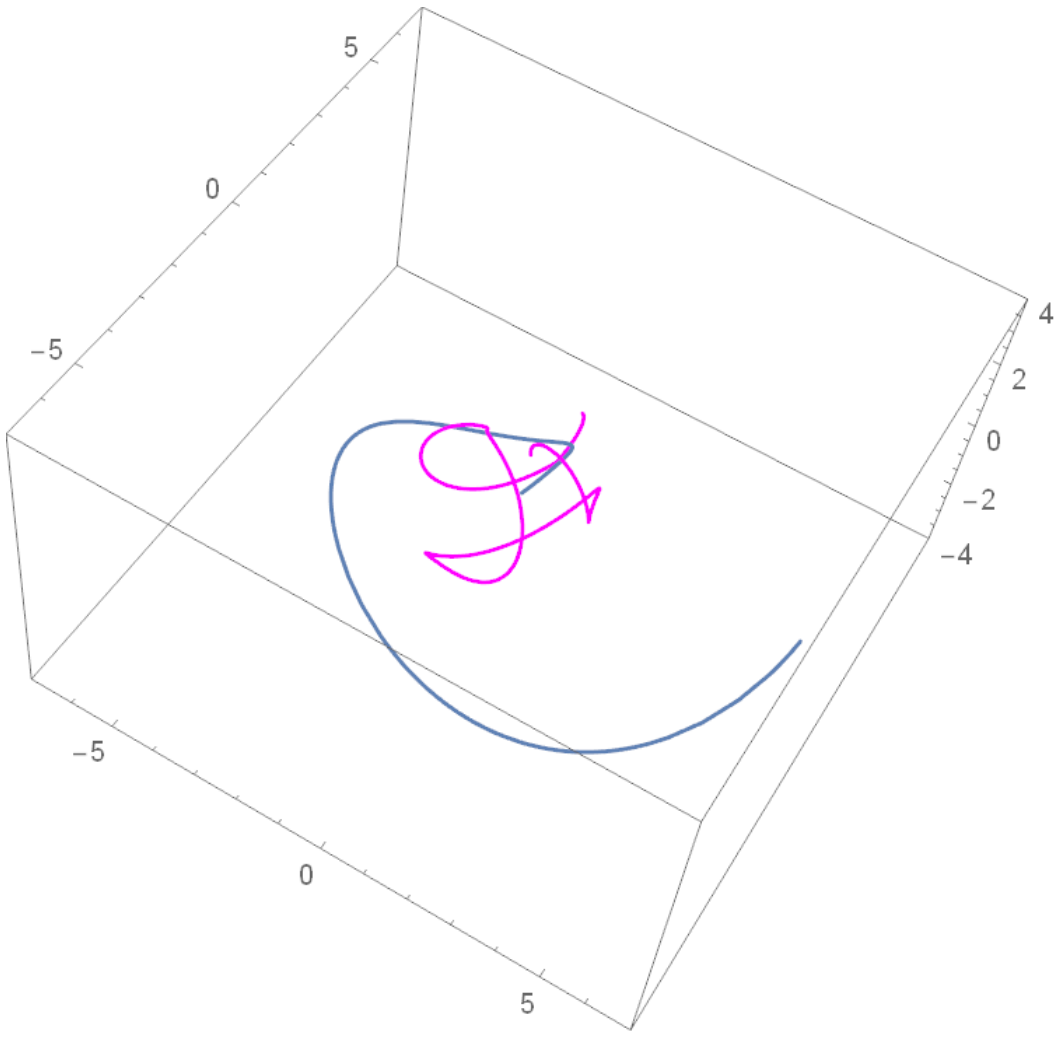
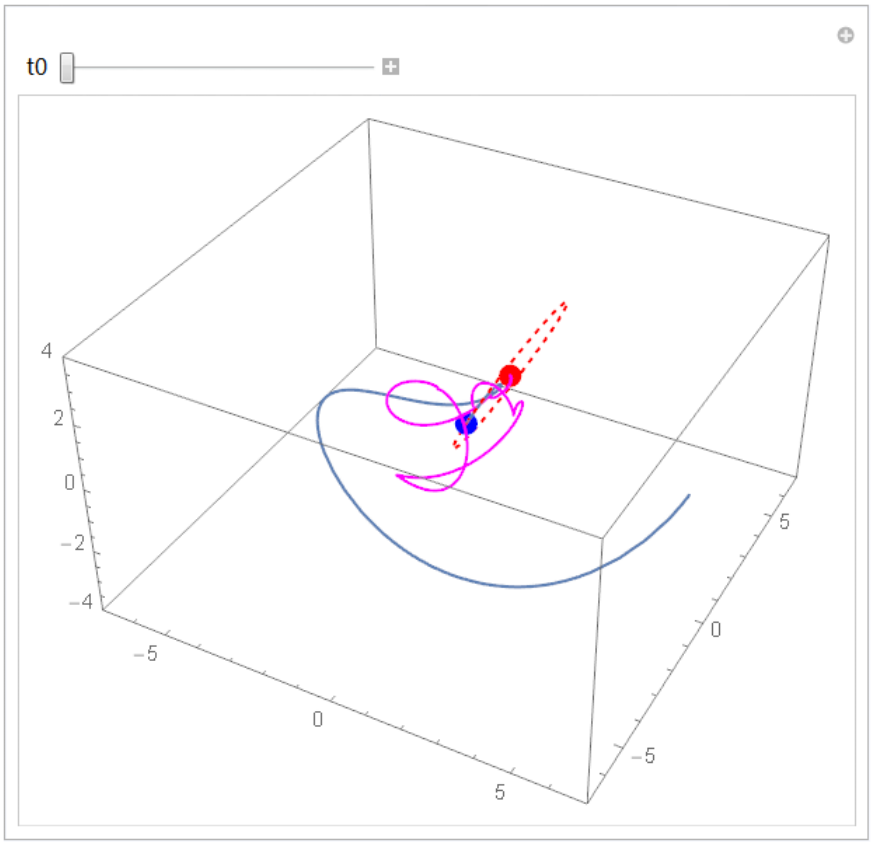


Figure 2.32. The curve r(t)=(t cos t,t sin t,sin 2t) and its evolute.

```
Module[{α,β},

α=curvePlot2;

β=r;

Manipulate[Show[α,point[β,t0],oscCircle[β,t0],oscCenter[β,t0],PlotRan
ge->{{-7,7},{-7,7},{-4,4}}],{{t0,0},0,2π,2π/20}]

]
```



Animation 2.33. The curve r(t)=(t cos t,t sin t,sin 2t) and its evolute; with the osculating circles.

2.3 Some applications

Using the mathematical tools we have studied earlier, we shall apply them to some applications in physical sciences.

2.3.1 Equations of motion in curvilinear coordinates

In Section 1.8 of Chapter 1 we study some curvilinear coordinates such as polar coordinates in the plane, cylindrical and spherical coordinates in three-dimensional space. Having defined velocity and acceleration vector we can now derive expression for these vectors in curvilinear coordinates.

2.3.1.1 Polar coordinates

The position vector r in polar coordinates is represented by equation (1.21):

$$\mathbf{r} = r\,\mathbf{e}_r \ .$$

Using Definition 2.2 the velocity vector becomes

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r \ .$$

By equation (1.22),

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta \ ,$$

where $v_r \equiv \dot{r}$ is the radial velocity component, and $v_\theta \equiv r\,\dot{\theta}$ is the angular velocity component.

The speed is

$$v = \sqrt{v_r^2 + v_\theta^2}$$

.

Differentiating again with respect to time, we obtain the acceleration vector:

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + r\,\ddot{\theta}\,\mathbf{e}_\theta + r\,\dot{\theta}\,\dot{\mathbf{e}}_\theta$$

.

Using equation (1.22) again we obtain

$$\mathbf{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\mathbf{e}_r + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\mathbf{e}_\theta$$

,

where $a_r \equiv \left(\ddot{r} - r\,\dot{\theta}^2\right)$ is the radial acceleration component, and $a_\theta \equiv \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)$ is the angular acceleration component.

The acceleration is

$$a = \sqrt{a_r^2 + a_\theta^2}$$

.

By Newton's Second Law of Motion F=m a, where m is the mass of the particle, we obtain the equations of motion in component forms:

$$F_r = m\,a_r = m\left(\ddot{r} - r\,\dot{\theta}^2\right),$$
$$F_\theta = m\,a_\theta = m\left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right).$$

2.3.1.2 Cylindrical coordinates

Recall from equation (1.27), a particle's position vector in cylindrical coordinates is written as

$$\mathbf{r} = r\,\mathbf{e}_r + z\,\mathbf{k}$$.

Using Definition 2.2 we differentiate the position vector with respect to time:

$$\mathbf{v} = \dot{\mathbf{r}} = r\,\dot{\mathbf{e}}_r + \dot{r}\,\mathbf{e}_r + z\,\dot{\mathbf{k}} + \dot{z}\,\mathbf{k}$$

.

Since k is constant,

$$\mathbf{v} = r\,\dot{\mathbf{e}}_r + \dot{r}\,\mathbf{e}_r + \dot{z}\,\mathbf{k}$$,

By equation (1.26),

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta + \dot{z}\,\mathbf{k}$$.

Taking another differentiation with respect to time, we obtain

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + r\,\ddot{\theta}\,\mathbf{e}_\theta + r\,\dot{\theta}\,\dot{\mathbf{e}}_\theta + \ddot{z}\,\mathbf{k}$$

.

Again by equation (1.26) we get

$$\mathbf{a} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + r\,\ddot{\theta}\,\mathbf{e}_\theta - r\,\dot{\theta}\,\dot{\theta}\,\mathbf{e}_r + \ddot{z}\,\mathbf{k}$$

.

Finally, the acceleration vector:

$$\mathbf{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\mathbf{e}_r + \left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\,\mathbf{k}$$

.

The equations of motion in component forms are

$$F_r = m\left(\ddot{r} - r\,\dot{\theta}^2\right),$$
$$F_\theta = m\left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right),$$
$$F_z = m\,\ddot{z}.$$

2.3.1.3 Spherical coordinates

The position vector r of a particle in spherical coordinates is represented by equation (1.31):

$$\mathbf{r} = r\,\mathbf{e}_r \ .$$

Differentiating with respect to time gives the velocity vector:

$$\mathbf{v} = \dot{\mathbf{r}} = r\,\dot{\mathbf{e}}_r + \dot{r}\,\mathbf{e}_r \ .$$

Substituting

$$\dot{\mathbf{e}}_r = \dot{\theta}\,\mathbf{e}_\theta + \sin\theta\,\dot{\phi}\ \mathbf{e}_\phi$$

from equation (1.33) into v we obtain

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta + r\sin\theta\,\dot{\phi}\,\mathbf{e}_\phi$$

.

To obtain the acceleration vector, we differentiate the velocity vector with respect to time:

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + r\,\ddot{\theta}\,\mathbf{e}_\theta + r\,\dot{\theta}\,\dot{\mathbf{e}}_\theta$$
$$+ \dot{r}\sin\theta\,\dot{\phi}\,\mathbf{e}_\phi + r\cos\theta\,\dot{\theta}\,\dot{\phi}\,\mathbf{e}_\phi + r\sin\theta\,\ddot{\phi}\,\mathbf{e}_\phi + r\sin\theta\,\dot{\phi}\,\dot{\mathbf{e}}_\phi$$

.

Substituting the derivatives of the unit vectors (1.33), (1.34) and (1.35):

$$\dot{\mathbf{e}}_r = \dot{\theta}\,\mathbf{e}_\theta + \sin\theta\,\dot{\phi}\;\mathbf{e}_\phi,$$
$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\,\mathbf{e}_r + \cos\theta\,\dot{\phi}\;\mathbf{e}_\phi,$$
$$\dot{\mathbf{e}}_\phi = -\sin\theta\,\dot{\phi}\,\mathbf{e}_r - \cos\theta\,\dot{\phi}\;\mathbf{e}_\theta.$$

into a, we obtain

$$\mathbf{a} = \ddot{r}\,\mathbf{e}_r + \dot{r}\left(\dot{\theta}\,\mathbf{e}_\theta + \sin\theta\,\dot{\phi}\;\mathbf{e}_\phi\right) + \dot{r}\,\dot{\theta}\,\mathbf{e}_\theta + r\,\ddot{\theta}\,\mathbf{e}_\theta + r\,\dot{\theta}\left(-\dot{\theta}\,\mathbf{e}_r + \cos\theta\,\dot{\phi}\;\mathbf{e}_\phi\right)$$
$$+ \dot{r}\sin\theta\,\dot{\phi}\,\mathbf{e}_\phi + r\cos\theta\,\dot{\theta}\,\dot{\phi}\,\mathbf{e}_\phi$$
$$+ r\sin\theta\,\ddot{\phi}\,\mathbf{e}_\phi + r\sin\theta\,\dot{\phi}\left(-\sin\theta\,\dot{\phi}\,\mathbf{e}_r - \cos\theta\,\dot{\phi}\;\mathbf{e}_\theta\right).$$

After rearranging,

$$\mathbf{a} = \left(\ddot{r} - r\,\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2\right)\mathbf{e}_r$$
$$+ \left(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2\right)\mathbf{e}_\theta$$
$$+ \left(2\,\dot{r}\sin\theta\,\dot{\phi} + 2\,r\cos\theta\,\dot{\theta}\,\dot{\phi} + r\sin\theta\,\ddot{\phi}\right)\mathbf{e}_\phi.$$

Sometimes, it is convenient to work with different expression of the acceleration vector.

Notice that

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 2\dot{r}\dot{\theta} + r\ddot{\theta},$$

and

$$\begin{aligned}
&\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right) \\
&= \frac{2r\dot{r}\dot{\phi}\sin^2\theta}{r\sin\theta} + \frac{r^2\ddot{\phi}\sin^2\theta}{r\sin\theta} + \frac{2r^2\dot{\phi}\sin\theta\cos\theta\dot{\theta}}{r\sin\theta} \\
&= 2\dot{r}\sin\theta\dot{\phi} + r\sin\theta\ddot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi}.
\end{aligned}$$

Thus, we can rewrite a as

$$\begin{aligned}
\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2\right)\mathbf{e}_r \\
&+ \left(\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) - r\sin\theta\cos\theta\dot{\phi}^2\right)\mathbf{e}_\theta \\
&+ \left(\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right)\right)\mathbf{e}_\phi.
\end{aligned}$$

So, the equations of motion in component forms are

$$\begin{aligned}
F_r &= m\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2\right), \\
F_\theta &= m\left(\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) - r\sin\theta\cos\theta\dot{\phi}^2\right), \\
F_\phi &= m\left(\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right)\right).
\end{aligned}$$

2.3.2 Kepler's Laws of Planetary Motion

In this section, we derive Kepler's three laws of planetary motion.

Kepler's three laws can be summarized below:

1. Planets move in elliptical orbits about the Sun, with the Sun at a focus of the ellipse.

2. The area per unit time swept out by a position vector from the Sun to a planet is constant.

3. The square of a planet's period T is proportional to the cube of the semimajor axis of the ellipse.

(2.31) $$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3$$,

where

a: the semimajor axis of the ellipse

G: the universal gravitational constant

M: the mass of the Sun

We assume that the Sun and the planet be point masses and the mass of the Sun is much larger than the planet so as to ignore the Sun's motion and assume that the planet revolves around the center of the Sun. We also assume that the only physical effect is the gravitational attraction between the Sun and the planet and so ignore the gravitation attraction of other planets on each other; the so-called two-body problem.

First we show that the motion is planar and the Sun lies in the planet's plane of motion.

Without loss of generality, let the Sun be at the origin of the coordinate system. Let r=r(t) be the position vector of the planet. Then the velocity vector of the planet is

v(t)=r'(t).

By Newton's Universal Law of Gravitation, the Sun attracts the planet with a gravitational force

(2.32) $$\mathbf{F}(r) = -\frac{GMm}{r^2}\mathbf{e}_r ,$$

where m is the mass of the plane, r the distance between the Sun and the planet.

$\mathbf{e}_r$ is the unit radial vector given by

$$\mathbf{e}_r = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} .$$

$$\mathbf{r}(t) = \|\mathbf{r}(t)\|\,\mathbf{e}_r .$$

Observe that the force is central, that is, it depends only on the distance r, acting along the line connecting the two bodies.

By Newton's Second Law of Motion we have

$$m\,\mathbf{a} = -\frac{GMm}{r^2}\mathbf{e}_r ,$$

where a is the acceleration vector of the planet.

Thus,

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{e}_r .$$

Note that the position vector and the acceleration vector are parallel but in opposite direction.

Proposition 2.9 Let r and m be the position vector and mass of the planet, respectively. The angular momentum vector defined as

(2.33) $\mathbf{L} = \mathbf{r} \times \mathbf{p}$,

is constant.

$\mathbf{p} \equiv m\,\mathbf{v}$ is called the linear momentum vector of the planet, where v the velocity vector.

Proof.

Taking the derivative of L, and by using the product rule of cross products we get

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\,\mathbf{v}) = m\,\frac{d\mathbf{r}}{dt} \times \mathbf{v} + m\,\mathbf{r} \times \frac{d\mathbf{v}}{dt}$$
$$= m\,\mathbf{v} \times \mathbf{v} + m\,\mathbf{r} \times \mathbf{a}$$

Since cross product of parallel vectors vanishes, the first and second terms on the right-hand side vanish. Hence, L is constant. In physics, this is known as the conservation of the angular momentum.

By Proposition 2.9 L is perpendicular to both r and v since m is a scalar. Because L is constant, both r and v lie in a plane perpendicular to vector L. Hence, the motion is planar and the Sun lies in the planet's plane of motion. For discussions below we assume motion in the xy-plane.

In polar coordinates (r,θ) the infinitesimal area swept out by position vector r from t_1 to t_2 (see Figure 2.34) is

$$dA = \frac{1}{2}\,\|\,\mathbf{r}(t_1) \times \mathbf{r}(t_2)\,\| = \frac{1}{2}\,\|\,\mathbf{r}(t_1) \times \mathbf{r}(t_1 + dt)\,\|$$

.

Using Taylor series and keeping only the first order term, we obtain

$$dA \simeq \frac{1}{2} \| \mathbf{r}(t_1) \times [\, \mathbf{r}(t_1) + \dot{\mathbf{r}}(t_1)\, dt \,] \|$$
$$\simeq \frac{1}{2} \| \mathbf{r}(t_1) \times \dot{\mathbf{r}}(t_1) \|\, dt$$

Since $\mathbf{v}(t_1) = \dot{\mathbf{r}}(t_1)$, by Proposition 2.9:

$$\| \mathbf{r}(t_1) \times \dot{\mathbf{r}}(t_1) \| = \| \frac{\mathbf{L}}{m} \| = \frac{1}{m} \| \mathbf{L} \|$$

, a constant.

So, the infinitesimal area becomes

$$dA = \frac{1}{2m} \| \mathbf{L} \|\, dt$$.

Thus,

(2.34) $\frac{dA}{dt} = \frac{1}{2m} \| \mathbf{L} \|$, a constant.

```
Module[{x,y,ellipse,vector,area,text},

x[θ_]:=5 Cos[θ]-2;

y[θ_]:=3Sin[θ];

ellipse=ParametricPlot[{x[θ],y[θ]},{θ,0,π/2},PlotRange->
{{0,3},Automatic}];

vector[1]=Graphics[Arrow[{{0,0}, {x[π/8],y[π/8]}}]];

vector[2]=Graphics[Arrow[{{0,0}, {x[π/5],y[π/5]}}]];

vector[3]=Graphics[Arrow[{{x[π/8],y[π/8]}, {x[π/5],y[π/5]}}]];

area=Table[Graphics[{GrayLevel[0.7],Line[{{0,0}, {x[θ],y[θ]}}]}],
{θ,π/8,π/5,π/300}];
```

```
text[1]=Graphics[Text["r(t1)",{2,0.7}]];

text[2]=Graphics[Text["r(t2)",{1.5,1.5}]];

text[3]=Graphics[Text["Δθ",{0.7,0.4}]];

Show[ellipse,Array[vector,3],area,Array[text,3]]

]
```

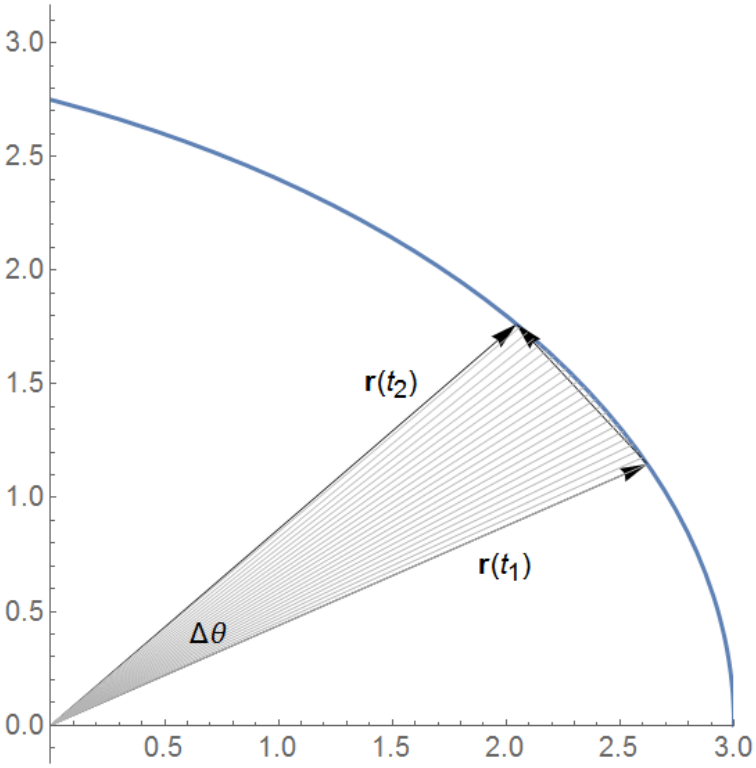


Figure 2.34. The infinitesimal area swept out by position vector r from t_1 to t_2 .

Alternatively, the area of circular sector of angle Δθ and radius r can be approximated by

$$\Delta A = \frac{1}{2} r^2 \Delta\theta$$.

By dividing the circular sector into n partition of each with a circular sector of radius r_j , the total area is approximated by the sum

$$A \approx \sum_{j=1}^{n} \Delta A_j = \frac{1}{2} \sum_{j=1}^{n} r_j^2 \Delta\theta_j$$

.

Taking n->∞, we get

$$A = \lim_{n\to\infty} \sum_{j=1}^{n} \Delta A_j = \frac{1}{2} \lim_{n\to\infty} \sum_{j=1}^{n} r_j^2 \Delta\theta_j$$

,

which is the Riemann sum for the integral of the area from θ to $\theta + \Delta\theta$:

$$A = \frac{1}{2} \int_{\theta}^{\theta+\Delta\theta} r^2 \, d\theta$$.

Hence, the area swept out by the planet from 0 to θ is

$$A = \frac{1}{2} \int_{0}^{\theta} r^2 \, d\theta$$.

By the Fundamental Theorem of Calculus, we have

(2.35) $\frac{dA}{d\theta} = \frac{1}{2} r^2$.

Since $\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt}$,

and using equation (2.34) and (2.35), we get

$$\frac{1}{2m} \| \mathbf{L} \| = \frac{1}{2} r^2 \theta' ,$$

where prime denotes differentiation with respect to t.

Hence,

(2.36) $L \equiv \| \mathbf{L} \| = m r^2 \theta'$, a constant.

Theorem 2.10 (Kepler's First Law)

Planets move in elliptical orbits about the Sun, with the Sun at a focus of the ellipse.

Proof.

Recall from the acceleration vector a

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\frac{GM}{r^2} \mathbf{e}_r .$$

But,

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{d\theta} \frac{d\theta}{dt} = -\frac{GM}{r^2} \mathbf{e}_r$$

$$\theta' \frac{d\mathbf{v}}{d\theta} = -\frac{GM}{r^2} \mathbf{e}_r$$

(2.37) $$d\mathbf{v} = -\frac{GM}{r^2 \theta'} \mathbf{e}_r \, d\theta .$$

Assuming the Sun is at the origin, we have then the unit radial vector $\mathbf{e}_r = \cos\theta \, \mathbf{i} + \sin\theta \, \mathbf{j}$, where θ is the angle between the position vector of the planet and the x-axis (see Section 1.8.1).

So, equation (2.37) becomes

$$d\mathbf{v} = -\frac{GM}{r^2\,\theta'}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\,d\theta$$

.

Substituting $\theta' = \frac{L}{mr^2}$ from equation (2.36) into dv gives

$$d\mathbf{v} = -\frac{GMm}{L}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\,d\theta$$

.

Integrating,

$$\int d\mathbf{v} = -\frac{GMm}{L}\int(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\,d\theta$$

.

(2.38)

$$\mathbf{v} = \frac{GMm}{L}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) + \mathbf{c}$$

.

where c is integration constant vector depends on the initial conditions of the planet.

Assume for the initial conditions at t=0 the planet is at the x-axis:

$$x(0) = x_0, \qquad y(0) = 0$$,

and the velocity vector points in positive y-direction such that

$$v_x(0) = 0, \qquad v_y(0) = v_{y0}$$.

Thus, with these initial conditions equation (2.38) becomes

$$v_{y0}\,\mathbf{j} = \frac{GMm}{L}\,\mathbf{j} + \mathbf{c}$$,

which yields $$\mathbf{c} = \left(v_{y0} - \frac{GMm}{L}\right)\mathbf{j}$$.

Write $\mathbf{c} \equiv c\,\mathbf{j}$, where $$c = \left(v_{y0} - \frac{GMm}{L}\right)$$.

Recall from Section 1.8.1 equation (1.20):

$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$.

Thus,

(2.39) $$\mathbf{v} = \frac{GMm}{L}\,\mathbf{e}_\theta + c\,\mathbf{j}$$,

where $$c = \left(v_{y0} - \frac{GMm}{L}\right)$$.

So the angular momentum vector becomes

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = r\,\mathbf{e}_r \times m\left(\frac{GMm}{L}\,\mathbf{e}_\theta + c\,\mathbf{j}\right)$$

,

$$\mathbf{L} = \frac{GMm^2 r}{L}\,(\mathbf{e}_r \times \mathbf{e}_\theta) + r\,m\,(\mathbf{e}_r \times c\,\mathbf{j})$$

.

But, by equation (1.20):

$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{k}$ and $\mathbf{e}_r \times c\,\mathbf{j} = c\cos\theta\,\mathbf{k}$,

where k is the unit vector in z-axis.

Thus,

$$\mathbf{L} = \frac{GMm^2 r}{L}\left(1 + \frac{cL}{GMm}\cos\theta\right)\mathbf{k}$$

.

Write

(2.40) $$\mathbf{L} \equiv \frac{kr}{L}(1 + e\cos\theta)\,\mathbf{k}$$,

where $k \equiv GMm^2$ and $e \equiv \frac{cL}{GMm}$ are constants.

The magnitude of the angular momentum is

$L = \frac{kr}{L}(1 + e\cos\theta)$, a constant;

and we obtain the polar equation for the planet's orbit:

(2.41) $$r = \frac{L^2}{k(1 + e\cos\theta)}$$.

Depending on the values of e which is known as eccentricity of the orbit, the trajectory can be an ellipse, circle, parabola or hyperbola. Only the first two trajectories are bounded. Since a circle is a special case of an ellipse, so if the orbit is bounded, then the trajectory must be an ellipse.

Theorem 2.11 (Kepler's Second Law)

The area per unit time swept out by a position vector from the Sun to a planet is constant.

```
Module[{x,y,ellipse,section},

x[θ_]:=5 Cos[θ]-2;

y[θ_]:=3Sin[θ];

ellipse=ParametricPlot[{x[θ],y[θ]},{θ,0,2π}];

section[1]=Table[Graphics[{GrayLevel[0.7],Line[{{0,0}, {x[θ],y[θ]}}]}],
{θ,0,π/4,π/200}];

section[2]=Table[Graphics[{GrayLevel[0.7],Line[{{0,0}, {x[θ],y[θ]}}]}],
{θ,5π/7,17 π/20, π/200}];

Show[ellipse,Array[section,2]]

]
```

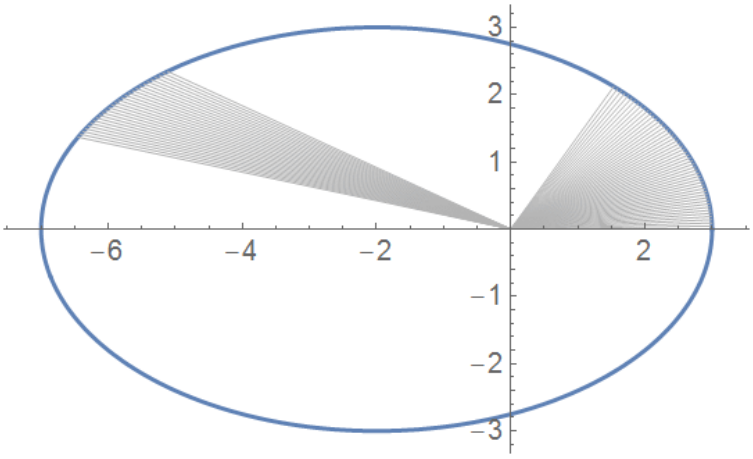


Figure 2.35. The area per unit time swept out by a position vector from the Sun to a planet is constant (Kepler's Second Law).

Proof.

Recall from our previous result (equation 2.34):

$\frac{dA}{dt} = \frac{1}{2m} \|\mathbf{L}\|$ constant.

Thus, the area swept out by the planet is the constant $\frac{1}{2m} \|\mathbf{L}\|$ times the time interval Δt.

Theorem 2.12 (Kepler's Third Law)

The square of a planet's period T is proportional to the cube of the semimajor axis of the ellipse.

Proof.

First, we compute the total area swept out by the planet for a complete orbit in time T, which is called the period of the orbit. Recall from geometry, the area of an ellipse with semimajor and semiminor axes having lengths a and b, respectively, is known to be $\pi a b$.

Thus,

$$\pi a b = \int_0^T \frac{dA}{dt} dt$$.

Substituting $\frac{dA}{dt}$ of equation (2.34) into the integrand above gives

$$\pi a b = \frac{1}{2m} \|\mathbf{L}\| \int_0^T dt = \frac{L}{2m} T$$

.

So, the period T is given by

$$T = \frac{2\pi a b m}{L}$$,

(2.42) $$T^2 = \frac{4\pi^2 a^2 b^2 m^2}{L^2} .$$

Recall from the polar equation of the orbit, equation (2.41):

$$r = \frac{L^2}{k(1 + e\cos\theta)} .$$

But the polar equation of the ellipse (see [Zwill] page 228, equation 4.10.12) is also given by

(2.43) $$r = \frac{l}{1 + e\cos\theta} ,$$

where e is the eccentricity of the orbit and l is the half of the latus rectum.

The lengths of the semimajor axis and semiminor axis are related by

$$l = \frac{b^2}{a} .$$

Thus,

$$\frac{b^2}{a} = \frac{L^2}{k} ,$$

$$b^2 = \frac{a L^2}{k} .$$

Substituting b^2 into equation (2.42) gives

$$T^2 = \frac{4\pi^2 a^2 a L^2 m^2}{k L^2} ,$$

$$T^2 = \frac{4\pi^2 a^3 m^2}{k} = \frac{4\pi^2 a^3 m^2}{G M m^2} = \left(\frac{4\pi^2}{G M}\right) a^3$$

.

Note that this relation between the period of the orbit and the length of the semimajor axis depends solely on the mass of the Sun M.

2.3.3 The Lorentz force

Consider the motion of a point particle of mass m and charge q in the presence of electric field vector E and the magnetic field vector B. The force exerted on the particle according to Lorentz force law is given by

(2.44) $$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right),$$

where c is the speed of light and $v \ll c$.

Suppose the initial position of the particle is at the origin and with initial velocity $\mathbf{v}(0) = (v_{x0}, v_{y0}, v_{z0})$. Let the magnetic field be uniform and lie in the direction of z-axis

$\mathbf{B} = B\,\mathbf{k}$.

Consider the following cases:

Case 1. There is no electric field. The particle enters the magnetic field perpendicular to it.

The Lorentz force becomes

$$\mathbf{F} = \frac{q}{c}\mathbf{v} \times \mathbf{B}$$.

Observe that in this case, the Lorentz force is perpendicular to both v and B; where B lies in the direction of z-axis and the velocity of the

particle is parallel to the xy-plane.

Write

$$\mathbf{v}(t) = v_x(t)\,\mathbf{i} + v_y(t)\,\mathbf{j}\ .$$

So,

$$\mathbf{F} = \frac{q}{c}\,[v_x(t)\,\mathbf{i} + v_y(t)\,\mathbf{j}] \times B\,\mathbf{k}$$

,

$$\mathbf{F} = \frac{q}{c}\,(v_y\,B\,\mathbf{i} - v_x\,B\,\mathbf{j})\ .$$

With these conditions, note that the motion of the particle is in the xy-plane.

According to Newton's Second Law of Motion

$$m\,(\ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j}) = \frac{q}{c}\,(\dot{y}\,B\,\mathbf{i} - \dot{x}\,B\,\mathbf{j})$$

,

where dot denotes differentiation with respect to t.

Equating like vector components leads to a system of coupled differential equations:

(2.45)
$$\begin{cases} m\,\ddot{x} = q\,B\,\dot{y}, \\ m\,\ddot{y} = -q\,B\,\dot{x}. \end{cases}$$

where for simplicity we set c=1.

Suppose at t=0, the velocity does not have component along x-axis but y-component such that

$$v_{y0} \equiv \dot{y}(0) = \dot{y}_0 , \qquad v_{x0} \equiv \dot{x}(0) = \dot{x}_0 = 0 ,$$

and

$$x(0) = x_0 , \qquad y(0) = y_0 .$$

The initial speed is then

$$v = \sqrt{{v_{x0}}^2 + {v_{y0}}^2} = \sqrt{{\dot{y}_0}^2} = \dot{y}_0$$

.

Using DSolve command in Mathematica we find the solutions to the above differential equations with initial conditions:

```
Clear[x,y,x0,y0,q,B]

sol=DSolve[{m x''[t]==q B y'[t],m y''[t]==-q B
x'[t],x'[0]==0,y'[0]==vy0,x[0]==x0,y[0]==y0},{x[t],y[t]},t]
```

We obtain the solutions:

(2.46)

$$\begin{cases} x(t) - x_0 = \frac{\dot{y}_0 m}{q B}\left(1 - \cos\frac{q B}{m} t\right), \\ y(t) - y_0 = \frac{\dot{y}_0 m}{q B} \sin\frac{q B}{m} t. \end{cases}$$

The solutions are parametric equations of a circle with constant radius (see Figure 2.36):

$$r = \frac{\dot{y}_0\, m}{q\, B}$$.

Since a particle moving in a circle has constant speed, write:

(2.47) $$r = \frac{v\, m}{q\, B} = \frac{v}{\omega_c}$$,

where $\omega_c \equiv \frac{q\, B}{m}$ is known as the cyclotron frequency (see [Cyclo] and [ThoMa]).

```
{x[t],y[t]}/.sol[[1]]
```

```
ParametricPlot[{x[t],y[t]}/.sol[[1]]/.{x0->0,y0->0,vy0->1,m->1,q->1,B->1},
{t,0,2π},ImageSize->300]
```

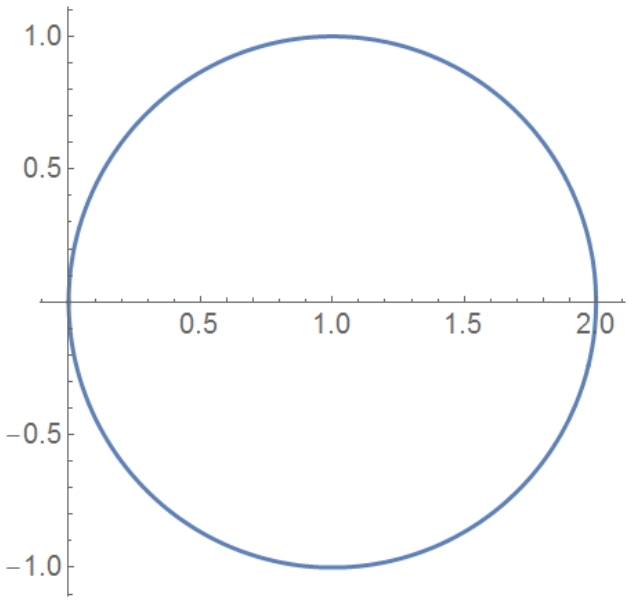


Figure 2.36. The trajectory of the particle under Lorentz force is a circle under certain conditions.

The time to complete one orbit is calculated using the equation:

(2.48) $$T = \frac{2\pi r}{v} = \frac{2\pi v m}{v q B} = \frac{2\pi m}{q B} .$$

It is found that the period is independent of the radius (see [Cyclo]).

Case 2. The uniform electric field E lies in the yz-plane.

The force on a particle of mass m that carries a charge q and has a velocity v is given by

$$\mathbf{F} = q\,\mathbf{E} + q\,\mathbf{v} \times \mathbf{B} ,$$

where for simplicity we set c=1.

In this case

$$\mathbf{E} = E_y\,\mathbf{j} + E_z\,\mathbf{k} , \qquad \mathbf{B} = B\,\mathbf{k} .$$

According to Newton's Second Law of Motion:

$$m\left(\ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}\right) = q\left(E_y\,\mathbf{j} + E_z\,\mathbf{k}\right) + q\left(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}\right) \times B\,\mathbf{k}$$

,

$$m\left(\ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}\right) = q\left(E_y\,\mathbf{j} + E_z\,\mathbf{k}\right) + q\,B\,v_y\,\mathbf{i} - q\,B\,v_x\,\mathbf{j}$$

.

Equating like components, we obtain a set of coupled differential equations:

$$\begin{cases} m\ddot{x} = qB\dot{y}, \\ m\ddot{y} = qE_y - qB\dot{x}, \\ m\ddot{z} = qE_z. \end{cases} \tag{2.49}$$

Observe that the first two equations are coupled in x and y, but the third equation is solely in z.

We first solve the third differential equation of (2.49) by integration:

$$\int_{\dot{z}_0}^{\dot{z}} d\dot{z} = \frac{q}{m} E_z \int_0^t dt,$$
$$\dot{z}(t) - \dot{z}_0 = \frac{qE_z}{m} t,$$
$$\dot{z}(t) = \dot{z}_0 + \frac{qE_z}{m} t.$$

Integrating once more,

$$\int_{z_0}^{z} dz = z(t) - z_0 = \dot{z}_0 t + \frac{qE_z}{2m} t^2$$

.

So,

$$z(t) = z_0 + \dot{z}_0 t + \frac{qE_z}{2m} t^2 \quad . \tag{2.50}$$

Next, we use DSolve command to find the velocities $v_x(t)$ and $v_y(t)$ of equation (2.49):

(* Find the velocity solutions for vx and vy of equation (2.49) *)

```
Clear[x,y,m,q,B,vsol,vx,vy]

vsol=DSolve[{m vx'[t]==q B vy[t],m vy'[t]==q Ey-q B
vx[t],vx[0]==vx0,vy[0]==vy0},{vx[t],vy[t]},t]//Simplify
```

The velocity solutions of the first two differential equations of (2.49) are obtained from Mathematica:

(2.51)

$$\dot{x}(t) = \frac{1}{B}\left[(B\,\dot{x}_0 - E_y)\cos\left(\frac{B\,q\,t}{m}\right) + B\,\dot{y}_0 \sin\left(\frac{B\,q\,t}{m}\right) + E_y\right]$$

,

(2.52)

$$\dot{y}(t) = \frac{1}{B}\left[(E_y - B\,\dot{x}_0)\sin\left(\frac{B\,q\,t}{m}\right) + B\,\dot{y}_0 \cos\left(\frac{B\,q\,t}{m}\right)\right]$$

.

Note that both solutions are oscillatory with the same period:

(2.53) $$T = \frac{2\,\pi\,m}{B\,q}$$.

```
vx[t_]=vx[t]/.vsol[[1]]

vy[t_]=vy[t]/.vsol[[1]]
```

The average velocity of $\dot{x}$ over one complete period is defined as

$$\langle \dot{x} \rangle = \frac{1}{T} \int_0^T \dot{x}(t)\, dt$$

,

$$\langle \dot{x} \rangle = \frac{Bq}{2\pi m} \frac{1}{B} \int_0^{\frac{2\pi m}{Bq}} \left[(B\dot{x}_0 - E_y) \cos\left(\frac{Bqt}{m}\right) + B\dot{y}_0 \sin\left(\frac{Bqt}{m}\right) + E_y \right] dt$$

.

It is found to be:

(2.54) $$\langle \dot{x} \rangle = \frac{E_y}{B}$$.

Similarly, the average velocity of $\dot{y}$ over one complete period is

$$\langle \dot{y} \rangle = \frac{1}{T} \int_0^T y(t)\, dt$$

,

$$\langle \dot{y} \rangle = \frac{Bq}{2\pi m} \frac{1}{B} \int_0^{\frac{2\pi m}{Bq}} \left[(E_y - B\dot{x}_0) \sin\left(\frac{Bqt}{m}\right) + B\dot{y}_0 \cos\left(\frac{Bqt}{m}\right) \right] dt$$

,

(2.55) $$\langle \dot{y} \rangle = 0$$.

(* Compute the average velocity of vx *)

Integrate[(B q)/(2π m) vx[t],{t,0,(2π m)/(B q)}]

(* Compute the average velocity of vy *)

Integrate[(B q)/(2π m) vy[t],{t,0,(2π m)/(B q)}]

Consider the following initial conditions:

$$x_0 \equiv x(0) = -\frac{A}{\omega_c}, \qquad v_{x0} \equiv \dot{x}(0) = \frac{E_y}{B}$$

,

$$y_0 \equiv y(0) = 0, \qquad v_{y0} \equiv \dot{y}(0) = A$$

,

$$z_0 \equiv z(0) = 0, \qquad v_{z0} \equiv \dot{z}(0) = C$$

.

where A, C are constants, B the magnitude of the magnetic field and ω_c is the cyclotron frequency.

Using the above initial conditions we include them in the DSolve command to find the x and y solutions of the first two equations of (2.49):

(* Find x and y solutions of equation (2.49) *)

Clear[x,y,m,q,A,B,sol]

sol=DSolve[{m x''[t]==q B y'[t],m y''[t]==q Ey-q B x'[t],x'[0]==Ey/B,y'[0]==A,x[0]==-A/ ωc,y[0]==0},{x[t],y[t]},t]//Simplify

We obtain from Mathematica the solution:

$$x(t) = -\frac{A\,m}{B\,q}\cos\left(\frac{B\,q\,t}{m}\right) + \frac{A\,m}{B\,q} + \frac{E_y}{B}\,t - \frac{A}{\omega_c}$$

.

Since the cyclotron frequency is defined by

$$\omega_c = \frac{q\,B}{m}$$,

so

$$x(t) = -\frac{A}{\omega_c}\cos\omega_c\,t + \frac{A}{\omega_c} + \frac{E_y}{B}\,t - \frac{A}{\omega_c}$$

,

(2.56)

$$x(t) = -\frac{A}{\omega_c}\cos\omega_c\,t + \frac{E_y}{B}\,t$$

.

And

$$y(t) = \frac{A\,m}{B\,q}\sin\left(\frac{B\,q\,t}{m}\right)$$,

(2.57) $$y(t) = \frac{A}{\omega_c}\sin\omega_c\,t$$.

The z(t) solution is found earlier in equation (2.50):

$$z(t) = C\,t + \frac{q\,E_z}{2\,m}\,t^2$$.

Therefore, the trajectory of the particle is

(2.58)

$$\begin{cases} x(t) = -\frac{A}{\omega_c}\cos\omega_c\,t + \frac{E_y}{B}\,t, \\ y(t) = \frac{A}{\omega_c}\sin\omega_c\,t, \\ z(t) = C\,t + \frac{q\,E_z}{2\,m}\,t^2, \end{cases}$$

where $\omega_c = \frac{q\,B}{m}$ is the cyclotron frequency.

```
Clear[x,y,z]

x[t_]:=-(A/ωc)Cos[ωc t]+Ey/B t

y[t_]:=A/ωc Sin[ωc t]

z[t_]:=cCap t+(q Ez)/(2m) t^2
```

Notice that we use cCap instead of C to avoid using the reserved symbol C in Mathematica.

For plotting purposes we set all constants equal to one.

```
ParametricPlot[{x[t],y[t]}/.{A->1,ωc->1,Ey->1,B->1},{t,0,6π},AxesLabel->
{"x","y"},AspectRatio->1/4,ImageSize->500]
```

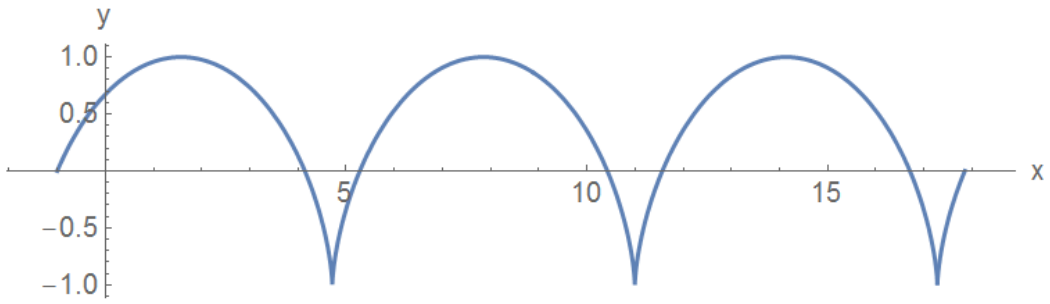


Figure 2.37. The trajectory of the particle under Lorentz force traces out a cycloid under certain conditions.

Note that the curve is similar to a cycloid we encountered in Example 2.11.

Let us plot again the curve y versus x for several values of A:

```
ParametricPlot[{x[t],y[t]}/.{A->#,ωc->1,Ey->1,B->1},{t,0,6π},AxesLabel->
{"x","y"},PlotLabel->"A="<>ToString[#,TraditionalForm],AspectRatio-
>1/4,ImageSize->400]& /@ {2,1/2,1}
```

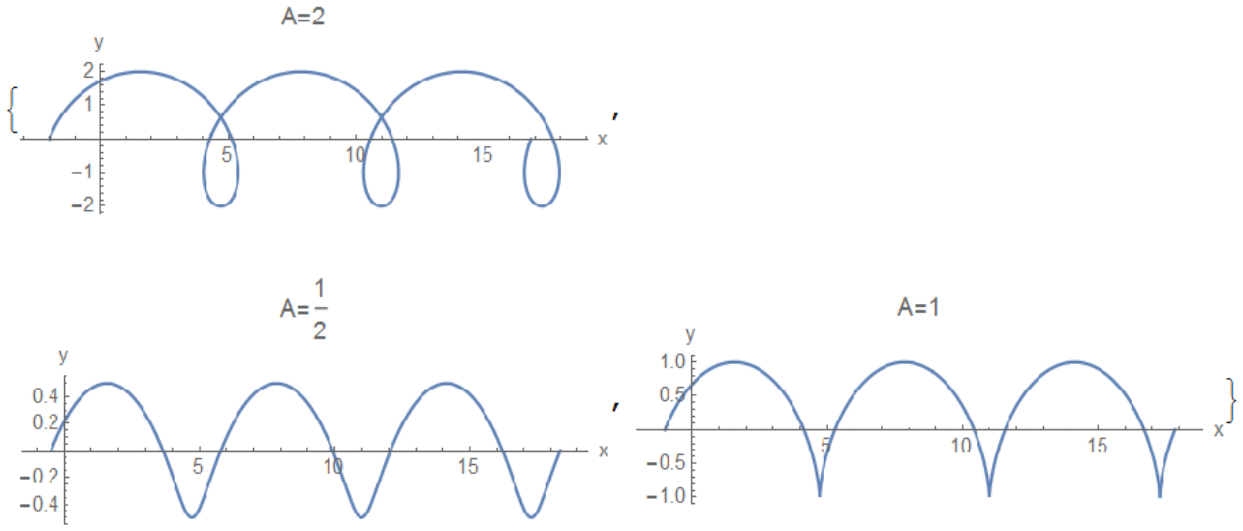


Figure 2.38. The trajectories of the particle under Lorentz force for A=2, A=1/2 and A=1.

```
Plot[z[t]/.{cCap->1,Ez->1,q->1,m->1},{t,0,6π}]
```

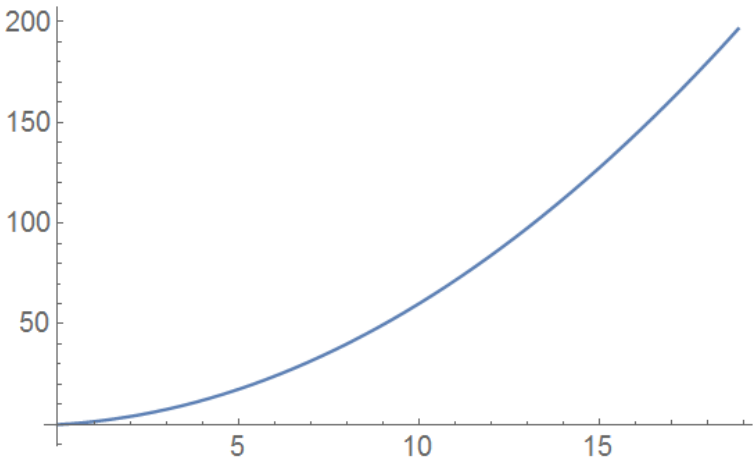


Figure 2.39. The path of the particle in z-axis.

Now, we plot the trajectory of the charged particle when it enters the electromagnetic field E and B:

```
Clear[trajectory]

trajectory[t_]:={x[t],y[t],z[t]}

trajectoryPlot=ParametricPlot3D[trajectory[t]/.{A->1,ωc->1,Ey->1,B->1,cCap->1,Ez->1,q->1,m->1}//Evaluate,{t,0,6π},AxesLabel->{"x","y","z"},PlotRange->{{-2,20},{-2,2},{0,200}},BoxRatios->{1,1,1}]
```

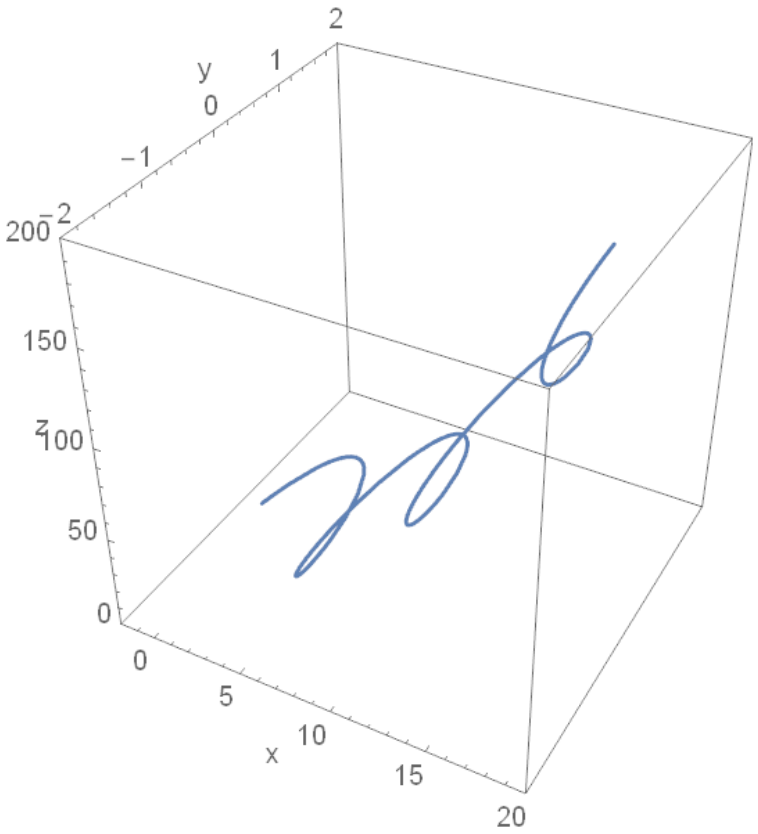


Figure 2.40. The path of the charged particle under Lorentz force.

The curvature and torsion can be computed as in the following:

(* The curvature of the path of the particle *)

κ[trajectory][t]//Simplify//PowerExpand

(* The torsion of the path of the particle *)

τ[trajectory][t]//Simplify

```
Plot[κ[trajectory][t]/.{A->1,ωc->1,Ey->1,B->1,cCap->1,Ez->1,q->1,m->1}//Evaluate,{t,0,#},AxesLabel->{"t","κ"},ImageSize->300]&/@{6π,60π}
```

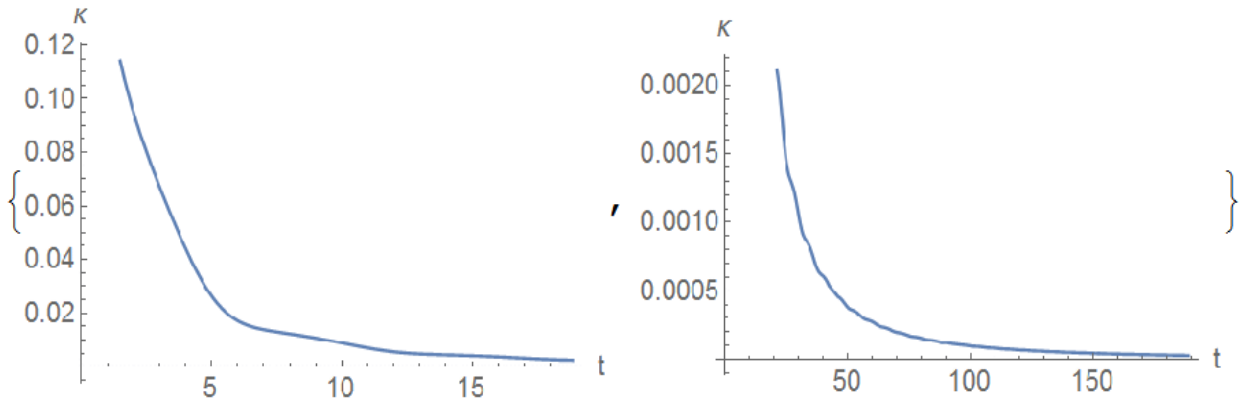


Figure 2.41. The curvature of the path of the particle under Lorentz force.

```
Plot[τ[trajectory][t]/.{A->1,ωc->1,Ey->1,B->1,cCap->1,Ez->1,q->1,m->1}//Evaluate,{t,0,#},AxesLabel->{"t","τ"},PlotRange->All,ImageSize->300]&/@{6π,60π}
```

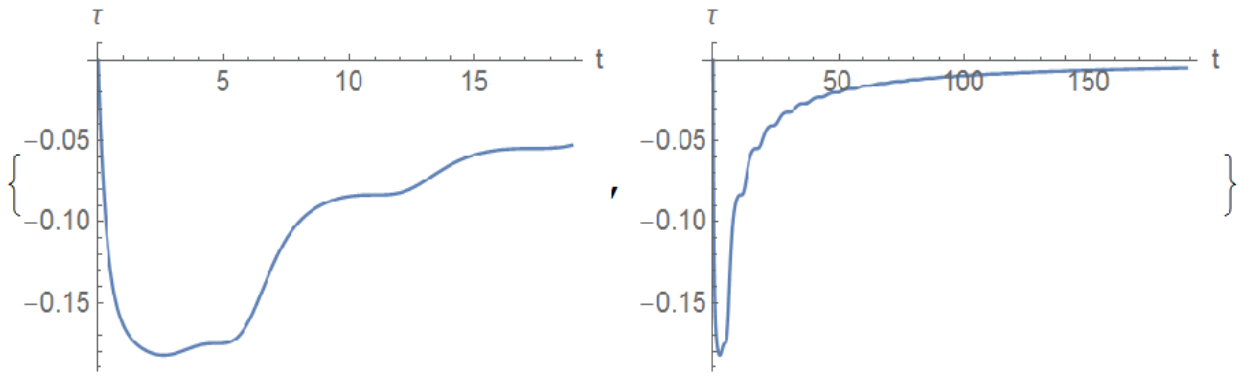


Figure 2.42. The torsion of the path of the particle under Lorentz force.

As time becomes larger the curvature and the torsion of the path approach zero. This implies that the trajectory of the particle becomes linear and lies in a plane.

(* The curvature of the path approaches zero as t approaches infinity *)

```
Limit[κ[trajectory][t]/.{A->1,ωc->1,Ey->1,B->1,cCap->1,Ez->1,q->1,m->1},t->∞]
```

(* The torsion of the path approaches zero as t approaches infinity *)

```
Limit[τ[trajectory][t]/.{A->1,ωc->1,Ey->1,B->1,cCap->1,Ez->1,q->1,m->1},t->∞]
```

Chapter 3. Differentiation in Several Variables

The concepts and techniques in single variable calculus can be extended to differential calculus of several variables. The differentiation of a function of several variables involves partial derivatives, that is, taking one for each variable by fixing the values of other variables. The partial derivatives are components of a vector defined as the gradient vector of the function.

In this chapter we will consider some of the key geometric properties of the gradient vector, which provides valuable insight into the function's behavior. We shall consider only functions of two and three variables and learn how to use Mathematica's graphing tools to visualize such functions.

Initialization

```
Clear["Global`*"]

SetOptions[ParametricPlot3D,PlotTheme->"Classic",PlotStyle-
>AbsoluteThickness[2]];

SetOptions[Plot3D,PlotTheme->"Classic",PlotStyle-
>AbsoluteThickness[2]];

SetOptions[ContourPlot,PlotTheme->"Classic",ContourStyle-
>AbsoluteThickness[2]];

SetOptions[ContourPlot3D,PlotTheme->"Classic",ContourStyle-
>AbsoluteThickness[2]];
```

3.1 Visualizing Functions

3.1.1 Functions of Two Variables

In general a function of two variables can be defined by a single equation:

z=f(x, y),

which is a scalar-valued function of two variables, x and y. The graph consisted of a set of points (x, y, f(x, y)) is a surface in three-dimensional space.

To graph a surface in Mathematica we can use Plot3D command.

Example 3.1. Plot the function $f(x, y) = y\, e^{x}$ on the rectangle -2<=x<=2, -2<=y<=2.

```
Clear[f,x,y]

f[x_,y_]:=y E^x

Plot3D[f[x,y],{x,-2,2},{y,-2,2},PlotRange->Full]
```

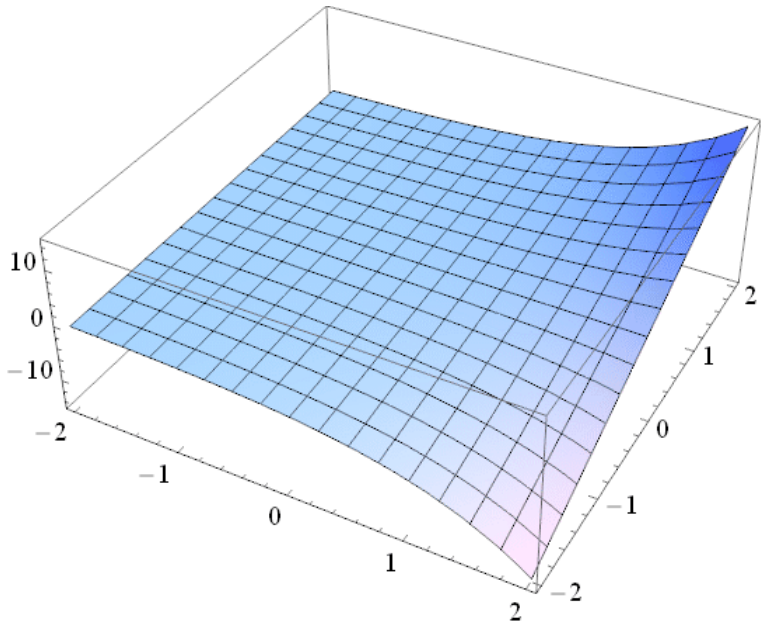


Figure 3.1. The graph of f(x,y)=y E^x.

Example 3.2. Plot the function g(x,y)=(sin (2 - x^2 - y^2))/(x^2 + y^2 + 1) on the rectangle -π<=x<=π, -π<=y<=π.

```
Clear[g,x,y]

g[x_,y_]:=Sin[2-x^2-y^2]/(x^2+y^2+1)

Plot3D[g[x,y],{x,-π,π},{y,-π,π},PlotRange->Full]
```

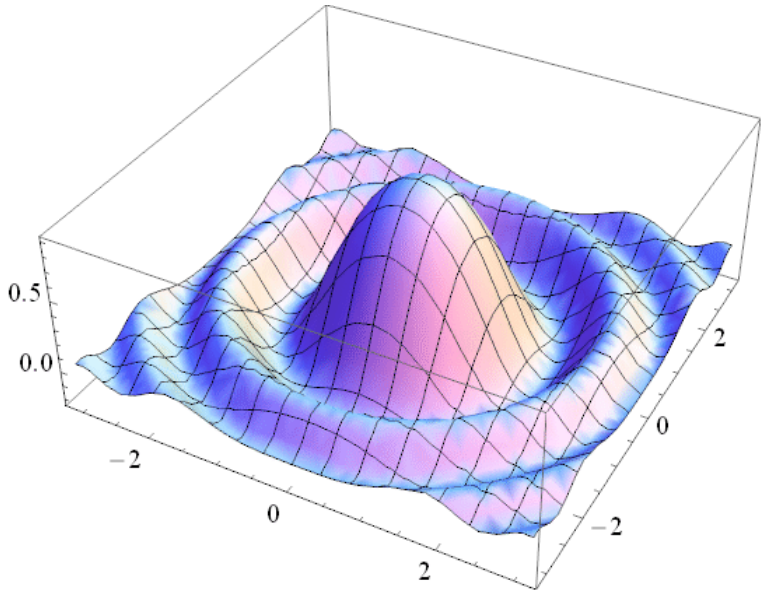


Figure 3.2. The graph of g(x,y)=(sin (2 - x^2 - y^2))/(x^2 + y^2 + 1).

Special curves that lie on the surface z=f(x, y) are called level curves. These curves are obtained by intersecting the surface with horizontal planes z=c, where c are constants.

The level curves for the surfaces in Example 3.1 and 3.2 can be drawn using ContourPlot command which are illustrated in Example 3.3 and 3.4, respectively.

Example 3.3. Draw the level curves at z=1 and z=2 for the surface function $f(x, y) = y\, e^{x}$ on the rectangle -2<=x<=2, -2<=y<=2.

```
ContourPlot[f[x,y],{x,-2,2},{y,-2,2},Contours->{1,2},ContourShading->None,ContourStyle->{{AbsoluteThickness[2],Blue},
{AbsoluteThickness[2],Red}}]
```

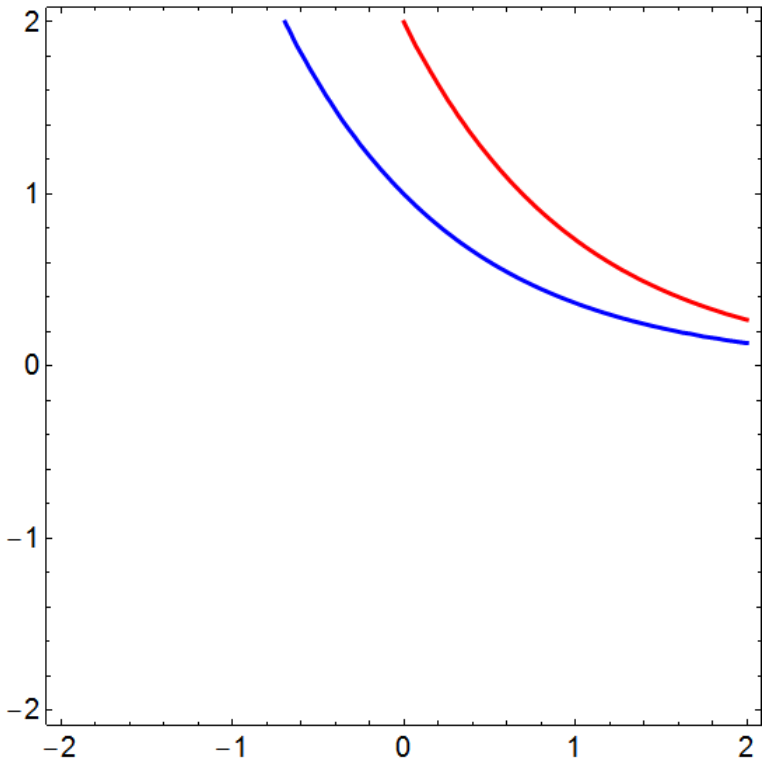


Figure 3.3. The level curves at z=1 (blue) and z=2 (red) for the surface function $f(x, y) = y\, e^x$.

The two level curves above correspond to the intersecting curves between the surface $f(x, y) = y\, e^x$ and the plane z=1 and z=2, respectively.

Example 3.4. Draw the level curves at z=0 and z=0.2 for the surface function g(x,y)=(sin (2 - x^2 - y^2))/(x^2 + y^2 + 1) on the rectangle -π<=x<=π, -π<=y<=π.

ContourPlot[g[x,y],{x,-π,π},{y,-π,π},Contours->{0,0.2},ContourShading->None,ContourStyle->{{AbsoluteThickness[2],Blue},
{AbsoluteThickness[2],Red}}]

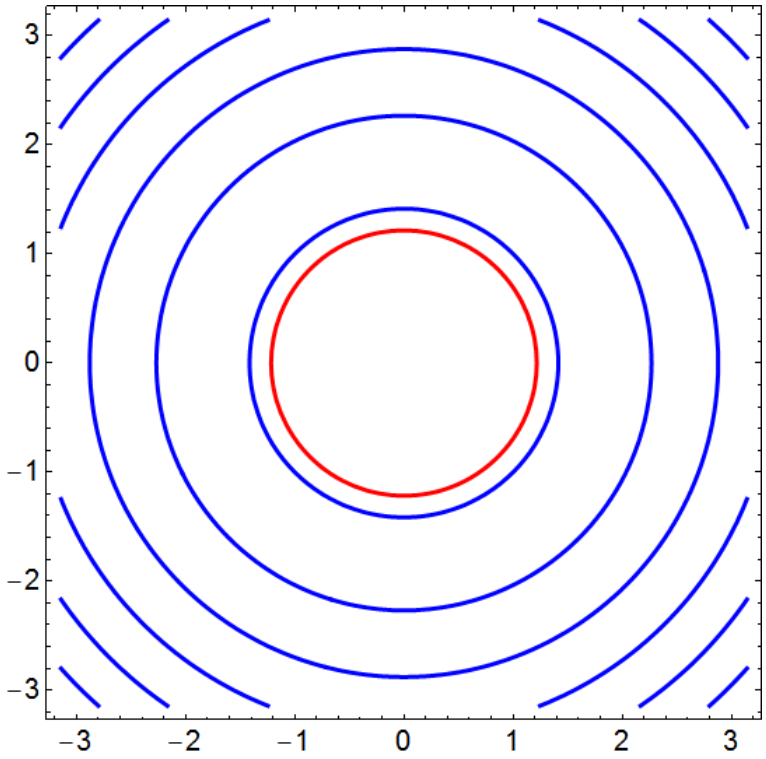


Figure 3.4. The level curves at z=0 (blue) and z=0.2 (red) for the surface function g(x,y)=(sin (2 - x^2 - y^2))/(x^2 + y^2 + 1).

In this case, the intersecting curves between the surface g(x,y)=(sin (2 - x^2 - y^2))/(x^2 + y^2 + 1) and the plane z=0 and z=0.2 are concentric circles centered at the origin. There are many intersecting circles between the plane z=0 with the surface. However, for the plane z=0.2, there is only one intersecting circle.

3.1.2 Functions of Three Variables

For functions of three variables x, y, z the graphs would consist of a set of points (x, y, z, f(x, y, z)) in four-dimensional space. Therefore, it is not possible to draw the graph. However, we can draw the surfaces of a

function f(x, y, z)=c, where c are constants. These surfaces are known as level surfaces.

Let us plot some well-known level surfaces.

Example 3.5. The ellipsoid:

f(x,y,z)=x^2/a^2+y^2/b^2+z^2/c^2=1.

Notice that if a=b=c, then the ellipsoid is a sphere of radius a. If one of the variable is zero then the ellipsoid is intersected with one of the coordinate planes, then we obtain a standard ellipse.

```
ContourPlot3D[x^2/a^2+y^2/b^2+z^2/c^2==1/.{a->1,b->2,c->1}//Evaluate,{x,-2,2},{y,-2,2},{z,-2,2}]
```

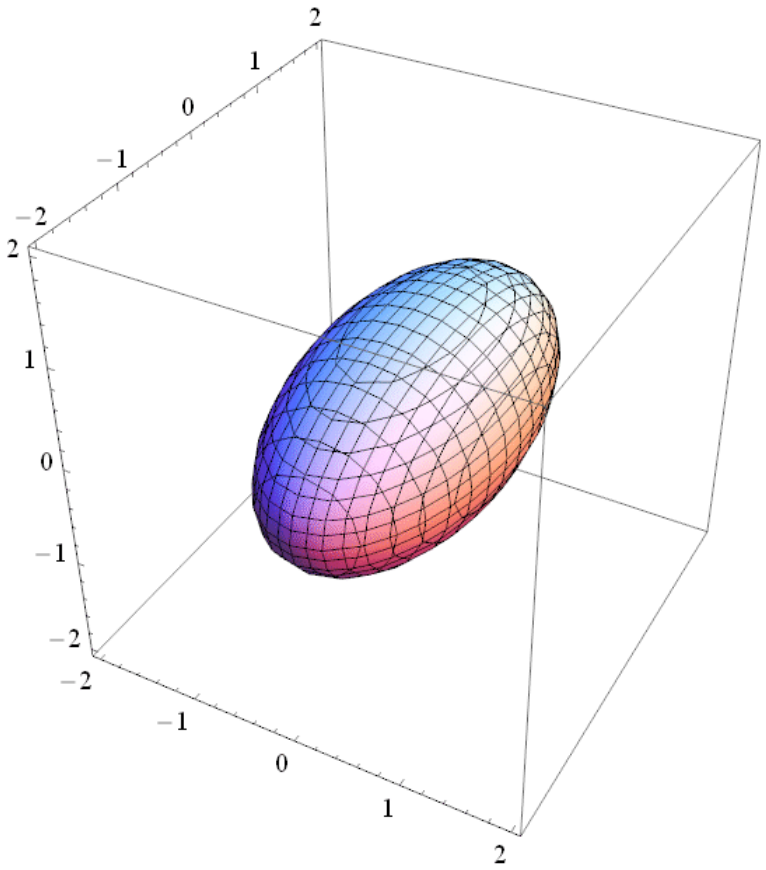


Figure 3.5. The ellipsoid x^2/a^2+y^2/b^2+z^2/c^2=1 for a=1, b=2, c=1.

Example 3.6. The elliptic cone:

f(x,y,z)=x^2/a^2+y^2/b^2-z^2/c^2=0.

ContourPlot3D[x^2/a^2+y^2/b^2-z^2/c^2==0/.{a->1,b->2,c->1}//Evaluate, {x,-4,4},{y,-4,4},{z,-2,2},PlotRange->All]

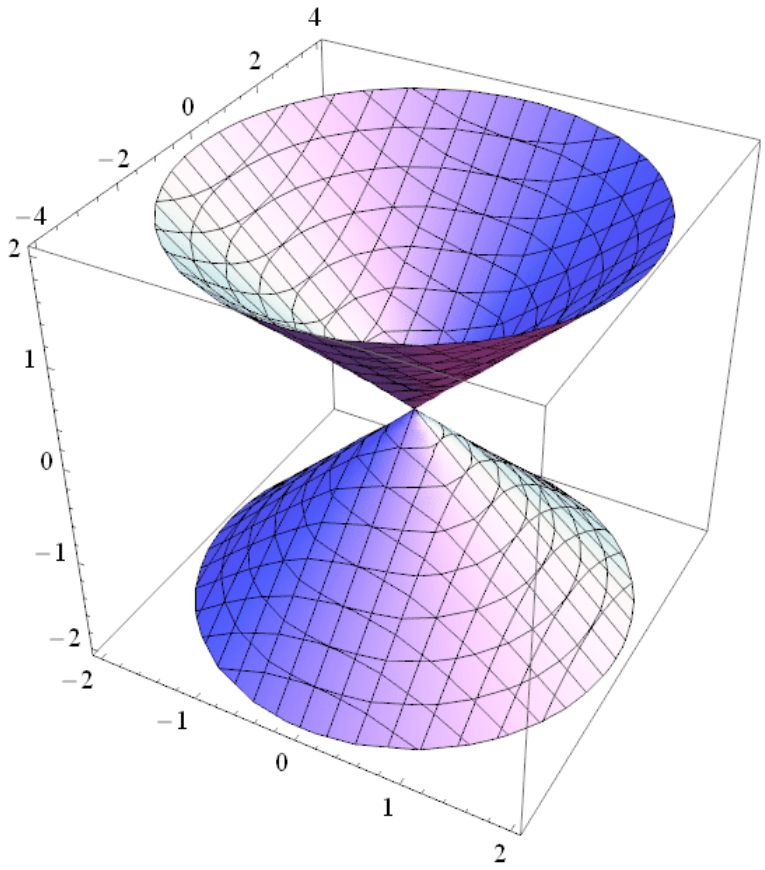


Figure 3.6. The elliptic cone x^2/a^2+y^2/b^2-z^2/c^2=0 for a=1, b=2, c=1.

Example 3.7. The elliptic paraboloid:

f(x,y,z)=x^2/a^2+y^2/b^2-z/c=0.

ContourPlot3D[x^2/a^2+y^2/b^2-z/c==0/.{a->1,b->2,c->1}//Evaluate, {x,-4,4},{y,-4,4},{z,-4,4},PlotPoints->40,PlotRange->All]

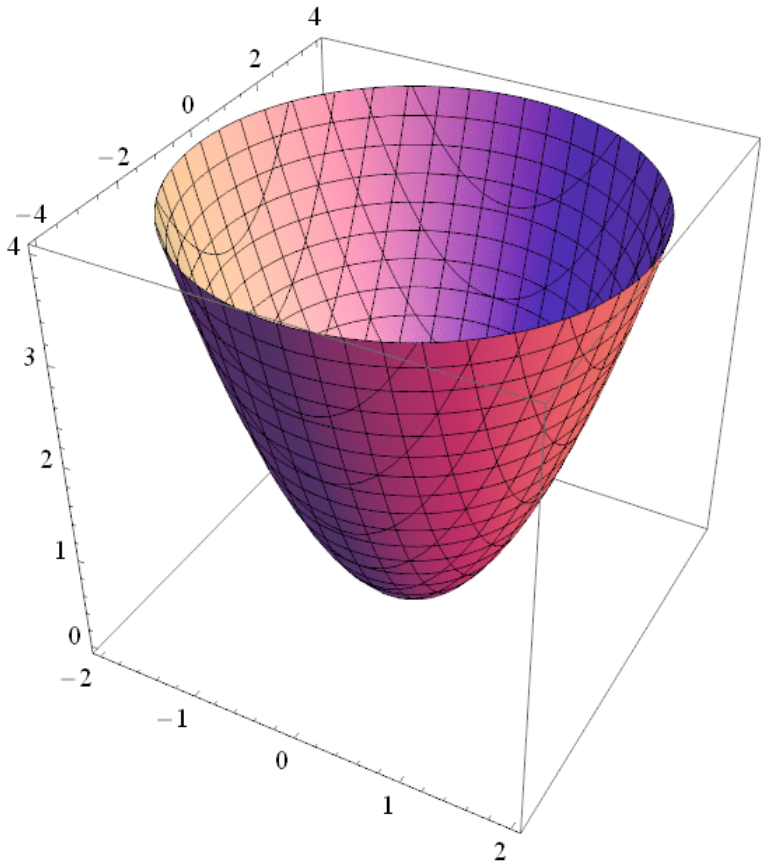


Figure 3.7. The elliptic paraboloid x^2/a^2+y^2/b^2-z/c=0 for a=1, b=2, c=1.

Example 3.8. The hyperbolic paraboloid:

f(x,y,z)=-(x^2/a^2)+y^2/b^2-z/c=0.

ContourPlot3D[-(x^2/a^2)+y^2/b^2-z/c==0/.{a->1,b->2,c->1}//Evaluate, {x,-4,4},{y,-4,4},{z,-4,4},PlotRange->All]

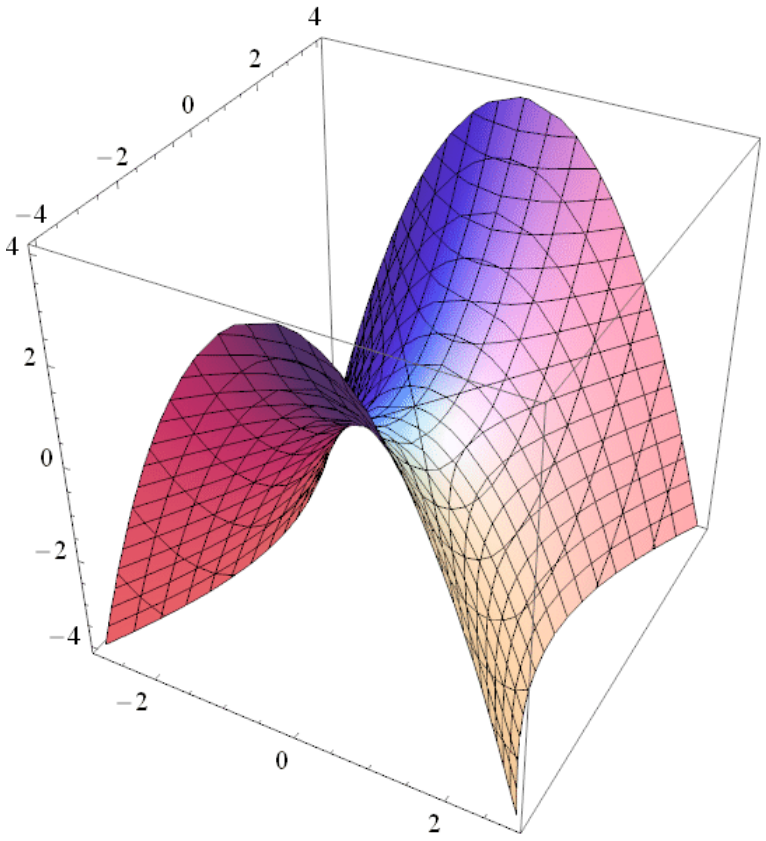


Figure 3.8. The hyperbolic paraboloid -(x^2/a^2)+y^2/b^2-z/c=0 for a=1, b=2, c=1.

Example 3.9. The hyperboloid of one sheet:

f(x,y,z)=x^2/a^2+y^2/b^2-z^2/c^2=1.

ContourPlot3D[x^2/a^2+y^2/b^2-z^2/c^2==1/.{a->1,b->2,c->1}//Evaluate, {x,-5,5},{y,-5,5},{z,-2,2},PlotRange->All]

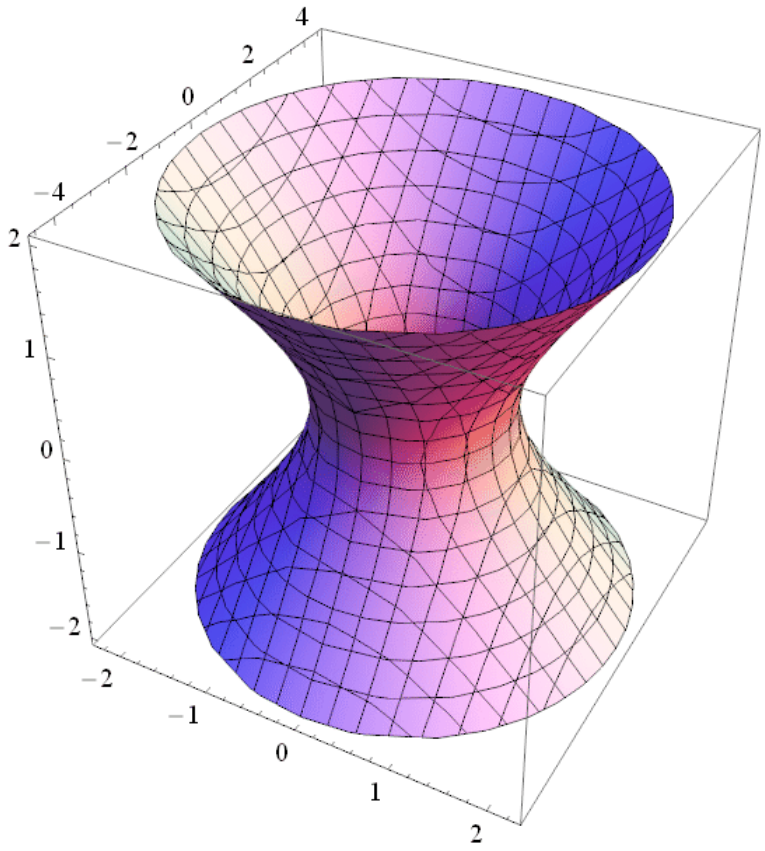


Figure 3.9. The hyperboloid of one sheet x^2/a^2+y^2/b^2-z^2/c^2=1 for a=1, b=2, c=1.

Example 3.10. The hyperboloid of two sheet:

f(x,y,z)=-(x^2/a^2)-y^2/b^2+z^2/c^2=1

ContourPlot3D[-(x^2/a^2)-y^2/b^2+z^2/c^2==1/.{a->1,b->2,c->1}//Evaluate,{x,-4,4},{y,-4,4},{z,-2,2}]

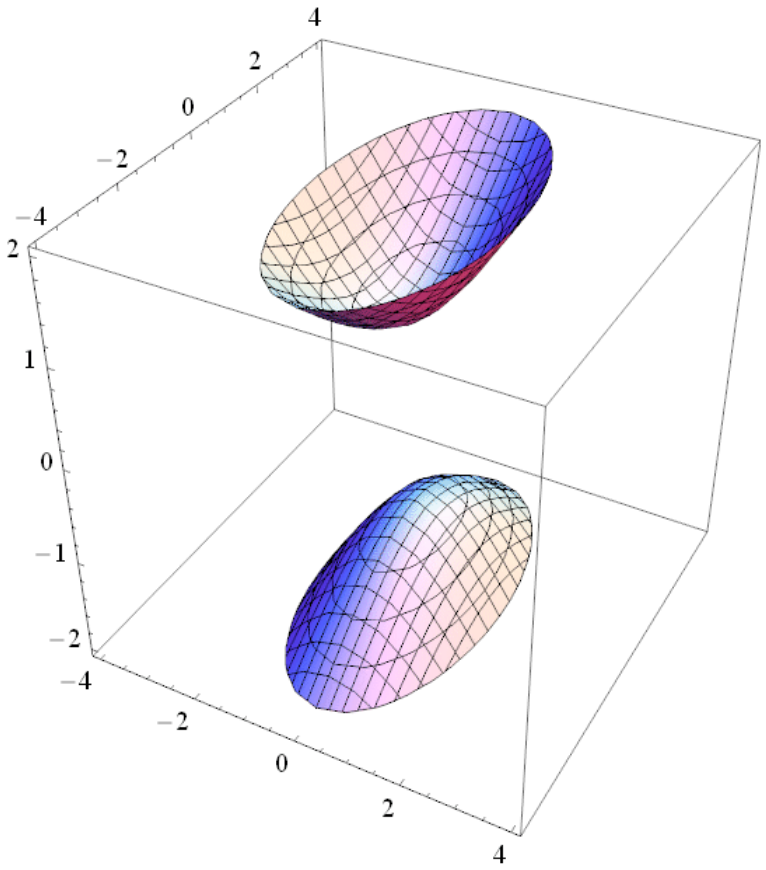


Figure 3.10. The hyperboloid of two sheet -(x^2/a^2)-y^2/b^2+z^2/c^2=1 for a=1, b=2, c=1.

3.1.3 Parametrized Surfaces

In previous sections we study surfaces of the form z=f(x, y) for two variables and level surfaces of the form f(x, y, z)=c for three variables. However, not all surfaces are of these forms. A surface can also be described by a parametric equation. We have studied in Section 1.3 parametrized curves, being one-dimensional, requires a single parameter t. A parametrized surface thus depends on two parameters which is two-dimensional. Let us denote the parameters by s and t. Thus, a surface in $\mathbb{R}^3$ can be described by the parametric equations

$$\begin{cases} x = x(s, t), \\ y = y(s, t), \\ z = z(s, t), \end{cases}$$

where the set of (s, t) define a region in $\mathbb{R}^2$ which could be open, bounded, or unbounded.

We give here as examples the parametric equations of some of the level surfaces we have encountered in Section 3.1.2 and plot the surfaces using ParametricPlot3D command.

Example 3.11. The parametric equation of the ellipsoid is given by

ellipsoid[a, b, c](s, t)=(a cos s cos t, b sin s cos t, c sin t).

```
ellipsoid[a_,b_,c_][s_,t_]:={a Cos[s] Cos[t],b Sin[s] Cos[t],c Sin[t]}
```

```
ParametricPlot3D[ellipsoid[1,2,1][s,t],{s,0,2π},{t,0,2π}]
```

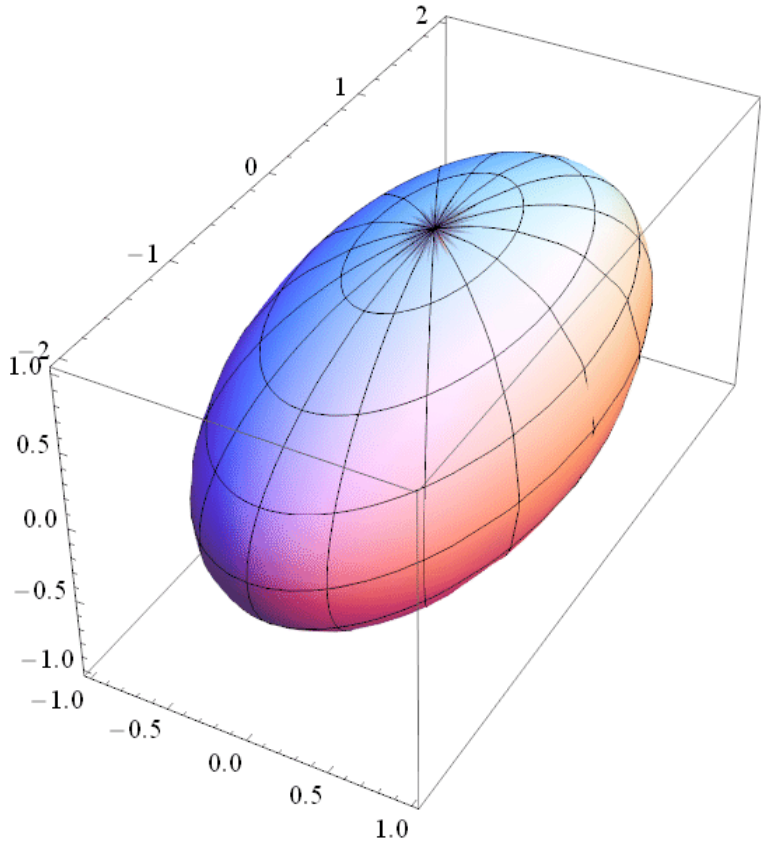


Figure 3.11. The ellipsoid by parametric equation ellipsoid[a, b, c](s, t)=(a cos s cos t, b sin s cos t, c sin t) for a=1, b=2, c=1.

Example 3.12. The parametric equation of the elliptic cone is given by

ellipticcone[a, b, c](s, t)=(a s cos t, b s sin t, c s).

```
ellipticcone[a_,b_,c_][s_,t_]:={a s Cos[t],b s Sin[t],c s}

ParametricPlot3D[ellipticcone[1,2,1][s,t],{s,-2,2},{t,0,2π},PlotRange->
{{-2,2},{-4,4},{-2,2}},BoxRatios->{1, 1, 1}]
```

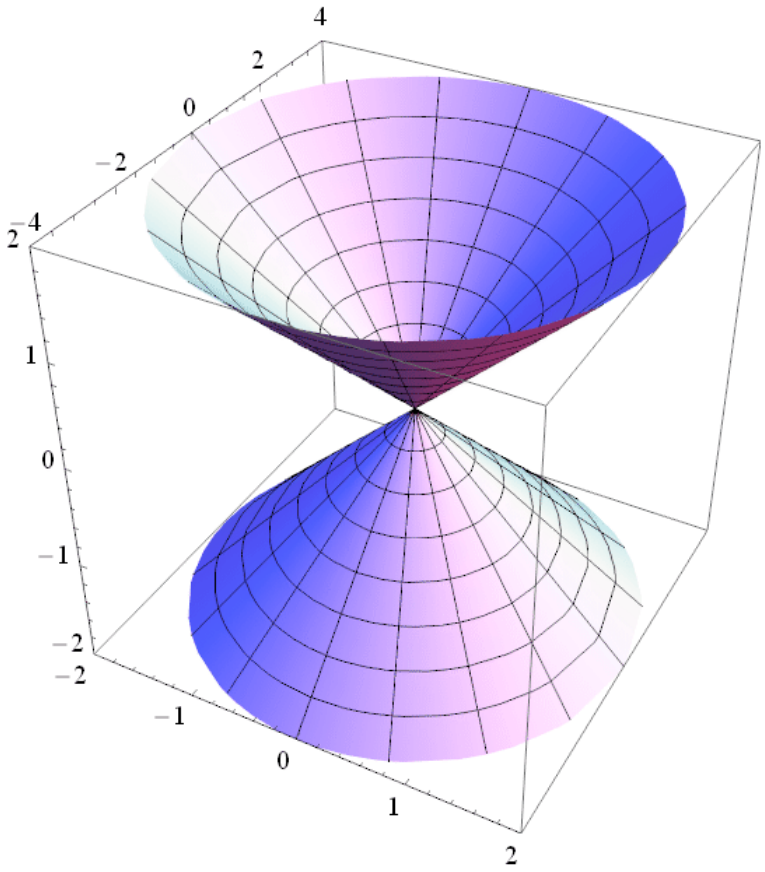


Figure 3.12. The elliptic cone by parametric equation ellipticcone[a, b, c](s, t)=(a s cos t, b s sin t, c s) for a=1, b=2, c=1.

Example 3.13. The parametric equation of the elliptic paraboloid is given by

ellipticparaboloid[a, b, c](s, t)=(a s cos t, b s sin t, c s^2).

ellipticparaboloid[a_,b_,c_][s_,t_]:={a s Cos[t],b s Sin[t],c s^2}

ParametricPlot3D[ellipticparaboloid[1,2,1][s,t],{s,-2,2},{t,0,2π},BoxRatios->{1, 1, 1}]

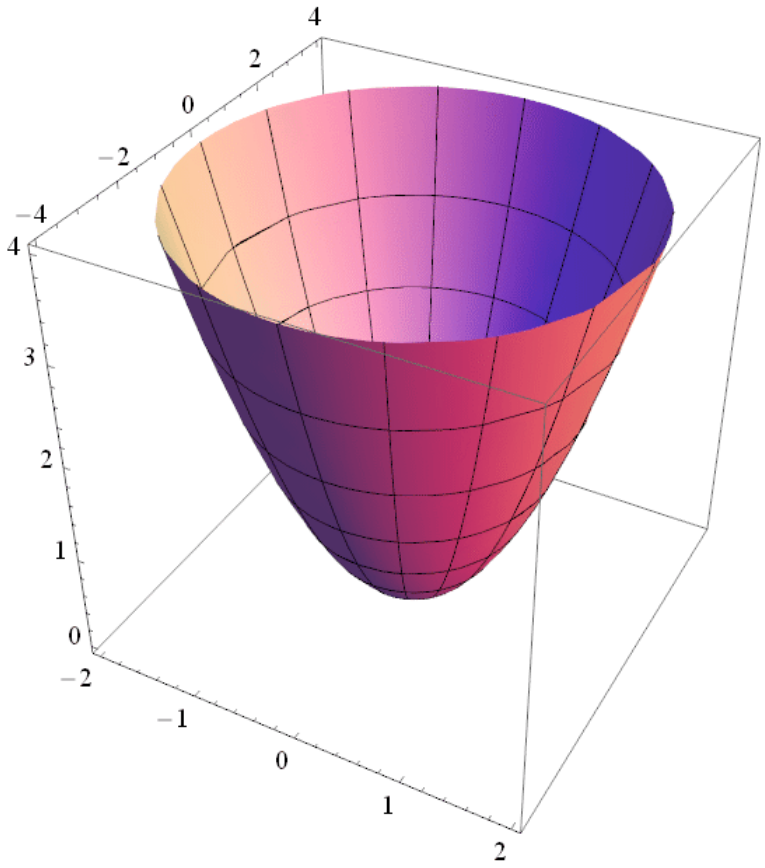


Figure 3.13. The elliptic paraboloid by parametric equation ellipticparaboloid[a, b, c](s, t)=(a s cos t, b s sin t, c s^2) for a=1, b=2, c=1.

Example 3.14. The parametric equation of the hyperbolic paraboloid is given by

hyperbolicparaboloid[a, b, c](s, t)=(s, t, c (t^2/b^2-s^2/a^2)).

hyperbolicparaboloid[a_,b_,c_][s_,t_]:={s,t,c(t^2/b^2-s^2/a^2)}

ParametricPlot3D[hyperbolicparaboloid[1,2,1][s,t],{s,-3,3},
{t,-4,4},BoxRatios->{1, 1, 1}]

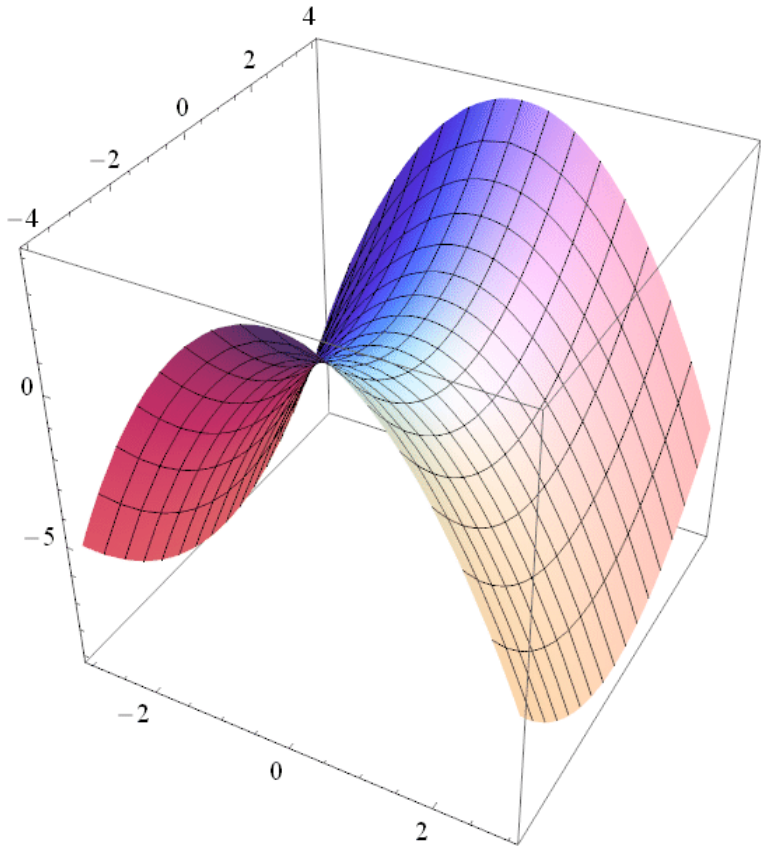


Figure 3.14. The hyperbolic paraboloid by parametric equation hyperbolicparaboloid[a, b, c](s, t)=(s, t, c (t^2/b^2-s^2/a^2)) for a=1, b=2, c=1.

Example 3.15. The parametric equation of the hyperboloid of one sheet is given by

hyperboloid1[a, b, c](s, t)=(a cos s cosh t, b sin s cosh t, c sinh t).

hyperboloid1[a_,b_,c_][s_,t_]:={a Cos[s]Cosh[t],b Sin[s]Cosh[t],c Sinh[t]}

ParametricPlot3D[hyperboloid1[1,2,1][s,t],{s,0,2π},{t,-1,1},PlotRange->All,BoxRatios->{1, 1, 1}]

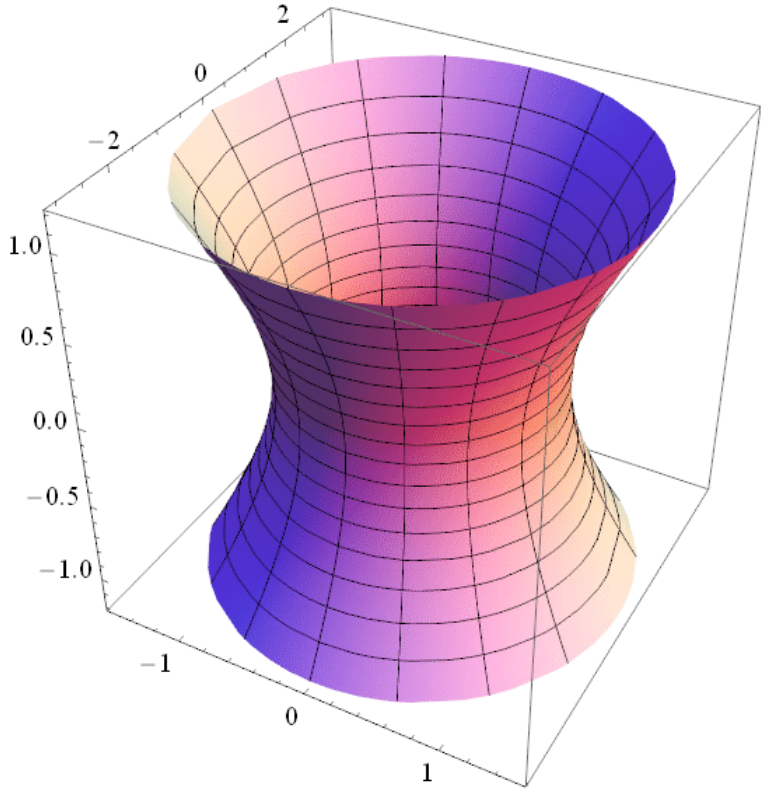


Figure 3.15. The hyperboloid of one sheet by parametric equation hyperboloid1[a, b, c](s, t)=(a cos s cosh t, b sin s cosh t, c sinh t) for a=1, b=2, c=1.

Example 3.16. The parametric equation of the hyperboloid of two sheet is given by

hyperboloid2[a, b, c](s, t)=(a cos s sinh t, b sin s sinh t, c cosh t).

hyperboloid2[a_,b_,c_][s_,t_]:={a Cos[s]Sinh[t],b Sin[s]Sinh[t],c Cosh[t]}

ParametricPlot3D[{hyperboloid2[1,2,1][s,t],hyperboloid2[1,2,-1][s,t]},
{s,0,2π},{t,-2,2},PlotRange->{{-4,4},{-4,4},{-2,2}},BoxRatios->{1, 1, 1}]

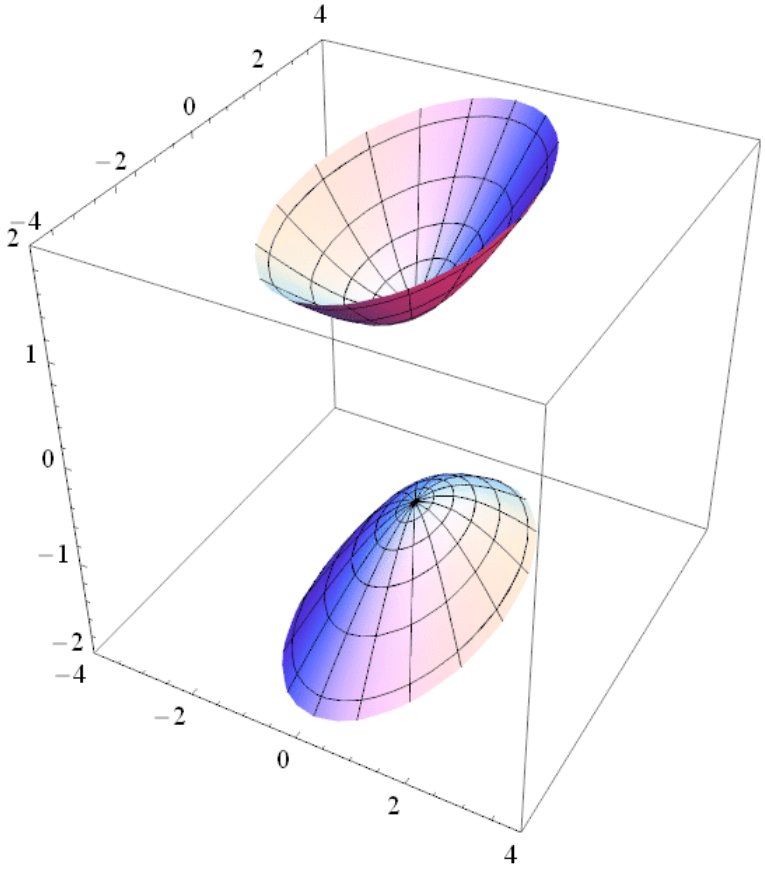


Figure 3.16. The hyperboloid of two sheet by parametric equation hyperboloid2[a, b, c](s, t)=(a cos s sinh t, b sin s sinh t, c cosh t) for a=1, b=2, c=1.

Note that for plotting hyperboloid of two sheets in parametric equation, the lower half is obtained by setting c<0.

3.2 The Derivative

The derivative of multivariable functions is similar to the one in single-variable function. However, for multivariable functions, we have to take derivative with respect to a specified variable when all other variables are held fixed. This is defined as partial derivative with respect to the variable concerned. Technically, taking the partial derivative of a function of more than one variable with respect to a specified variable is similar to derivative in single-variable function by treating other variables as constants. The partial derivatives of a function f with respect to x and y are denoted by f_x and f_y , respectively. The other notations are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. In general, Mathematica uses the following notation to denote partial derivatives of function f with respect to first variable and second variable by $f^{(1,0)}$ and $f^{(0,1)}$, respectively.

In Mathematica, we can use either the D command or the ∂ notation to compute partial derivatives of a multivariable function.

Notes:

- To generate the ∂ sign, press the Esc key, then type the word "pd" and followed by another Esc key.

- To generate ∂ with subscripts such as ∂_x , press the Esc key, then type the word "pd" and followed by another Esc key. Holding down the Ctrl key, press the - key. Then type "x".

Here are some examples:

Example 3.17. Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function of two variables

(a) f(x,y)=ln (x^2 + y)

(b) g(x,y)=(x^2 + 3y^2) E^(1-x^2-y^2)

(c) h(x,y)=(sin (2 - x^2 - y^2))/(x^2 + y^2 + 1)

Solution:

(a)

Clear[f,x,y]

f[x_,y_]:=Log[x^2+y]

D[f[x,y],x]

D[f[x,y],y]

or, alternatively

```
∂x f[x, y]
∂y f[x, y]
```

(b)

Clear[g,x,y]

g[x_,y_]:=(x^2+3y^2)E^(1-x^2-y^2)

D[g[x,y],x]

D[g[x,y],y]

or, alternatively

```
∂x g[x, y]
∂y g[x, y]
```

(c)

Clear[h,x,y]

h[x_,y_]:=Sin[2-x^2-y^2]/(x^2+y^2+1)

D[h[x,y],x]//Simplify

D[h[x,y],y]//Simplify

or, alternatively

```
∂x h[x, y] // Simplify
∂y h[x, y] // Simplify
```

Example 3.18. Given a function f(x,y,z), find the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

(a) f(x,y,z)=x^3 y z^2

(b) f(x,y,z)=(x + 2y + 3z)/(1 + x^2 + y^2 + z^2)^(3/2)

(c) f(x,y,z)=E^(a x) sin b y + E^(a z) cos b x

Solution:

(a)

```
Clear[f,x,y,z]

f[x_,y_,z_]:=x^3 y z^2

D[f[x,y,z],x]

D[f[x,y,z],y]

D[f[x,y,z],z]
```

(b)

```
Clear[f,x,y,z]

f[x_,y_,z_]:=(x+2y+3z)/(1+x^2+y^2+z^2)^(3/2)
```

```
D[f[x,y,z],x]//Simplify

D[f[x,y,z],y]//Simplify

D[f[x,y,z],z]//Simplify
```

(c)

```
Clear[f,x,y,z,a,b]

f[x_,y_,z_]:=E^(a x) Sin[b y]+E^(a z) Cos[b x]

D[f[x,y,z],x]

D[f[x,y,z],y]

D[f[x,y,z],z]
```

Higher order partial derivatives can be done similarly as are illustrated in the following examples.

Example 3.19. Find the partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y\,\partial x}$, $\frac{\partial^2 f}{\partial x\,\partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ of the functions

(a) f(x,y,z)=Sqrt[3x^2+y^2-2z^2]

(b) f(x,y,z)=E^-x sin (y z)

(c) f(x,y,z)=2x^3 y^2 z^3-x^5 y z

Solution:

(a)

Clear[f,x,y,z]

f[x_,y_,z_]:=Sqrt[3x^2+y^2-2z^2]

D[f[x,y,z],{x,2}]//Simplify

D[f[x,y,z],x,y]

D[f[x,y,z],y,x]

D[f[x,y,z],{y,2}]//Simplify

or, alternatively

```
∂{x,2} f[x, y, z] // Simplify
∂y,x f[x, y, z]
∂x,y f[x, y, z]
∂{y,2} f[x, y, z] // Simplify
```

(b)

```
Clear[f,x,y,z]

f[x_,y_,z_]:=E^-x Sin[y z]


D[f[x,y,z],{x,2}]

D[f[x,y,z],x,y]

D[f[x,y,z],y,x]

D[f[x,y,z],{y,2}]
```

or, alternatively

$$\partial_{\{x,2\}} f[x, y, z]$$
$$\partial_{y,x} f[x, y, z]$$
$$\partial_{x,y} f[x, y, z]$$
$$\partial_{\{y,2\}} f[x, y, z]$$

(c)

```
Clear[f,x,y,z]

f[x_,y_,z_]:=2x^3 y^2 z^3-x^5 y z


D[f[x,y,z],{x,2}]

D[f[x,y,z],x,y]

D[f[x,y,z],y,x]
```

D[f[x,y,z],{y,2}]

or, alternatively

$\partial_{\{x,2\}} f[x, y, z]$
$\partial_{y,x} f[x, y, z]$
$\partial_{x,y} f[x, y, z]$
$\partial_{\{y,2\}} f[x, y, z]$

So far, we have considered functions of the form $f(x, y) \in \mathbb{R}$ and $f(x, y, z) \in \mathbb{R}$ which are scalar-valued functions of two and three variables, respectively. We have also seen that the derivatives of these functions depend on which variable is being taken for differentiation while fixing other variables. This means that the derivative of f depend on the choice of direction. As a matter of fact, there are infinitely many directions to choose from to describe the derivative of f. This leads us to the definition of gradient and derivative of f.

Definition 3.1 (the gradient and the derivative)

The gradient of a scalar-valued function of two variables $f : (x, y) \to \mathbb{R}$ at a point P is the vector

(3.1)

$$\nabla f_P = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)_P = \left(\frac{\partial f}{\partial x}\right)_P \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_P \mathbf{j}$$

,

and the derivative of f(x,y) at P denoted by $D f_P$ is the row matrix whose entries are the components of ∇f_P , that is

(3.2) $$Df_P = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right)_P .$$

Similarly, for scalar-valued function of three variables, $f(x, y, z) \in \mathbb{R}$, the gradient is

(3.3)

$$\nabla f_P = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)_P = \left(\frac{\partial f}{\partial x} \right)_P \mathbf{i} + \left(\frac{\partial f}{\partial y} \right)_P \mathbf{j} + \left(\frac{\partial f}{\partial z} \right)_P \mathbf{k}$$

,

and the derivative is the row matrix:

(3.4) $$Df_P = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right)_P .$$

The definition above can be generalized to scalar-valued functions of more than three variables. However, in this book we shall focus on two and three variables only.

Example 3.20. Find the gradient and derivative of

(a) f(x,y)=x/(x^2 + y^2)

(b) f(x,y,z)=x y z^-2

Solution:

(a)

```
Clear[f,x,y]

f[x_,y_]:=x/(x^2+y^2)
```

We use Grad command to compute the gradient of f.

In doing so, we need to inform Mathematica to use as coordinates x and y and to find the gradient in Cartesian coordinates system. It is entered as in the following:

```
Grad[f[x,y],{x,y},"Cartesian"]//Simplify
```

Thus, the gradient of f(x,y)=x/(x^2 + y^2) is

$$\nabla f(x, y) = \frac{-x^2 + y^2}{\left(x^2 + y^2\right)^2}\,\mathbf{i} - \frac{2\,x\,y}{\left(x^2 + y^2\right)^2}\,\mathbf{j}$$

.

and the derivative of f(x,y)=x/(x^2 + y^2) is a 1x2 row matrix:

$$D f(x, y) = \left(\frac{-x^2 + y^2}{\left(x^2 + y^2\right)^2} \quad -\frac{2\,x\,y}{\left(x^2 + y^2\right)^2} \right)$$

.

(b)

Clear[f,x,y,z]

f[x_,y_,z_]:=x y z^-2

We need to enter into Grad command, the three coordinates x, y and z and to specify the Cartesian coordinates system to be used.

Grad[f[x,y,z],{x,y,z},"Cartesian"]

Thus, the gradient of f(x,y,z)=x y z^-2

$$\nabla f(x, y, z) = \frac{y}{z^2}\,\mathbf{i} + \frac{x}{z^2}\,\mathbf{j} - \frac{2xy}{z^3}\,\mathbf{k}$$

.

and the derivative of f(x,y,z)=x y z^-2 is a 1x3 row matrix:

$$D f(x, y) = \begin{pmatrix} \frac{y}{z^2} & \frac{x}{z^2} & -\frac{2xy}{z^3} \end{pmatrix}$$

.

In single variable calculus, the derivative of a scalar-valued function leads us to the definition of tangent line. Similarly, the scalar-valued function of two and three variables leads us to the definition of tangent plane and tangent hyperplane, respectively. Just as a tangent line is a good linear approximation of the curve at a point, a tangent plane or a tangent hyperplane is a good linear approximation of the surface and hypersurface, respectively.

We state the following theorems without proof.

Theorem 3.2. If there exists a tangent plane to a surface function z=f(x,y) at point P=(a,b,f(a,b)), then the equation of the tangent plane at point P is given by

(3.5)

$$h(x, y) = f(a, b) + \frac{\partial f(a, b)}{\partial x}(x - a) + \frac{\partial f(a, b)}{\partial y}(y - b)$$

.

It is not difficult to generalize Theorem 3.2 to three (or more) variables.

Theorem 3.3. If there exists a tangent hyperplane to a hypersurface function w=f(x,y,z) at point P=(a,b,c,f(a,b,c)), then the equation of the tangent hyperplane at point P is given by

(3.6)

$$h(x, y, z) = f(a, b, c) + \frac{\partial f(a, b, c)}{\partial x}(x - a) + \frac{\partial f(a, b, c)}{\partial y}(y - b) + \frac{\partial f(a, b, c)}{\partial z}(z - c)$$

.

Using vector notation and by Definition 3.1 we can write the tangent equation as

(3.7)

$$h(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

or

(3.8)

$$h(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

,

where x=(x,y,z) and a=(a,b,c).

Obviously, the vector notation provides a compact form in describing the tangent equation of function of several variables.

Example 3.21. A surface function is given by z=x^2-6x+y^3. Find an equation of the plane tangent to this surface at point (1,2,3).

Solution:

```
Clear[f,x,y]

f[x_,y_]:=x^2-6x+y^3

D[f[x,y],x]

D[f[x,y],x]/.{x->1,y->2}

D[f[x,y],y]
```

D[f[x,y],y]/.{x->1,y->2}

Write f(x,y)=z=x^2-6x+y^3, so

$$\frac{\partial f}{\partial x} = 2x - 6, \qquad \frac{\partial f}{\partial y} = 3y^2,$$
$$\frac{\partial f}{\partial x}(1, 2) = -4, \qquad \frac{\partial f}{\partial y}(1, 2) = 12.$$

By Theorem 3.2 the tangent plane to the surface at the point (1,2,3) is given by

$$h(x, y) = f(1, 2) + \frac{\partial f(1, 2)}{\partial x}(x - 1) + \frac{\partial f(1, 2)}{\partial y}(y - 2)$$

,

h(x,y)=3-4(x-1)+12(y-2),

h(x,y)=-4x+12y-17.

Here is a Mathematica module to find the tangent plane of a surface function given by z=f(x,y) at point P=(a,b,f(a,b)) as described in Theorem 3.2:

(* The equation of the tangent plane to the surface

defined by z=f(x,y) at (x,y)=(a,b)

*)

tangentPlane[f_,a_]:=

Module[{u,v},

```
f[u,v]+Grad[f[u,v],{u,v},"Cartesian"].{x-a[[1]],y-a[[2]]}/.{u->a[[1]],v->a[[2]]}

]



tangentPlane[f,{1,2}]//Simplify



surface=Plot3D[f[x,y],{x,0,2},{y,1,3},PlotStyle->Opacity[0.6],BoxRatios->{1, 1, 1}];

tangentP=Plot3D[-4x+12y-17,{x,0,2},{y,1,3},Mesh->None,PlotStyle->{Green,Opacity[0.3]}];

point=Graphics3D[{Red,PointSize[0.03],Point[{1,2,3}]}];

Show[surface,tangentP,point,ViewPoint->{10,10,10}]
```

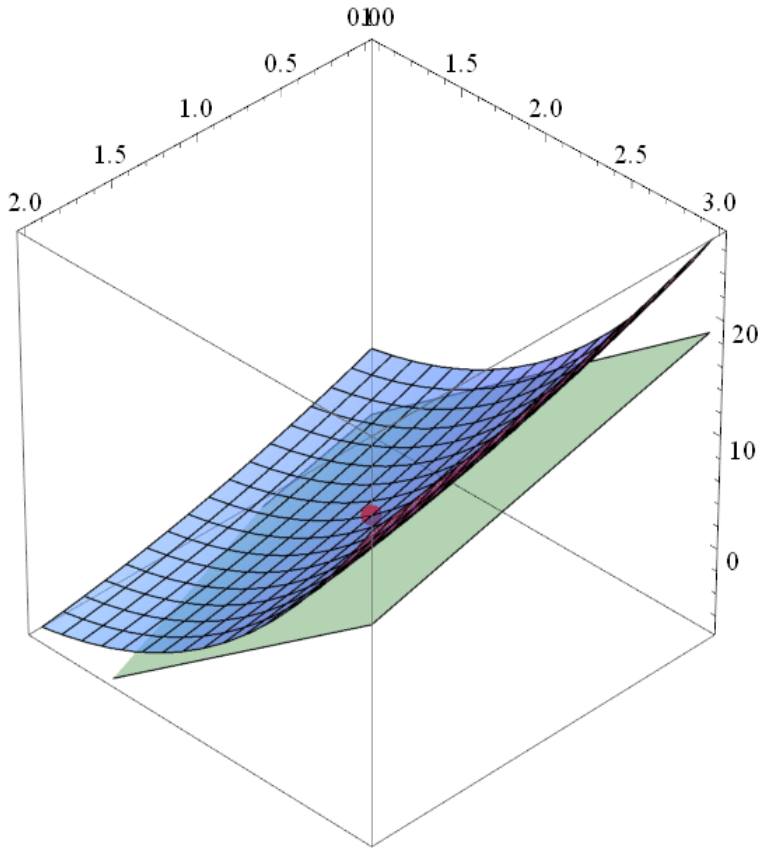


Figure 3.17. The surface z=x^2-6x+y^3 and the plane tangent to it at the point (1,2,3).

The Mathematica module to find a tangent hyperplane to a hypersurface function w=f(x,y,z) at point P=(a,b,c,f(a,b,c)) is given below:

```
(* The equation of the tangent hyperplane to the surface

defined by w=f(x,y,z) at (x,y,z)=(a,b,c)

*)

tangentHyperplane[f_,a_]:=

Module[{s,t,v},
```

f[s,t,v]+Grad[f[s,t,v],{s,t,v},"Cartesian"].{x-a[[1]],y-a[[2]],z-a[[3]]}/.{s->a[[1]],t->a[[2]],v->a[[3]]}

]

Example 3.22. Find an equation of the tangent plane to the surface at the given point:

(a) x z+2x^2 y+y^2 z^3=11, P=(2,1,1).

(b) x^2+z^2 E^(y-x)=13, P=(2,3,3/Sqrt[E]).

Solution:

(a)

We may consider the surface x z+2x^2 y+y^2 z^3=11 as a hypersurface defined by

w=f(x,y,z)=x z+2x^2 y+y^2 z^3-11.

Then by Theorem 3.3 the equation of the tangent hyperplane at point P is given by equation (3.6):

$$h(x, y, z) = f(a, b, c) + \frac{\partial f(a, b, c)}{\partial x}(x - a) + \frac{\partial f(a, b, c)}{\partial y}(y - b) + \frac{\partial f(a, b, c)}{\partial z}(z - c)$$

Clear[f]

f[x_,y_,z_]:=x z+2x^2 y+y^2 z^3-11

tangentHyperplane[f,{2,1,1}]//Simplify

As we have mentioned earlier in Section 3.1.2 the graph of f(x,y,z) consists of a set of points (x,y,z,f(x,y,z)) in four-dimensional space. So, we are not able to draw it. However, we may plot it as a level surface at height 0, as is illustrated in the figure below:

```
(* Plot the hypersurface as a level surface at height 0 *)

P={2,1,1};

surface=ContourPlot3D[x z+2x^2 y+y^2 z^3==11,{x,-8,8},{y,-8,8},
{z,-8,8},Mesh->None,ContourStyle->Opacity[0.6],AxesLabel->
{"x","y","z"}];

tangentP=ContourPlot3D[9 x+10 y+5 z==33,{x,-8,8},{y,-8,8},
{z,-8,8},Mesh->None,ContourStyle->{Green,Opacity[0.3]}];

point=Graphics3D[{Red,PointSize[0.03],Point[P]}];

Show[surface,tangentP,point,ViewPoint->{15,-8,20}]
```

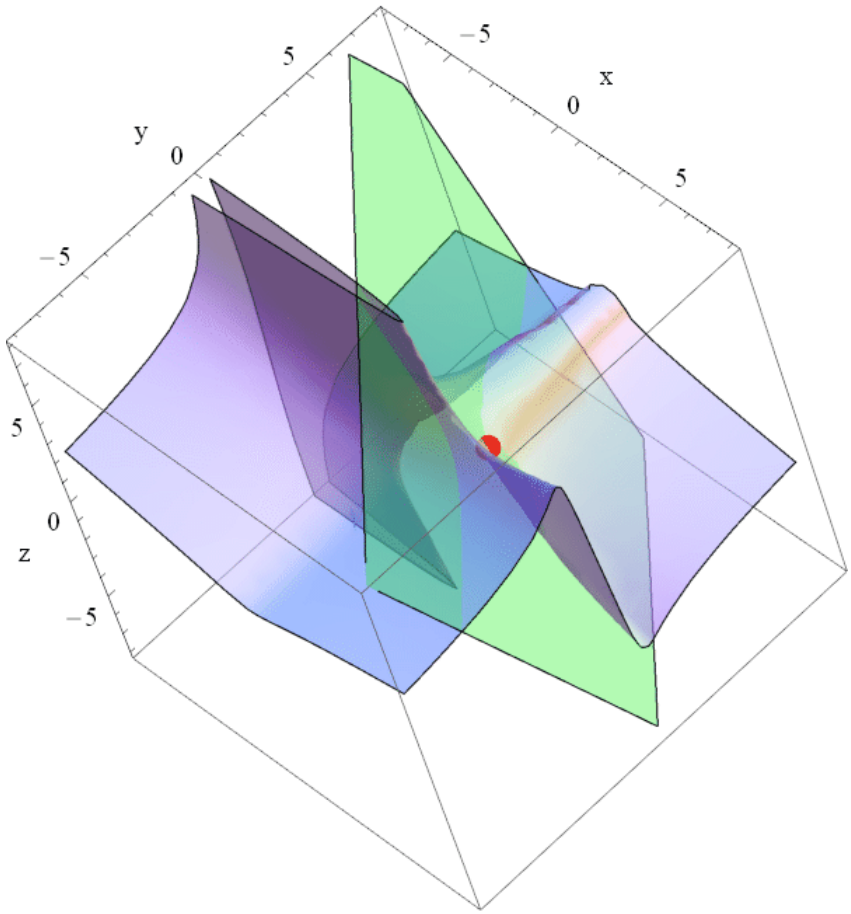


Figure 3.18. The surface x z+2x^2 y+y^2 z^3=11 and the tangent plane at P=(2,1,1).

(b)

Clear[f]

f[x_,y_,z_]:=x^2+z^2 E^(y-x)-13

tangentHyperplane[f,{2,3,3/Sqrt[E]}]//Simplify

```
(* Plot the hypersurface as a level surface at height 0 *)

P={2,3,3/Sqrt[E]};

surface=ContourPlot3D[x^2+z^2 E^(y-x)==13,{x,-8,8},{y,-8,8},
{z,-8,8},Mesh->None,ContourStyle->Opacity[0.6],AxesLabel->
{"x","y","z"}];

tangentP=ContourPlot3D[-5 x+9 y+6 Sqrt[E] z==35,{x,-8,8},{y,-8,8},
{z,-8,8},Mesh->None,ContourStyle->{Green,Opacity[0.3]}];

point=Graphics3D[{Red,PointSize[0.03],Point[P]}];

Show[surface,tangentP,point,ViewPoint->{15,-8,20}]
```

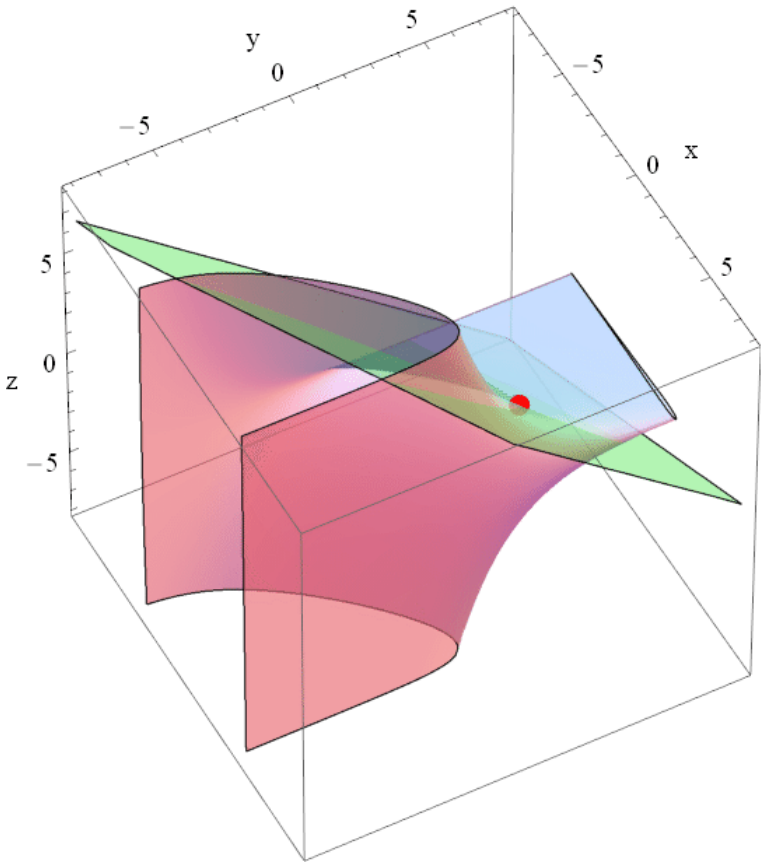


Figure 3.19. The surface x^2+z^2 E^(y-x)=13 and the tangent plane at P= (2,3,3/Sqrt[E]).

Up to this point, we have discussed the gradient and derivative of scalar-valued functions of several variables. The results can be generalized to vector-valued functions of several variables as well. However, since a vector-valued function consists of scalar components is a vector itself we shall not define its gradient but only the derivative.

Definition 3.4 (matrix of partial derivatives)

Let $\mathbf{f}:(x_1, x_2, \ldots x_n) \to \mathbb{R}^m$ be a vector-valued function of n variables. The matrix of partial derivatives of f, denoted Df is a $m \times n$ matrix whose ij-th entry is $\frac{\partial f_i}{\partial x_j}$, where $f_i:(x_1, x_2, \ldots x_n) \to \mathbb{R}$ is the i-th component function of f. The derivative of f is defined as

(3.9)

$$D\mathbf{f}(x_1, x_2, \ldots, x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

.

Note that when m=1, the matrix reduces to a row matrix which is the same as the one in the case for derivative of a scalar-valued function as defined in

Definition 3.1. The i-th row of Df is just Df_i in which the entries are the components of the gradient vector ∇f_i .

Example 3.23. Find the matrix of partial derivatives of the function f at the point (2,1) given by

f(x,y)=(x y^2, x^2+y, sin π x y).

Solution:

This is a three-dimensional vector-valued function of two variables x and y, where

$$f_1(x, y) = x\,y^2, \qquad f_2(x, y) = x^2 + y, \qquad f_3(x, y) = \sin \pi\, x\, y$$

.

So, we find the matrix of partial derivatives using Definition 3.4:

$$D\mathbf{f}(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix} = \begin{pmatrix} y^2 & 2\,x\,y \\ 2\,x & 1 \\ \pi\, y \cos \pi\, x\, y & \pi\, x \cos \pi\, x\, y \end{pmatrix}$$

.

The matrix of partial derivatives of the function f at the point (2,1) is then

$$D\mathbf{f}(2, 1) = \begin{pmatrix} 1 & 4 \\ 4 & 1 \\ \pi & 2\,\pi \end{pmatrix}.$$

```
Clear[f,x,y]

f[x_,y_]={x y^2,x^2+y,Sin[π x y]}
```

Mathematica has a nice tool to compute the matrix of partial derivatives using the Outer command.

```
(* Compute Df using Outer command *)

Outer[D,f[x,y],{x,y}]//MatrixForm

Outer[D,f[x,y],{x,y}]/.{x->2,y->1}//MatrixForm
```

Example 3.24. Find the matrix of partial derivatives of the function f at the point (2,0,-1) given by

```
f(x,y,z)=(2x-y+3z,Sqrt[x^2+y^2+z^2])
```

Solution:

The function defined in this example is two-dimensional vector-valued function of three variables, where

$$f_1(x, y, z) = 2x - y + 3z, \qquad f_2(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

.

So, the matrix of partial derivatives of f is

$$D\mathbf{f}(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{2}{x} & \frac{-1}{y} & \frac{3}{z} \\ \sqrt{x^2+y^2+z^2} & \sqrt{x^2+y^2+z^2} & \sqrt{x^2+y^2+z^2} \end{pmatrix}$$

.

$$D\mathbf{f}(2, 0, -1) = \begin{pmatrix} 2 & -1 & 3 \\ \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} \end{pmatrix}$$

.

```
Clear[f,x,y,z]

f[x_,y_,z_]:={2x-y+3z,Sqrt[x^2+y^2+z^2]}


(* Compute Df using Outer command *)

Outer[D,f[x,y,z],{x,y,z}]//MatrixForm

Outer[D,f[x,y,z],{x,y,z}]/.{x->2,y->0,z->-1}//MatrixForm
```

3.2.1 Directional Derivatives

Suppose f(x,y) be a scalar-valued function of two variables. We have seen earlier that the definition z=f(x,y) corresponds to a surface in three-dimensional space. Geometrically, the partial derivatives of f with respect to x, that is $\frac{\partial f}{\partial x}$ at a point (a,b,f(a,b)) can be viewed as the slope of the curve obtained as the intersection of the surface with the plane y=b. Similarly for

the other partial derivative $\frac{\partial f}{\partial y}$ at the same point as the slope of the curve obtained as the intersection of the surface with the plane x=a.

However, there are infinitely many planes that could intersect with the surface at the point (a,b,f(a,b)). Thus, this leads to the definition of directional derivatives.

Definition 3.5. (directional derivative)

Let $f : (x, y) \to \mathbb{R}$ be a scalar-valued function, and $\mathbf{a} \equiv (x_0, y_0)$ be a point in $\mathbb{R}^2$. If $\mathbf{u} = (u_0, u_1) \in \mathbb{R}^2$ is any unit vector, then the directional derivative of f at a in the direction of u, denoted $D_{\mathbf{u}} f(\mathbf{a})$ is

(3.10)

$$D_{\mathbf{u}} f(\mathbf{a}) = \frac{d}{dt} f(\mathbf{a} + t\,\mathbf{u}) \,|_{t=0}$$

, where $t \in \mathbb{R}$.

Definition 3.5 can be generalized to scalar-valued function of more than two variables.

Note that the partial derivatives with respect to x and y are the directional derivatives with respect to the standard unit vectors i=(1,0) and j=(0,1), respectively.

Thus,

$D_{\mathbf{i}} f(\mathbf{a}) = f_x(\mathbf{a})$ and $D_{\mathbf{j}} f(\mathbf{a}) = f_y(\mathbf{a})$.

Suppose x and y are functions of t, and let r(t)=(x(t),y(t)) parametrizes a curve in the domain of f.

Write r(t)=a+t u, where $\mathbf{a} \equiv (x_0, y_0)$ a constant. Then by the chain rule and Definition 3.1 we have

$$\frac{d}{dt} f(\mathbf{a} + t\,\mathbf{u}) = D f(x, y)\, \frac{d}{dt} \mathbf{r}(t)$$

.

(3.11)

$$\frac{d}{dt} f(\mathbf{a} + t\,\mathbf{u}) = \nabla f(x, y) \cdot \mathbf{r}'(t)$$

.

Thus, by equation (3.10)

(3.12)

$$D_{\mathbf{u}} f(\mathbf{a}) = \frac{d}{dt} f(\mathbf{a} + t\,\mathbf{u}) \,|_{t=0} = \nabla f(x_0, y_0) \cdot \mathbf{r}'(0)$$

.

Equation (3.12) tells us that the directional derivative of f in the direction of u is the dot product of the gradient of f with the velocity vector of the curve at t=0, where u=r'(t).

Suppose that a curve parametrized by c(t)=(x(t),y(t)) is a level curve of f(x,y).

We see in Section 3.1.1 a level curve of z=f(x,y) is constant. So, f(c(t))=constant.

Taking the derivative of f(c(t)) with respect to t and using the chain rule we get

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 0$$

.

Thus, we have $\nabla f(x, y) \cdot \mathbf{c}'(t) = 0$ for all t, that is, the gradient of f is perpendicular to the tangent vector to the level curve, and thus to the level curve itself.

Another consequence of the directional derivative is due to the property of the dot product at the right-hand side of equation (3.12). The value of $D_{\mathbf{u}} f(\mathbf{a})$ is maximum when u=r'(0) points in the same direction as ∇f and is minimum when u points in the opposite direction.

The maximum and minimum values of $D_{\mathbf{u}} f(\mathbf{a})$ are $\| \nabla f(\mathbf{a}) \|$ and $- \| \nabla f(\mathbf{a}) \|$, respectively.

Finally, if the point $\mathbf{a} \equiv (x_0, y_0)$ is a critical point of f, then $\nabla f(x_0, y_0) = (0, 0)$. Thus, $D_{\mathbf{u}} f(\mathbf{a}) = 0$ implies that all directional derivatives of f vanish at a. At such point however, there may not have a well-defined tangent vector to the level curve.

As illustrations, let us consider some examples below.

Example 3.25. Find the directional derivative of the given function f at the point a in the direction parallel to the unit vector u.

(a) f(x,y)=E^y cos x, a=(π/3,0), u=(i + 2j)/Sqrt[5].

(b) f(x,y,z)=E^-(x^2 + y^2 + z^2), a=(1,1,1), u=(i + j + k)/Sqrt[3].

Solution:

(a)

Using equation (3.10) from Definition 3.5 the directional derivative of f at a=(π/3,0) in the direction of u=(i + 2j)/Sqrt[5], denoted $D_{\mathbf{u}} f(\mathbf{a})$, is

$$D_{\mathbf{u}} f(\mathbf{a}) = \frac{d}{dt} f(\mathbf{a} + t\,\mathbf{u}) \,|_{t=0}$$

,

$$D_{\mathbf{u}} f\left(\frac{\pi}{3}, 0\right) = \frac{d}{dt} f\left(\left(\frac{\pi}{3}, 0\right) + t\,\frac{1}{\sqrt{5}}\,(1, 2)\right) \,|_{t=0}$$

,

which yields

$$D_{\mathbf{u}} f\left(\frac{\pi}{3}, 0\right) = \frac{2-\sqrt{3}}{2\sqrt{5}}$$.

Clear[f,a,u,t]

f[{x_,y_}]:=E^y Cos[x]

a={π/3,0};

u=1/Sqrt[5] {1,2};

D[f[a+t u],t]/.t->0//Simplify

Similarly, we can also use the gradient of f to find the directional derivative of f at a=(π/3,0) in the direction of u=(i + 2j)/Sqrt[5]. Recall equation (3.12)

$$D_{\mathbf{u}} f(\mathbf{a}) = \frac{d}{dt} f(\mathbf{a} + t\,\mathbf{u}) \,|_{t=0} = \nabla f(x_0, y_0) \cdot \mathbf{r}'(0)$$

.

The gradient of f(x,y) is

$$\nabla f(x, y) = (-e^y \sin x, e^y \cos x)$$

,

$$\nabla f\left(\frac{\pi}{3}, 0\right) = \left(-\frac{1}{2}\sqrt{3}, \frac{1}{2}\right)$$

.

The parametric equation is

r(t)=a+t u,

r(t)=(π/3,0)+t 1/Sqrt[5] (1,2)

r'(t)=1/Sqrt[5] (1,2), r'(0)=1/Sqrt[5] (1,2).

Thus, using equation (3.12):

$$D_{\mathbf{u}} f\left(\frac{\pi}{3}, 0\right) = \left(-\frac{1}{2}\sqrt{3}, \frac{1}{2}\right) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{2-\sqrt{3}}{2\sqrt{5}}$$

.

```
Grad[f[{x,y}],{x,y},"Cartesian"].u/.{x->π/3,y->0}//Simplify
```

(b)

Using the same method as in part (a) we obtain the directional derivative of f at a=(1,1,1) in the direction of u=(i + j + k)/Sqrt[3]:

$$D_{\mathbf{u}} f(1, 1, 1) = -\frac{2\sqrt{3}}{e^3}$$.

```
Clear[f,a,u,t]

f[{x_,y_,z_}]:=E^-(x^2+y^2+z^2)

a={1,1,1};

u=1/Sqrt[3] {1,1,1};


D[f[a+t u],t]/.t->0//Simplify
```

Example 3.26. A scalar-valued function is given by

f(x,y)=(x^2+y^2)^2-x^2+y^2.

(a) Graph the surface and the level curves of the function in the region -2<=x<=2, -2<=y<=2.

(b) Find the critical points of f, that is, points where $\nabla f = 0$.

(c) Draw the gradient of f and show that they are perpendicular to the level curves.

Solution:

(a)

```
Clear[f,x,y]

f[x_,y_]:=(x^2+y^2)^2-x^2+y^2

Plot3D[f[x,y],{x,-2,2},{y,-2,2},AxesLabel->{"x","y","z"},BoxRatios->
{1,1,1}]
```

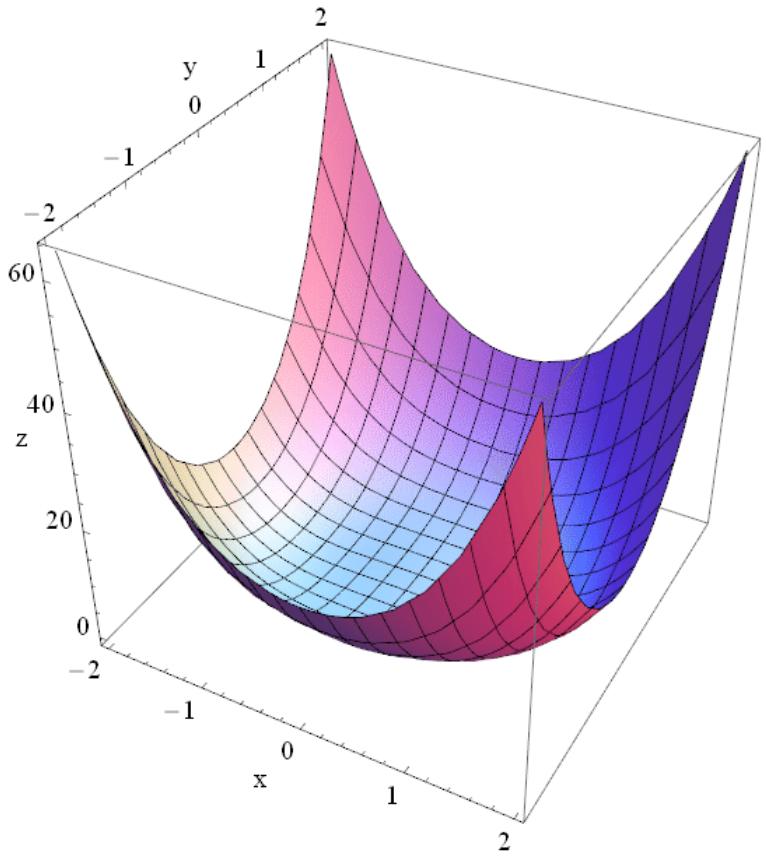


Figure 3.20. The surface of the function f(x,y)=(x^2+y^2)^2-x^2+y^2.

To draw the level curves of the function z=f(x,y) we use ContourPlot command; and to differentiate those level values which are negative, null, and positive, we use blue, red, and green color, respectively. For our purpose, we choose z=(-0.2,-0.1) to represent negative level curves, z=0 for null level curve, and z=(0.1,0.2,1,4,10) for positive level curves.

(* ContourPlot for negative values of f *)

```
contourPlotNeg=ContourPlot[f[x,y],{x,-2,2},{y,-2,2},ContourShading-
>False,Contours->{-0.2,-0.1},ContourStyle->
{{AbsoluteThickness[2],Blue}}];
```

```
(* ContourPlot for positive values of f *)

contourPlotPos=ContourPlot[f[x,y],{x,-2,2},{y,-2,2},ContourShading-
>False,Contours->{0.1,0.2,1,4,10},ContourStyle->
{{AbsoluteThickness[2],Green}}];

(* ContourPlot for special case f=0 *)

contourPlotNull=ContourPlot[f[x,y],{x,-2,2},{y,-2,2},ContourShading-
>False,Contours->{0},ContourStyle->{AbsoluteThickness[2],Red}];

Show[contourPlotNeg,contourPlotPos,contourPlotNull]
```

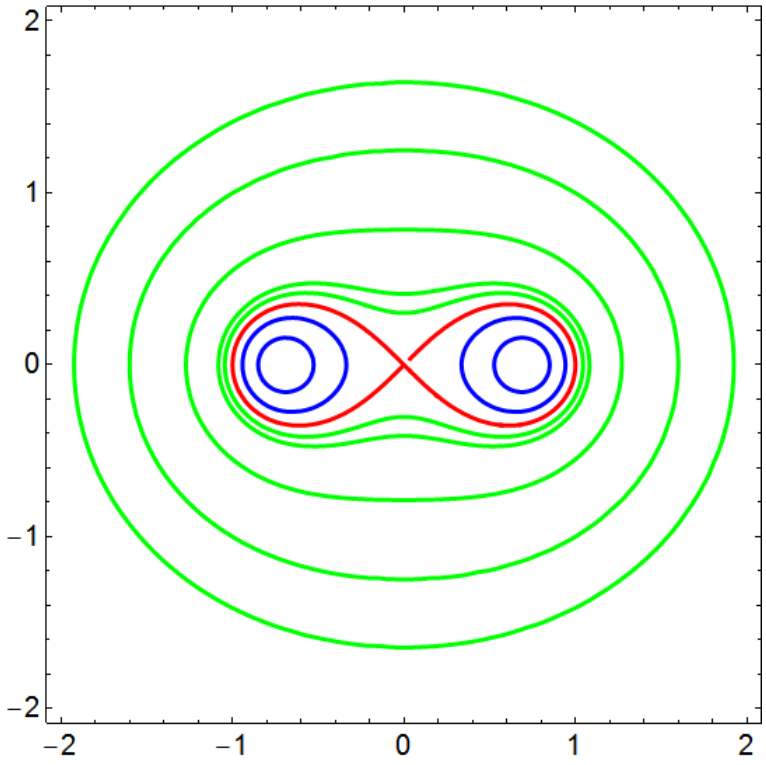


Figure 3.21. The level curves of the function z=f(x,y)=(x^2+y^2)^2-x^2+y^2. Blue curve corresponds to the value z=-0.2 and z=-0.1; red curve corresponds to z=0, and green curves correspond to z=0.1,0.2,1,4, and 10.

(b)

```
(* Compute the gradient of f *)

grad=Grad[f[x,y],{x,y},"Cartesian"]


(* Find the critical points of f *)

sol=Solve[grad=={0,0}//Thread,{x,y}]


(* These are the critical points of f *)

criticalPoints={x,y}/.sol
```

Out of five of the critical points obtained, we only use three for drawing purposes since the other two are complex values. So, these are

(0,0), (-(1/Sqrt[2]),0) and (1/Sqrt[2],0).

```
criticalPointsPlot=Graphics[{PointSize[0.02],{Point[{0,0}],Point[{-
(1/Sqrt[2]),0}],Point[{1/Sqrt[2],0}]}}];

Show[contourPlotNeg,contourPlotPos,contourPlotNull,criticalPointsPlot]
```

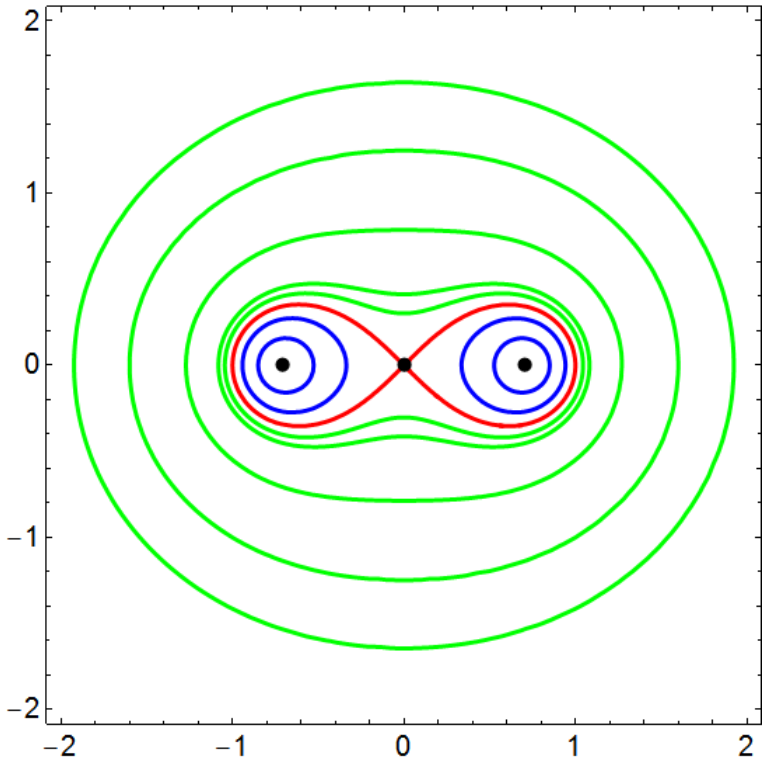


Figure 3.22. The level curves of f(x,y)=(x^2+y^2)^2-x^2+y^2 with three critical points at (0,0), (-(1/Sqrt[2]),0) and (1/Sqrt[2],0).

Notice that at these critical points all directional derivatives of f vanishes. Also note that the level curve z=0 passes through the origin. At this point, there is no well-defined tangent direction since the level curve crosses itself here.

(c)

To plot the gradient vector field of f, we can use StreamPlot command:

streamPlot=StreamPlot[{grad[[1]],grad[[2]]},{x,-2,2},{y,-2,2}]

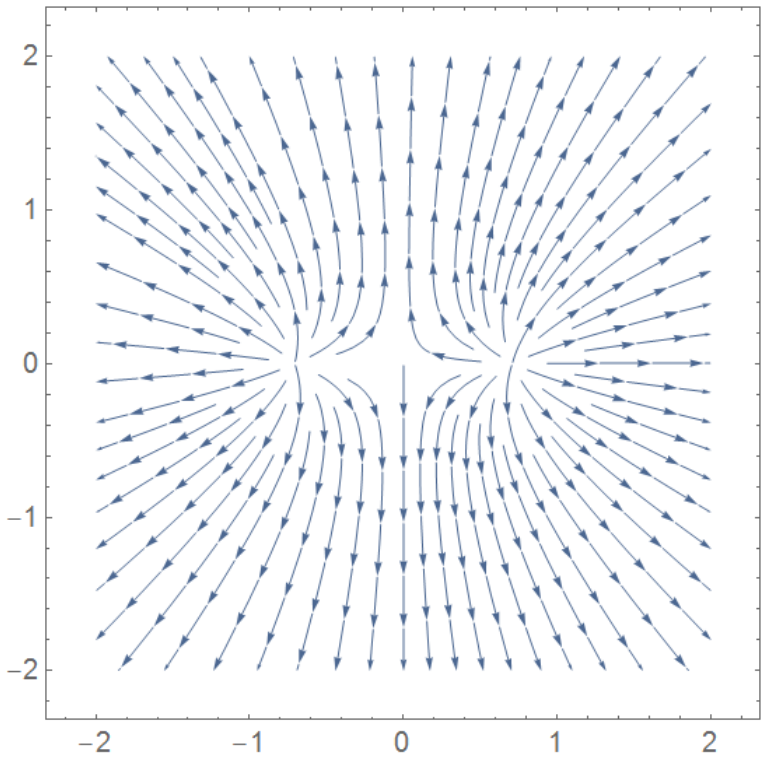


Figure 3.23. The gradient field of f(x,y)=(x^2+y^2)^2-x^2+y^2.

Show[contourPlotNeg,contourPlotPos,contourPlotNull,criticalPointsPlot,streamPlot]

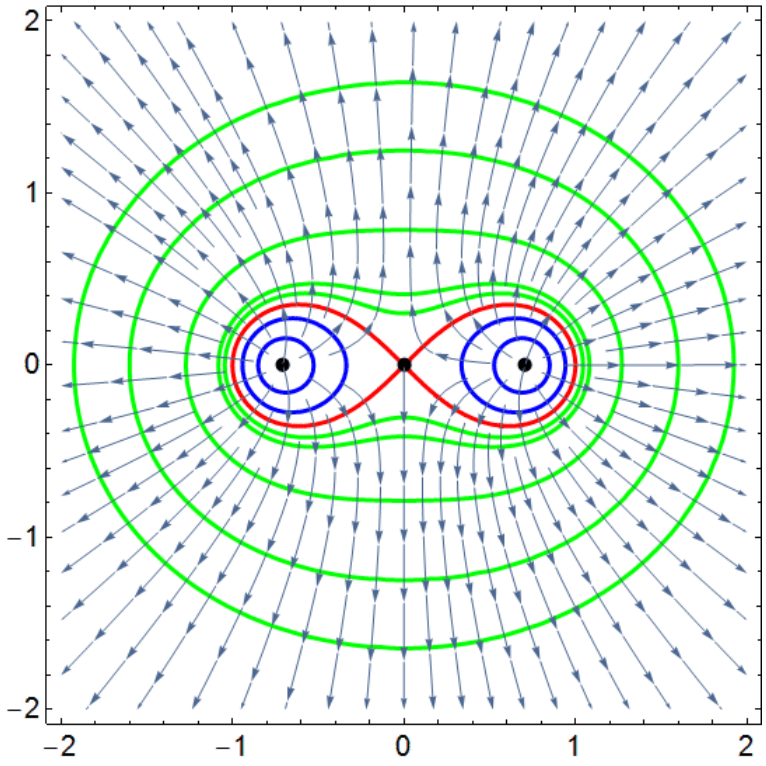


Figure 3.24. The level curves of f(x,y)=(x^2+y^2)^2-x^2+y^2 with the three critical points at (0,0), (-(1/Sqrt[2]),0) and (1/Sqrt[2],0), and the gradient field of f.

As can be seen from the figure above, the gradient vectors are perpendicular to these level curves, and all moving away from the critical points. However, at the origin (which is one of the critical points where $\nabla f = 0$), there are gradient vectors that point toward and outward of this point. This explains the self-crossing level curve at the origin.

Example 3.27. Find the curve y=g(x) passing through (0,1) that crosses each level curve of the graph f(x,y)=y sin x at a right angle. Plot some level curves of f(x,y) and the curve y=g(x).

Solution:

First we evaluate the gradient of f(x,y):

$$\nabla f(x, y) = (y \cos x, \sin x)$$

.

Grad[y Sin[x],{x,y},"Cartesian"]

Recall from Section 3.2.1 the gradient of f(x,y) is perpendicular to the level curves of f(x,y). Thus, the slope of y=g(x) is parallel to $\nabla f(x, y)$.
Therefore,

dy/dx=(tan x)/y.

Solving the differential equation above yields the general solutions:

y=±Sqrt[2c-ln (cos^2 x)],

where c is constant of integration.

DSolve[y'[x]==Tan[x]/y[x],y,x]

Since the curve passes through the point (0,1), the particular solution is y=Sqrt[2c-ln (cos^2 x)].

Substituting (x,y)=(0,1) into the equation above gives

1=Sqrt[2c-ln (cos^2 0)],

c=1/2.

Hence, the equation of the curve:

y=g(x)=Sqrt[1-ln (cos^2 x)].

```
plot[1]=ContourPlot[y Sin[x],{x,0,10},{y,0,5},ContourShading->False,Contours->{-1,0.2,0.5,1},ContourStyle->{Black,Green,Yellow,Red},Frame->None,Axes->True,AxesLabel->{"x","y"},PlotLegends->Automatic];

plot[2]=Plot[Sqrt[1-Log[Cos[x]^2]],{x,0,10}];

Show[Array[plot,2],AspectRatio->1/2]
```

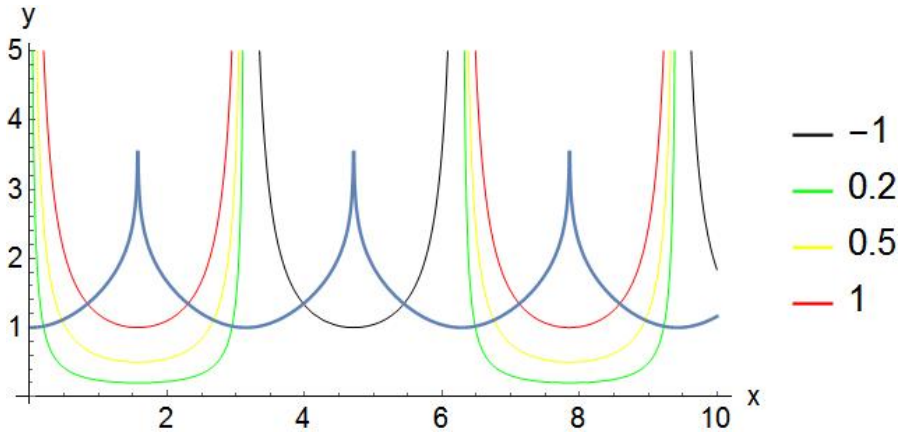


Figure 3.25. The level curves of the graph f(x,y)=y sin x and the curve y=g(x)=Sqrt[1-ln (cos^2 x)] that crosses each level curve at a right angle.

Example 3.28. Consider an ellipsoidal mountain with equation z=100-Sqrt[4x^2+y^2], where z>=0.

(a) It is raining and rainwater is running off the mountain. Plot the mountain and find the curves whose paths the raindrops must follow to slide down as rapidly as possible.

(b) Suppose a contractor is to build a road up the mountain to the top. According to the normal standard the rate of climb of the road should be a fixed reasonable constant 1/25 (this amounts to a 4% grade condition). Show the 4% grade condition is satisfied by the road if the path is parametrized by the following equations:

x(t)=a t cos (b ln t), y(t)=2a t sin (b ln t), z(t)=100-2a t, for 0<=t<=50/a,

where a=1/50, b=Sqrt[3599]. Plot the path of the road going up the mountain.

Solution:

(a)

The ellipsoidal mountain with equation z=100-Sqrt[4x^2+y^2], where z>=0 can be considered as level curves of the form f(x,y)=constant:

f(x,y)=100-Sqrt[4x^2+y^2]=c, where c is constant.

```
Clear[f]

f[x_,y_]:=100-Sqrt[4x^2+y^2]


mountain=Plot3D[f[x,y],{x,-100,100},{y,-100,100},Mesh-
>None,PlotRange->{Automatic,Automatic,{0,100}},AxesLabel->
{"x","y","z"},ImageSize->500]
```

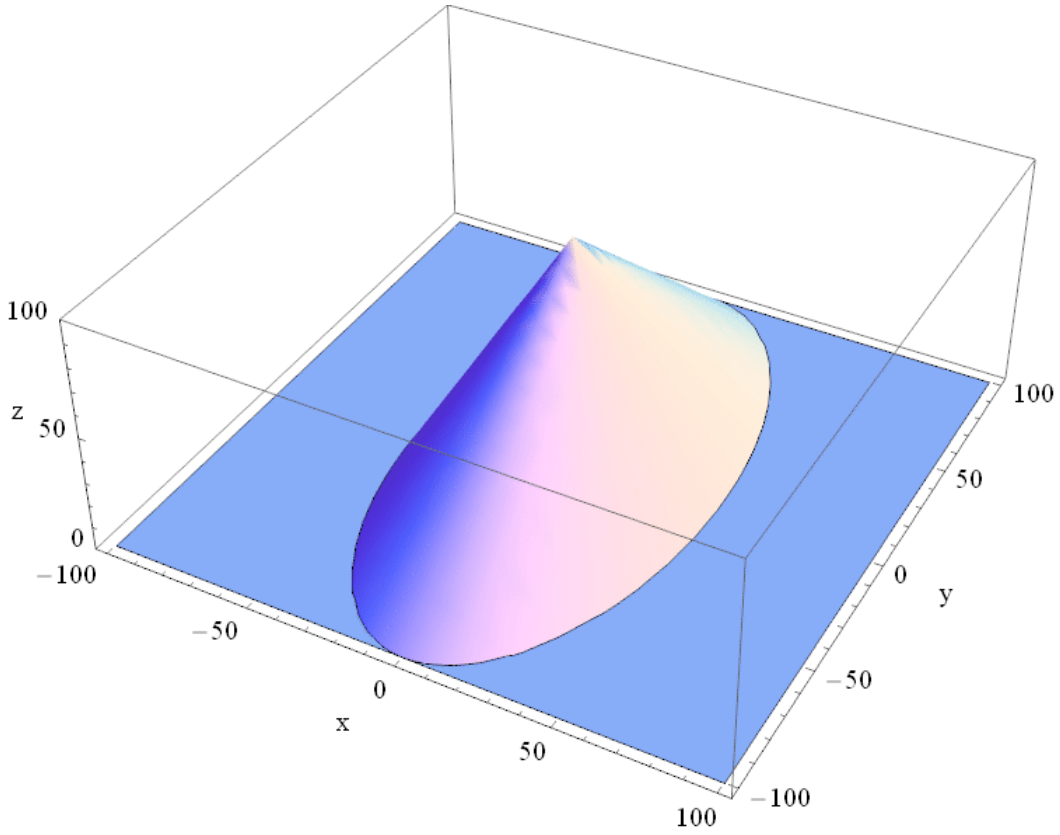


Figure 3.26. The graph of the mountain as represented by z=100-Sqrt[4x^2+y^2].

(* Find the gradient of f(x,y) *)

Grad[f[x,y],{x,y},"Cartesian"]

From Mathematica result above, the gradient of f(x,y)=100-Sqrt[4x^2+y^2] is

$$\nabla f(x, y) = \left(-\frac{4x}{\sqrt{4x^2+y^2}}, -\frac{y}{\sqrt{4x^2+y^2}} \right)$$

.

Recall from equation (3.12) the directional derivative of f(x,y) in the direction u is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

$D_{\mathbf{u}} f(x, y)$ **is maximum if u is in the direction of the gradient of f and minimum if u is in opposite direction.**

In this case, the raindrops are heading towards lower level curves while intersecting each level curve orthogonally. For the path of steepest descent, we consider the negative gradient of f:

$$-\nabla f(x, y) = \left(\frac{4x}{\sqrt{4x^2 + y^2}}, \frac{y}{\sqrt{4x^2 + y^2}} \right)$$

.

This means that the slope of the path dy/dx is

$$\frac{dy}{dx} = \frac{y}{\sqrt{4x^2 + y^2}} \Big/ \frac{4x}{\sqrt{4x^2 + y^2}} = \frac{y}{4x}$$

.

Hence, the equation of the level curve can be determined by solving the differential equation above.

DSolve[{y'[x]==y[x]/(4x)},y[x],x]//PowerExpand

The solution is y=c x^(1/4).

The path of the raindrops (x, y, f(x,y)) is then

(x, c x^(1/4), 100-Sqrt[4x^2+c^2 x^(1/2)]),

where c is any arbitrary constant.

```
Clear[r]

r[c_][t_]={t,c t^(1/4),100-Sqrt[4t^2+c^2 t^(1/2)]}


rainPaths=ParametricPlot3D[Table[r[c][t],{c,0,20,5}]//Evaluate,
{t,0,60},PlotStyle->Table[{Dashed,AbsoluteThickness[3],Hue[i]},
{i,0,1,0.2}]];
```

Here we show some paths of the raindrops with steepest descent depending on the value of c:

```
Show[mountain,rainPaths,ViewPoint->{100,-70,100}]
```

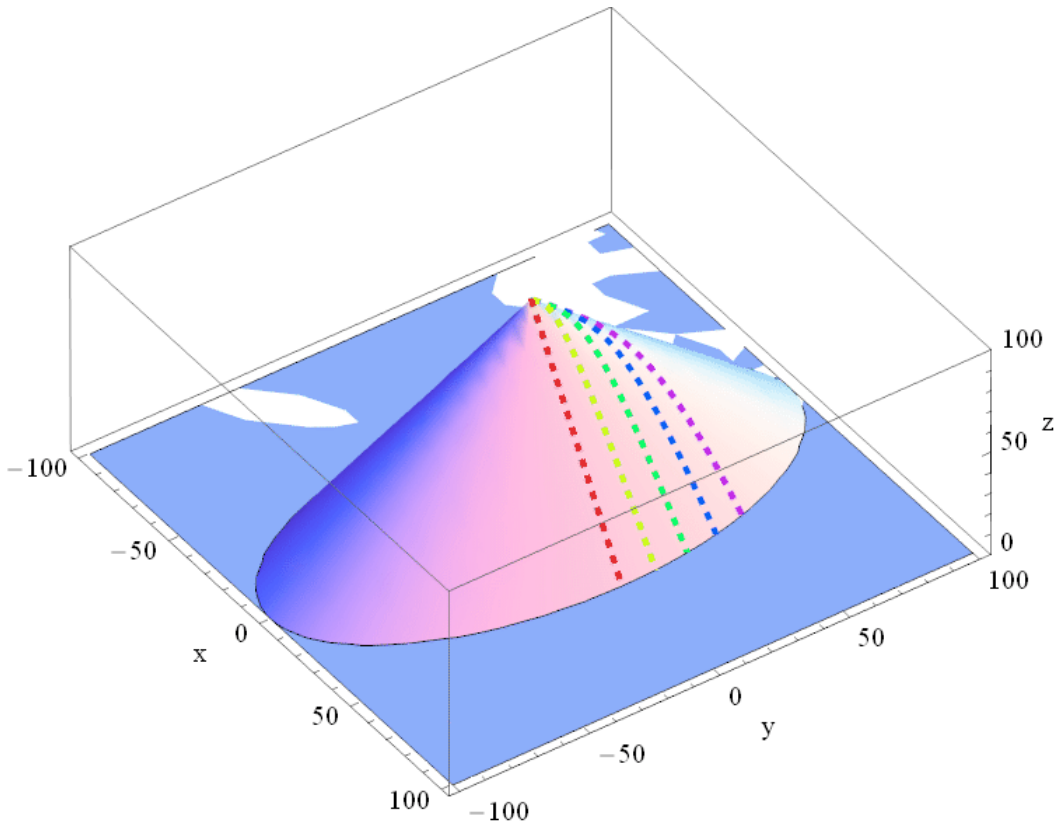


Figure 3.27. Some paths of the raindrops with steepest descent.

(b)

Using equation (3.11) the rate of climb of f in the direction of the tangent vector along the road must satisfy:

$$\| \nabla f \cdot \mathbf{r}' \| = \frac{1}{25} .$$

Clear[a,b]

x[t_]:=a t Cos[b Log[t]]

y[t_]:=2a t Sin[b Log[t]]

```
z[t_]:=100-Sqrt[4x[t]^2+y[t]^2]//Simplify//PowerExpand


r[t_]:={x[t],y[t],z[t]}


gradf[t_]:=Grad[f[x,y],{x,y,z},"Cartesian"]/.{x->x[t],y->y[t]}//Simplify//PowerExpand


(* The 4% grade condition is satisfied *)

Abs[gradf[t].r'[t]]/.{a->1/50,b->Sqrt[3599]}//Simplify


road=ParametricPlot3D[Evaluate[{x[t],y[t],z[t]}/.{a->1/50,b->Sqrt[3599]}],{t,0,2500},AxesLabel->{"x","y","z"},BoxRatios->{1, 1, 1}];


Show[mountain,road]
```

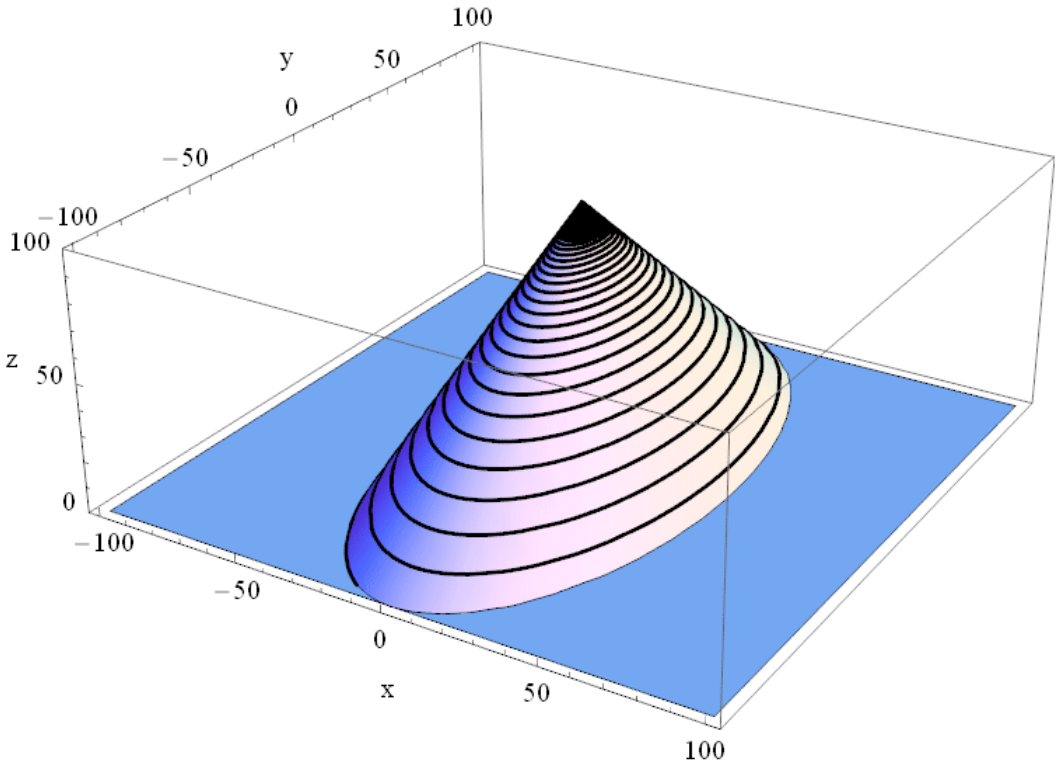


Figure 3.28. The path of the road going up the mountain with 4% grade condition.

Chapter 4. Surfaces

A surface is a 2-dimensional analog of a curve. Just as a 3-dimensional curve is considered as the image of a continuous function mapping an interval in $\mathbb{R}$ to $\mathbb{R}^3$, a surface is the image of a continuous function mapping an interval in $\mathbb{R}^2$ to $\mathbb{R}^3$. Since the domain of surfaces are 2-dimensional; in general surfaces are much more complicated to deal with than curves.

Surfaces in $\mathbb{R}^3$ can be described using the following methods:

1. as a graph of a function of two variables, that is, z=f(x, y).

So, surfaces are set of points (x, y, f(x, y)).

2. as a level set of function of three variables, that is, f(x, y, z)=c, where c is constant.

3. as a parametrized continuous function $\mathbf{X} : D \to \mathbb{R}^3$, where the region D is either a rectangle or a disk in $\mathbb{R}^2$ consists of a connected open set, either some or all of its boundary points.

Throughout the book we consider mostly with the parametrization of surfaces in $\mathbb{R}^3$. If u and v are coordinates on D, they are called parameters or generalized coordinates on the surface.

Thus for $(u, v) \in D$, the component functions of parametrized surface X can be written as

X(u, v)=(x(u, v), y(u, v), z(u, v)).

To plot surfaces in Mathematica we use Plot3D, ContourPlot3D, and ParametricPlot3D for first, second, and third method, respectively.

Initialization

```
Clear["Global`*"]

SetOptions[Plot3D,PlotTheme->"Classic"];

SetOptions[ContourPlot3D,PlotTheme->"Classic"];

SetOptions[Plot,PlotTheme->"Classic",PlotStyle->AbsoluteThickness[2]];

SetOptions[ParametricPlot3D,PlotTheme->"Classic",PlotStyle-
>AbsoluteThickness[2]];
```

Example 4.1. (Method 1) Plot the paraboloid given by the equation z=2x^2+2y^2.

Solution:

```
Plot3D[2x^2+2y^2,{x,-2,2},{y,-2,2}]
```

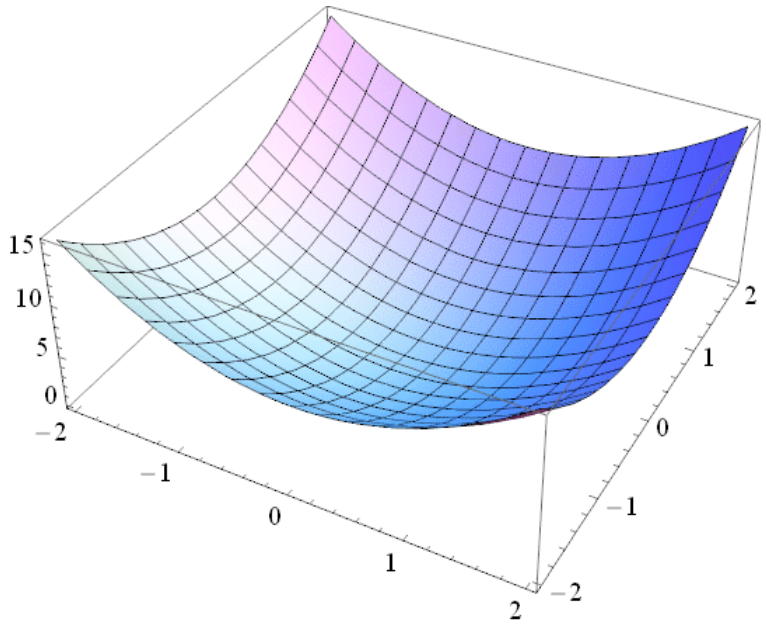


Figure 4.1. The graph of a paraboloid z=2x^2+2y^2.

Example 4.2. (Method 2) Plot the surface defined by the function

f(x,y,z)=x^2 y-z^2 y^3+ z^2=1.

Solution:

ContourPlot3D[x^2 y-z^2 y^3+z^2==1,{x,-5,5},{y,-5,5},{z,-5,5}]

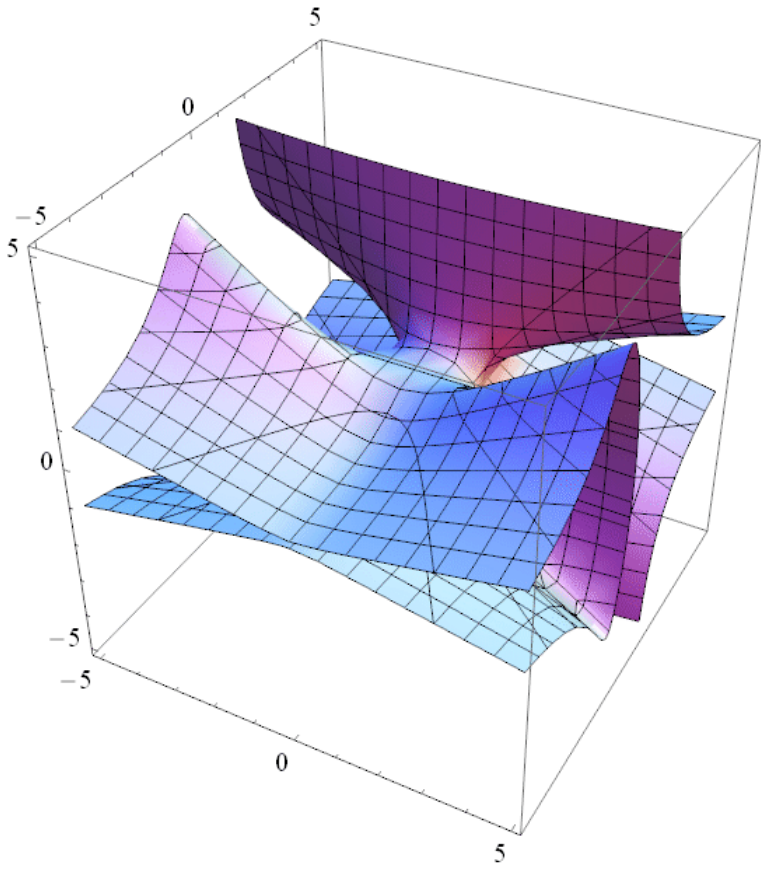


Figure 4.2. The graph of the surface f(x,y,z)= x^2 y-z^2 y^3+ z^2=1.

Example 4.3. (Method 3) Plot the ellipsoid defined in parametric form

ellipsoid[a, b, c](u,v)=(a cos v cos u, b cos v sin u, c sin v),

where $u \in [0, 2\pi]$, $v \in [0, 2\pi]$.

Solution:

Clear[ellipsoid,a,b,c,u,v]

ellipsoid[a_,b_,c_][u_,v_]:={a Cos[v]Cos[u],b Cos[v] Sin[u],c Sin[v]}

ParametricPlot3D[ellipsoid[1,2,1][u,v],{u,0,2π},{v,0,2π}]

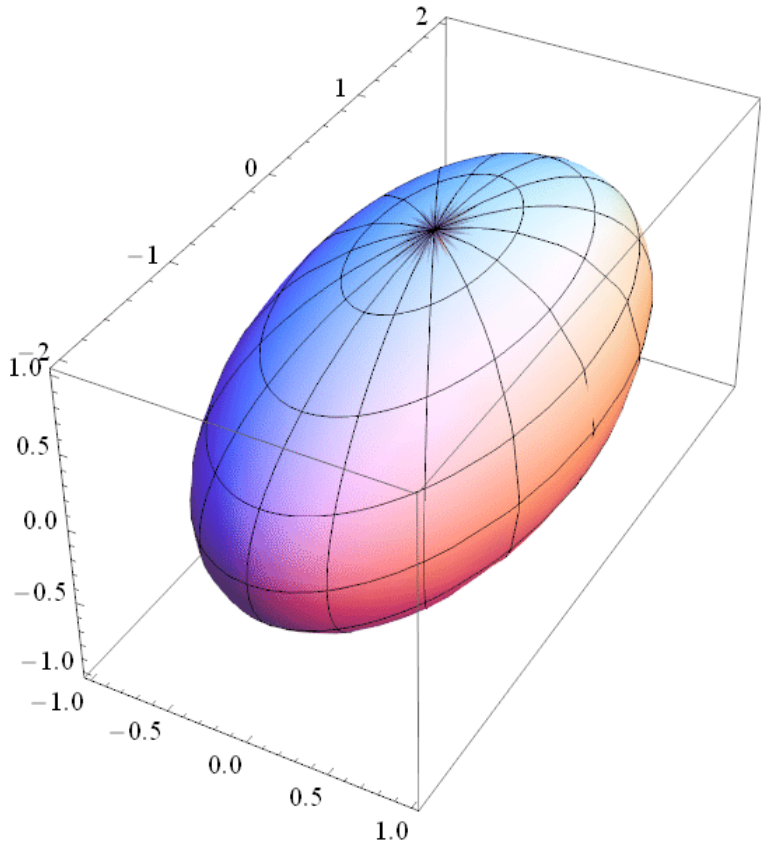


Figure 4.3. The graph of an ellipsoid parametrized by ellipsoid[a,b,c](u,v)= (a cos v cos u, b cos v sin u, c sin v).

4.1 Coordinate Curves, Tangent Planes, and Normal Vectors

Let $\mathbf{X} : D \to \mathbb{R}^3$ be a continuous parametrization of a surface. If we fix $v = v_0$ and let only u varies, we obtain a continuous function $\mathbf{X}(u, v_0)$. Similarly, if we fix $u = u_0$ and let only v varies, the continuous function be $\mathbf{X}(u_0, v)$.

We call $\mathbf{X}(u, v_0)$ and $\mathbf{X}(u_0, v)$, the u-coordinate curve at v_0 and v-coordinate curve at u_0, respectively.

The surface X is regular at a point (u_0, v_0) if its partial derivatives denoted $\mathbf{X}_u(u_0, v_0)$ and $\mathbf{X}_v(u_0, v_0)$ are continuous, nonzero, and noncollinear. Points where X is not regular are called singular points.

Since $\mathbf{X}_u(u_0, v_0)$ and $\mathbf{X}_v(u_0, v_0)$ are noncollinear, that is, they do not lie on the same line, they determine a tangent plane at the regular point (u_0, v_0). We can form a vector

(4.1)

$$\mathbf{N}(u_0, v_0) = \mathbf{X}_u(u_0, v_0) \times \mathbf{X}_v(u_0, v_0)$$

,

which is normal to this tangent plane, and hence to the surface itself at (u_0, v_0).

A regular surface is said to be smooth if $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v$ is nonzero at every point within its definition.

The equation for the plane tangent to this surface at the point (x_0, y_0, z_0) is then

(4.2)

$$(x - x_0, y - y_0, z - z_0) \cdot \mathbf{N} = 0$$

,

where

$$x_0 \equiv x(u_0, v_0),\ y_0 \equiv y(u_0, v_0),\ z_0 \equiv z(u_0, v_0)$$

.

We define modules to compute the normal vector and the equation for the tangent plane to the surface as follows:

```
normal[X_][u_,v_]:=

Module[{U,V},

Cross[D[X[U,V],U],D[X[U,V],V]]/.{U->u,V->v}//Simplify

]


tangentPlane[X_][u0_,v0_]:=

Module[{},

normal[X][u0,v0].({x,y,z}-X[u0,v0])==0//Simplify
```

]

Example 4.4. Let $\mathbf{X}: \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface given by

X(u, v)=(u^2 cos v, u^2 sin v, u), -3<=u<=3, 0<=v<=2π.

(a) Find a normal vector to this surface at (u,v)=(-1,0).

(b) Determine the plane tangent at the point (1,0,-1).

(c) Find an equation for the image of X in the form f(x, y, z)=0.

(d) Plot the tangent plane and the surface at point (1,0,-1).

Solution:

(a) To avoid confusing with the x-coordinate, we use capital X for the surface definition in Mathematica.

```
Clear[X]

X[u_,v_]:={u^2 Cos[v],u^2 Sin[v],u}

X[-1,0]
```

So, (u,v)=(-1,0) corresponds to the point (1,0,-1).

ParametricPlot3D[X[u,v]//Evaluate,{u,-3,3},{v,0,2π}]

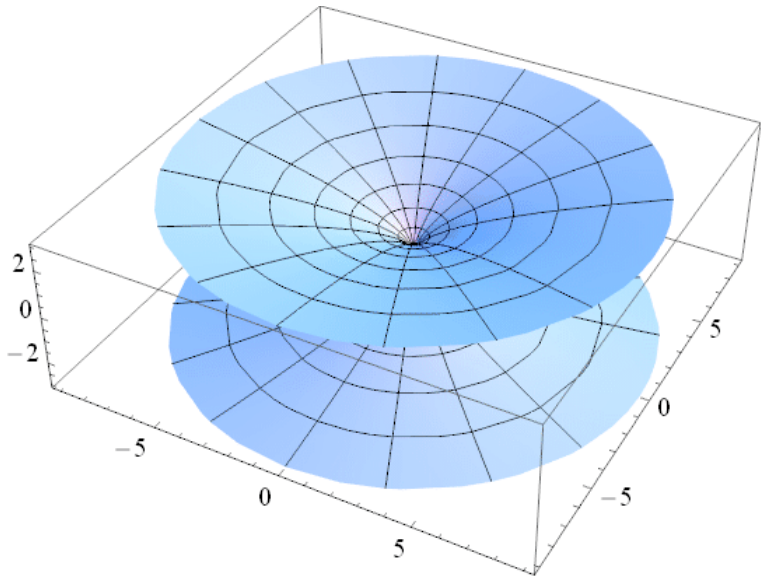


Figure 4.4. The graph of the surface X(u,v)=(u^2 cos v,u^2 sin v,u).

$$\mathbf{X}_u = (2u\cos v,\ 2u\sin v,\ 1),$$
$$\mathbf{X}_v = (-u^2\sin v,\ u^2\cos v,\ 0).$$

.

By equation (4.1) the normal vector is

$$\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v .$$

$$\mathbf{N} = (2u\cos v,\ 2u\sin v,\ 1) \times (-u^2\sin v,\ u^2\cos v,\ 0),$$
$$\mathbf{N} = (-u^2\cos v,\ -u^2\sin v,\ 2u^3).$$

The normal vector at (u, v)=(-1, 0) is

N(-1, 0)=(-1, 0, -2).

normal[X][u,v]

normal[X][-1,0]

(b)

Note that the point (1, 0, -1) corresponds to (u, v)=(-1, 0).

Using equation (4.2) the equation for the plane tangent to this surface at the point (1, 0, -1) is then

$$(x-1, y, z+1) \cdot \mathbf{N}(-1, 0) = 0,$$

$$(x-1, y, z+1) \cdot (-1, 0, -2) = 0,$$

$$x + 2z = -1.$$

tangentPlane[X][-1,0]

(c)

From the parametric equation write

x=u^2 cos v, y=u^2 sin v, z=u.

Hence,

x^2+y^2=u^4.

Thus,

z=u=(x^2+y^2)^(1/4), x^2+y^2=z^4.

So,

f(x,y,z)=x^2+y^2-z^4=0.

Note that this is a level surface of f(x,y,z) at height 0 as is discussed in Chapter 2.

(d)

```
u0=-1;v0=0;

Module[{x0,y0,z0},

{x0,y0,z0}=X[u0,v0];

plot1=ContourPlot3D[tangentPlane[X][u0,v0]//Evaluate,{x,-10,10},
{y,-10,10},{z,-3,3},PlotTheme->"Classic"];

plot2=ParametricPlot3D[X[u,v]//Evaluate,{u,-3,3},{v,0,2π},PlotTheme-
>"Classic"];

plot3=Graphics3D[{Red,PointSize[0.02],Point[X[u0,v0]]}];

Show[plot1,plot2,plot3,PlotRange->All]

]
```

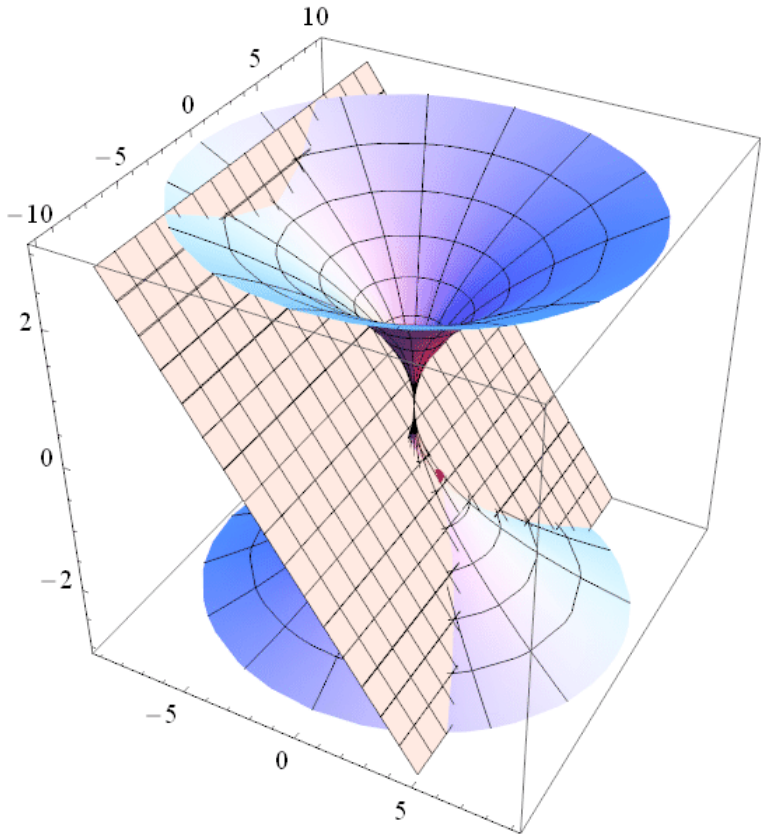


Figure 4.5. A tangent plane at point (1,0,-1) to the surface X(u,v)=(u^2 cos v,u^2 sin v,u).

Example 4.5. The parametric form of a torus is given by

X(u, v)=((3+2 cos v) cos u, (3+2cos v) sin u, 2sin v), 0<=u,v<=2π

(a) Find a normal vector to this surface at (u,v)=(π/3,π/3).

(b) Determine the plane tangent at (u,v)=(π/3,π/3).

(c) Find an equation for the image of X in the form f(x,y,z)=0.

(d) Plot the tangent plane and the surface at (u,v)=(π/3,π/3).

Solution:

(a)

```
Clear[X]

X[u_,v_]:={(3+2Cos[v])Cos[u],(3+2Cos[v])Sin[u],2Sin[v]}

ParametricPlot3D[X[u,v]//Evaluate,{u,0,2π},{v,0,2π}]
```

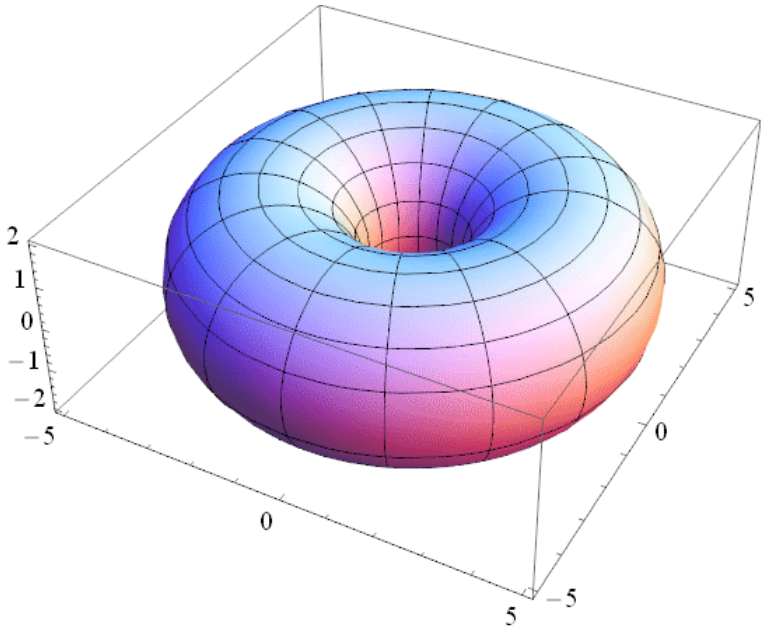


Figure 4.6. The graph of the torus X(u, v)=((3+2 cos v) cos u,(3+2cos v) sin u, 2sin v).

$$\mathbf{X}_u = (-(3 + 2\cos v)\sin u,\ (3 + 2\cos v)\cos u,\ 0),$$
$$\mathbf{X}_v = (-2\sin v\cos u,\ -2\sin v\sin u,\ 2\cos v).$$

The normal vector is given by

$$\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v \ .$$

$$\mathbf{N} = (-(3 + 2\cos v)\sin u,\ (3 + 2\cos v)\cos u,\ 0)$$
$$\times\ (-2\sin v\cos u,\ -2\sin v\sin u,\ 2\cos v),$$

,

$$\mathbf{N} = (2\,(3 + 2\cos v)\cos u\cos v,\ 2\,(3 + 2\cos v)\sin u\cos v,$$
$$2\,(3 + 2\cos v)\sin v).$$

The normal vector at (u,v)=(π/3,π/3) is

N(π/3, π/3)=(2, 2Sqrt[3], 4Sqrt[3]).

normal[X][u,v]

normal[X][π/3,π/3]

(b)

X[π/3,π/3]

Note that (u, v)=(π/3, π/3) corresponds to the point (2, 2Sqrt[3], Sqrt[3]).

The equation for the plane tangent to this surface at the point (2, 2Sqrt[3], Sqrt[3]) is then

$$\left(x - 2,\ y - 2\sqrt{3},\ z - \sqrt{3}\right) \cdot \mathbf{N}\left(\frac{\pi}{3},\ \frac{\pi}{3}\right) = 0$$

,

$$\left(x-2,\, y-2\sqrt{3}\,,\, z-\sqrt{3}\right)\cdot\left(2,\, 2\sqrt{3}\,,\, 4\sqrt{3}\right)=0$$

,

$$x+\sqrt{3}\,y+2\sqrt{3}\,z=14$$.

tangentPlane[X][π/3,π/3]

(c)

Recall X(u, v)=((3+2 cos v) cos u, (3+2cos v) sin u, 2sin v),

From the parametric equation write the components as

x=(3+2 cos v) cos u, y=(3+2cos v) sin u, z=2sin v.

Hence,

x^2+y^2=(3+2 cos v)^2,

Sqrt[x^2+y^2]=3+2 cos v,

4cos^2 v=(3 - Sqrt[x^2+y^2])^2,

4 - z^2=(3 - Sqrt[x^2+y^2])^2.

Thus, there is a relation between x,y, and z given by

z^2=4 - (3 - Sqrt[x^2+y^2])^2

So,

f(x,y,z)=4 - (3 - Sqrt[x^2+y^2])^2 - z^2=0.

(d)

u0=π/3;v0=π/3;

Module[{x0,y0,z0},

{x0,y0,z0}=X[u0,v0];

plot1=ContourPlot3D[tangentPlane[X][u0,v0]//Evaluate,{x,-5,5},{y,-5,5},{z,-3,3},PlotTheme->"Classic"];

plot2=ParametricPlot3D[X[u,v]//Evaluate,{u,0,2π},{v,0,2π},PlotTheme->"Classic"];

plot3=Graphics3D[{Red,PointSize[0.02],Point[X[u0,v0]]}];

Show[plot1,plot2,plot3,BoxRatios->Automatic]

]

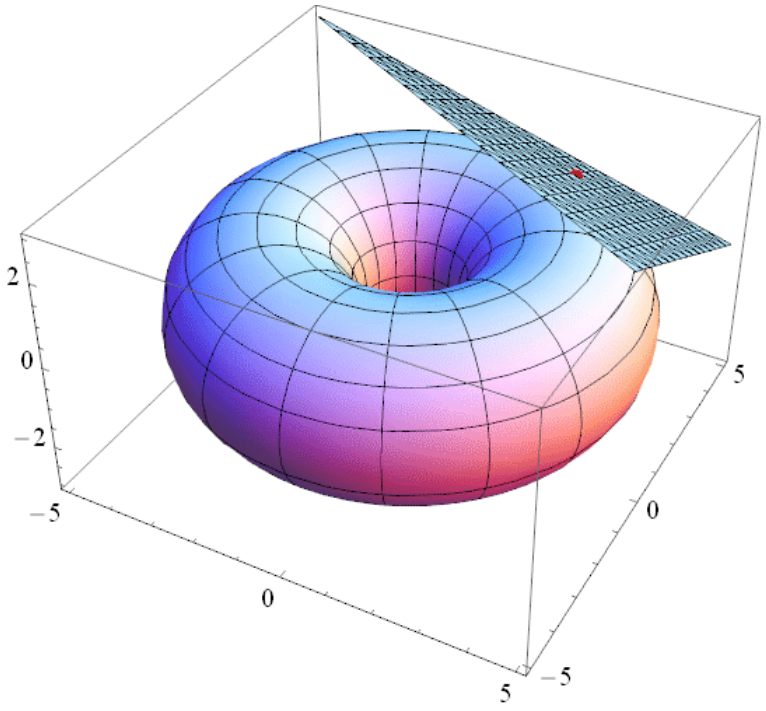


Figure 4.7. A tangent plane at point (2, 2Sqrt[3], Sqrt[3]) to the torus X(u, v)=((3+2 cos v) cos u, (3+2cos v) sin u, 2sin v).

The following figures show the u-coordinate curves and v-coordinate curves, respectively.

(* u-coordinate curves *)

ParametricPlot3D[Table[X[u,v]//Evaluate,{v,0,2π,π/6}],{u,0,2π},PlotStyle->{Blue,Dashed,AbsoluteThickness[2]}]

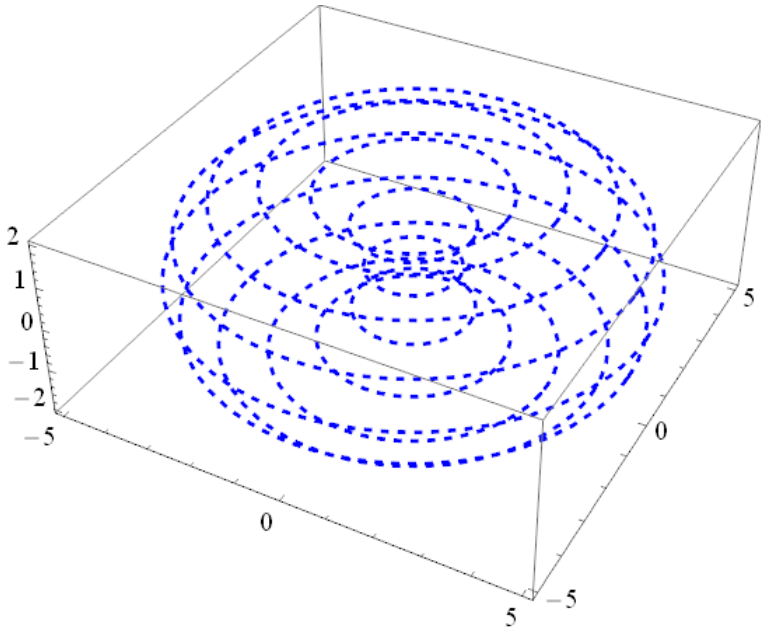


Figure 4.8. Some of the u-coordinate curves of the torus X(u, v)=((3+2 cos v) cos u, (3+2cos v) sin u, 2sin v).

(* v-coordinate curves *)

ParametricPlot3D[Table[X[u,v]//Evaluate,{u,0,2π,π/6}],{v,0,2π},PlotStyle->{Red,Dashed,AbsoluteThickness[2]}]

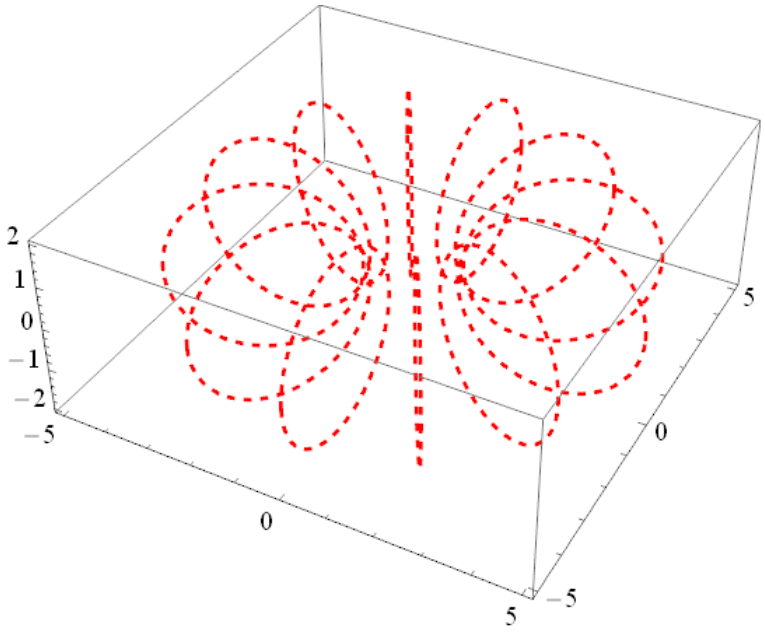


Figure 4.9. Some of the v-coordinate curves of the torus X(u, v)=((3+2 cos v) cos u, (3+2cos v) sin u, 2sin v).

4.2 Surface Integrals

4.2.1 Area of a Parametrized Surface

Definition 4.1

Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where D is an arbitrary region, then the surface area of X is defined by

(4.3)

$$\text{Area} = \iint_D \| \mathbf{X}_u \times \mathbf{X}_v \| \, du \, dv$$

.

Example 4.6. Find the surface area of the torus described in Example 4.5.

Solution:

Recall from Example 4.5 the parametric form of a torus is given by

X(u, v)=((3+2 cos v) cos u, (3+2cos v) sin u, 2sin v), 0<=u,v<=2π.

The normal vector is

$$\mathbf{N}(u, v) = \mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v)$$

,

$$\mathbf{N}(u, v) = (2(3 + 2\cos v)\cos u \cos v,\ 2(3 + 2\cos v)\sin u \cos v,$$
$$2(3 + 2\cos v)\sin v).$$

.

$$\| \mathbf{N}(u, v) \| = \| \mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v) \|$$
$$= 2(3 + 2\cos v).$$

Clear[X]

X[u_,v_]:={(3+2Cos[v])Cos[u],(3+2Cos[v])Sin[u],2Sin[v]}

Sqrt[Cross[D[X[u,v],u],D[X[u,v],v]].Cross[D[X[u,v],u],D[X[u,v],v]]]//Simplify//PowerExpand

By equation (4.3) the surface area is

$$\text{Area} = \int_{u=0}^{2\pi} \int_{v=0}^{2\pi} 2(3 + 2\cos v)\, du\, dv$$
$$= 24\pi^2$$

Integrate[2(3+2 Cos[v]),{u,0,2π},{v,0,2π}]

Proposition 4.2

Let a smooth surface be described by the graph of a differentiable function z=f(x,y), where $x, y \in D$. Then, the surface area of the graph of f(x,y) over D is given by

(4.4)

$$\text{Area} = \int\int_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dx\, dy$$

.

Proof.

Since the surface is described by the graph z=f(x,y), we can parametrized the surface by

X(u, v)=(u, v, f(u, v)),

where u=x, v=y.

This implies that

$$\mathbf{X}_u = \left(1, 0, \frac{\partial f}{\partial u}\right), \quad \mathbf{X}_v = \left(0, 1, \frac{\partial f}{\partial v}\right).$$

$$\mathbf{X}_u \times \mathbf{X}_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right).$$

Hence, by Definition 4.1:

$$\text{Area} = \int\int_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dx\, dy$$

.

Example 4.7. Find the area of a surface described by the graph

z=f(x,y)=1/(4 y^2) (x^2-6 x y+2 x y^2+6 y^3+y^4),

where the region is a rectangle 1/2<=x<=3 and 1/2<=y<=3.

Solution:

Clear[f,x,y]

f[x_,y_]:=1/(4y^2) (x^2-6x y+2x y^2+6y^3+y^4)

Plot3D[f[x,y]//Evaluate,{x,0.5,3},{y,0.5,3}]

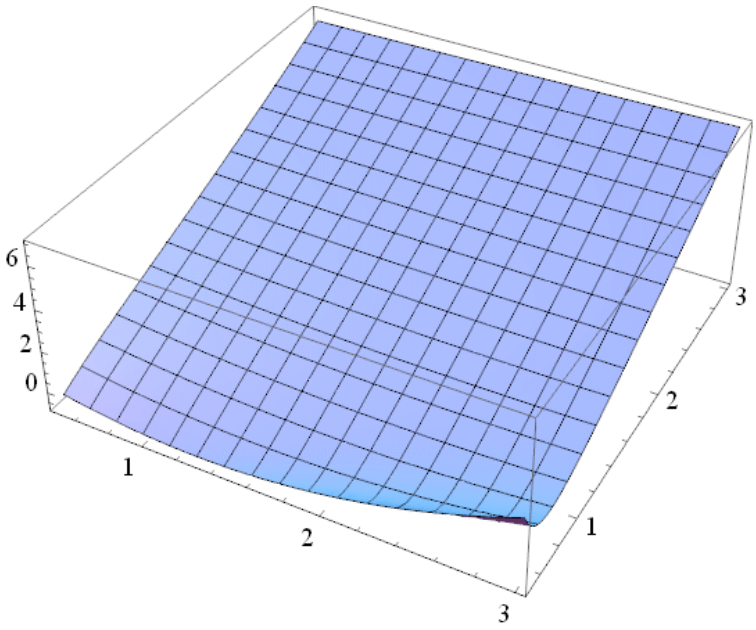


Figure 4.10. The graph of the surface z=1/(4 y^2) (x^2-6 x y+2 x y^2+6 y^3+y^4).

$$\frac{\partial f}{\partial x} = \frac{2x - 6y + 2y^2}{4y^2},$$

$$\frac{\partial f}{\partial y} = \frac{-x^2 + 3xy + 3y^3 + y^4}{2y^3}.$$

fx=D[f[x,y],x]

fy=D[f[x,y],y]//Together

Using equation (4.4), the surface area is

$$\int_{x=0.5}^{3}\int_{y=0.5}^{3}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\,dx\,dy \simeq 19.6428$$

.

NIntegrate[(fx^2+fy^2+1)^(1/2),{x,0.5,3},{y,0.5,3}]

Note that we have used numerical integration to compute the area since it may not have an analytical solution.

4.2.2 Scalar Surface Integrals

Suppose there is a continuous scalar function f along a smooth parametrized surface X whose domain includes the surface; we define the scalar surface integral, denoted $\int\int_X f\,dS$ given by

(4.5)

$$\int\int_X f\,dS = \int\int_D f(\mathbf{X})\,\|\mathbf{X}_u \times \mathbf{X}_v\|\,du\,dv$$

,

where D is a bounded arbitrary region.

Physically, if f represents mass or electrical charge density, then the scalar surface integral yields the total mass or total charge on the surface, respectively.

Example 4.8. Let f(x,y,z)=x^2+y^2+z^2 be the electrical charge density on the surface of a sphere of radius 1 defined by

X(u, v)=(cos u sin v, sin u sin v, cos v), 0<=u<=2π, 0<=v<=π.

Find the total surface charge on the sphere.

Solution:

```
Clear[X,u,v]

X[u_,v_]:={Cos[u]Sin[v],Sin[u]Sin[v],Cos[v]}

ParametricPlot3D[X[u,v],{u,0,2π},{v,0,π}]
```

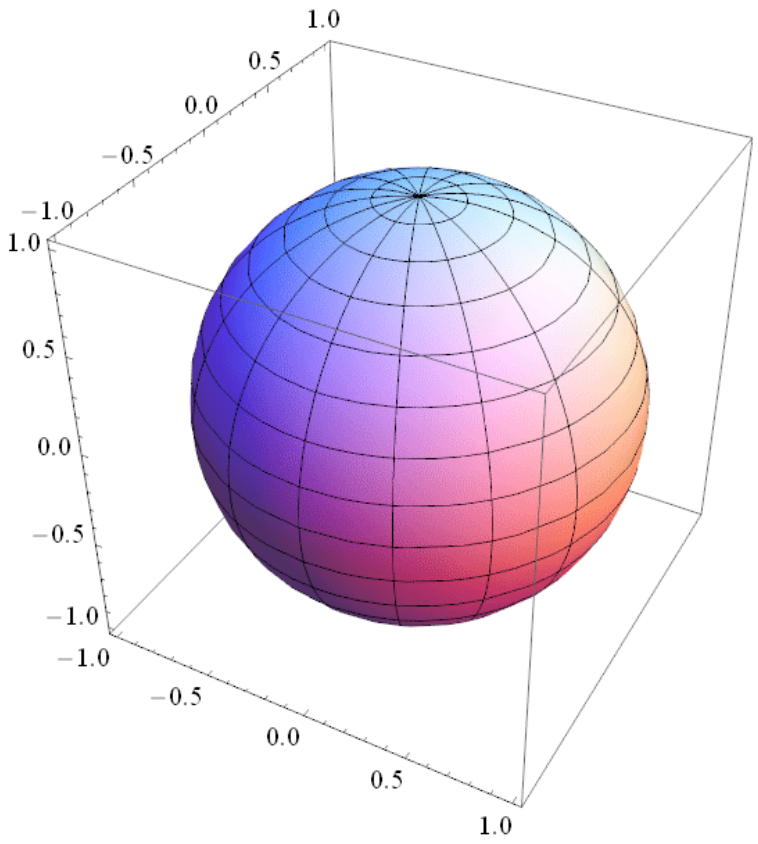


Figure 4.11. The graph of a unit sphere.

Using equation (4.5) the total surface charge can be computed as:

$$\int\int_X f \, dS = \int\int_D f(\mathbf{X}) \, \| \mathbf{X}_u \times \mathbf{X}_v \| \, du \, dv$$

.

$$\mathbf{X}_u = (-\sin u \sin v, \cos u \sin v, 0),$$
$$\mathbf{X}_v = (\cos u \cos v, \sin u \cos v, -\sin v),$$
$$\mathbf{X}_u \times \mathbf{X}_v = \left(-\cos u \sin^2 v, -\sin u \sin^2 v, -\cos v \sin v\right)$$

$$\| \mathbf{X}_u \times \mathbf{X}_v \| = \sin v \, .$$

Since

x=cos u sin v, y=sin u sin v, z=cos v,

we get

f(x,y,z)=(cos u sin v)^2+(sin u sin v)^2+(cos v)^2

=cos^2 u sin^2 v + sin^2 u sin^2 v + cos^2 v

=1

The total surface charge:

$$\int\int_D f(\mathbf{X}) \, \| \mathbf{X}_u \times \mathbf{X}_v \| \, du \, dv = \int_{u=0}^{2\pi} \int_{v=0}^{\pi} \sin v \, dv \, du$$
$$= 4\pi$$

Integrate[Sin[v],{u,0,2π},{v,0,π}]

Example 4.9. Let S be the surface given by

z=ln (5-x^2-y^2), for 0<=x<=1, 0<=y<=1.

(a) Plot the surface.

(b) Calculate the surface area of S.

(c) Calculate $\int\int_S x^2 y^3 \, dS$.

Solution:

(a)

Clear[f,x,y]

f[x_,y_]:=Log[5-x^2-y^2]

Plot3D[f[x,y],{x,0,1},{y,0,1}]

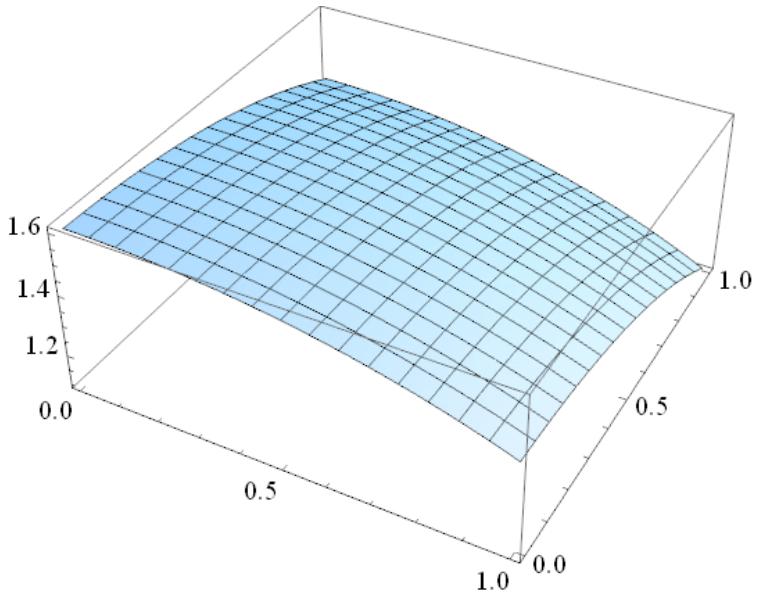


Figure 4.12. The graph of the surface z=ln (5-x^2-y^2).

(b)

$$\frac{\partial f}{\partial x} = -\frac{2x}{5 - x^2 - y^2},$$

$$\frac{\partial f}{\partial y} = -\frac{2y}{5 - x^2 - y^2}.$$

fx=D[f[x,y],x]

fy=D[f[x,y],y]

Using equation (4.4), the surface area is

$$\int_{x=0}^{1}\int_{y=0}^{1}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\,dx\,dy \simeq 1.07805$$

.

NIntegrate[(fx^2+fy^2+1)^(1/2),{x,0,1},{y,0,1}]

(c)

The parametric equation of the surface is given by

X(u, v)=(u, v, ln (5-u^2-v^2)), 0<=u<=1, 0<=v<=1,

where u=x, v=y.

The scalar surface integral can be computed using equation (4.5):

$$\iint_{\mathbf{X}} f\,dS = \iint_{D} f(\mathbf{X})\,\|\mathbf{X}_u \times \mathbf{X}_v\|\,du\,dv$$

.

$$\mathbf{X}_u = \left(1,\ 0,\ -\frac{2u}{5-u^2-v^2}\right),$$
$$\mathbf{X}_v = \left(0,\ 1,\ -\frac{2v}{5-u^2-v^2}\right),$$
$$\mathbf{X}_u \times \mathbf{X}_v = \left(-\frac{2u}{-5+u^2+v^2},\ -\frac{2v}{-5+u^2+v^2},\ 1\right)$$

.

$$\| \mathbf{X}_u \times \mathbf{X}_v \| = \sqrt{\frac{25 + u^4 - 6\,v^2 + v^4 + 2\,u^2\left(-3 + v^2\right)}{\left(-5 + u^2 + v^2\right)^2}}$$

.

Clear[X,u,v]

X[u_,v_]:={u,v,Log[5-u^2-v^2]}

D[X[u,v],u]

D[X[u,v],v]

Cross[D[X[u,v],u],D[X[u,v],v]]

Sqrt[Cross[D[X[u,v],u],D[X[u,v],v]].Cross[D[X[u,v],u],D[X[u,v],v]]]//Together//Simplify

$$\iint_{\mathbf{x}} f \, d\mathbf{l} \, S = \int_{x=0}^{1}\int_{y=0}^{1} u^2\, v^3 \sqrt{\frac{25 + u^4 - 6\,v^2 + v^4 + 2\,u^2\left(-3 + v^2\right)}{\left(-5 + u^2 + v^2\right)^2}}\, d\mathbf{l}\, u \, d\mathbf{l}\, v$$

$$\simeq 0.09814$$

NIntegrate[u^2 v^3*Sqrt[(25+u^4-6 v^2+v^4+2 u^2 (-3+v^2))/(-5+u^2+v^2)^2],{u,0,1},{v,0,1}]

4.2.3 Vector Surface Integrals

Suppose there is a continuous vector function F along a smooth parametrized surface X whose domain includes the surface; we define the vector surface integral, denoted $\int\int_X \mathbf{F} \cdot d\mathbf{S}$ given by

(4.6)

$$\int\int_X \mathbf{F} \cdot d\mathbf{S} = \int\int_D \mathbf{F}(\mathbf{X}) \cdot \mathbf{N}(u, v)\, du\, dv$$

,

where D is a bounded arbitrary region and the normal vector $\mathbf{N}(u, v) = \mathbf{X}_u \times \mathbf{X}_v$ is continuously consistent.

We write a module called vectorSurfIntegral to compute the vector surface integral of F(x,y,z) along the parametrized surface X as given by equation (4.6). Since the module uses Integrate command, we assume Mathematica has analytical solution. We will use this module again in Chapter 8 when we study Stokes's and Gauss's Theorems.

```
(* Compute the vector surface integral of F(x,y,z) along X

  sign_:1 indicates the default sign of normal vector

*)

vectorSurfIntegral[X_,F_,{s0_,s1_},{t0_,t1_},sign_:1]:=

Module[{u,v,F1,F2,F3,normalVector},

normalVector=sign*Cross[D[X[u,v],u],D[X[u,v],v]]/.{u->s,v->t};

{F1,F2,F3}={F[X[u,v][[1]],X[u,v][[2]],X[u,v][[3]]][[1]],F[X[u,v]
[[1]],X[u,v][[2]],X[u,v][[3]]][[2]],F[X[u,v][[1]],X[u,v][[2]],X[u,v][[3]]]
[[3]]}/.{u->s,v->t};

Integrate[Dot[{F1,F2,F3},normalVector],{s,s0,s1},{t,t0,t1}]
```

]

Note that by default the module expects the normal vector of X to be as is calculated. In the case that we need to reverse the direction of the normal vector, then we need to assign -1 to the sign argument in the module above.

A surface which normal vectors are continuously consistent is said to be orientable (or two-sided). Otherwise, the surface is called nonorientable (or one-sided).

Physically, the quantity $\int\int_X \mathbf{F} \cdot d\mathbf{S}$ is known as the flux of F across X(u,v). If F represents the velocity vector field of a fluid, then the flux may be considered as the rate of fluid transported across X per unit time. The normal vector at a point is chosen as outward-pointing vector in a continuously consistent manner.

The following example is an instance of a nonorientable surface.

Example 4.10. The surface parametrized by

$$\begin{cases} x = \left(a + \cos\frac{u}{2}\sin v - \sin\frac{u}{2}\sin 2v\right)\cos u, \\ y = \left(a + \cos\frac{u}{2}\sin v - \sin\frac{u}{2}\sin 2v\right)\sin u, \\ z = \sin\frac{u}{2}\sin v + \cos\frac{u}{2}\sin 2v, \end{cases}$$

,

for 0<=u<=2π, 0<=v<=2π, where a is a positive constant, is known as a Klein bottle.

```
Clear[X,X1,X2,X3,s,t]

X1[a_][u_,v_]:=(a+Cos[u/2]Sin[v]-Sin[u/2]Sin[2v])Cos[u]

X2[a_][u_,v_]:=(a+Cos[u/2]Sin[v]-Sin[u/2]Sin[2v])Sin[u]

X3[a_][u_,v_]:=Sin[u/2]Sin[v]+Cos[u/2]Sin[2v]

X[a_][u_,v_]:={X1[a][u,v],X2[a][u,v],X3[a][u,v]}


(* Plot the Klein bottle for a=1 *)

ParametricPlot3D[X[1][u,v]//Evaluate,{u,0,2π},{v,0,2π}]
```

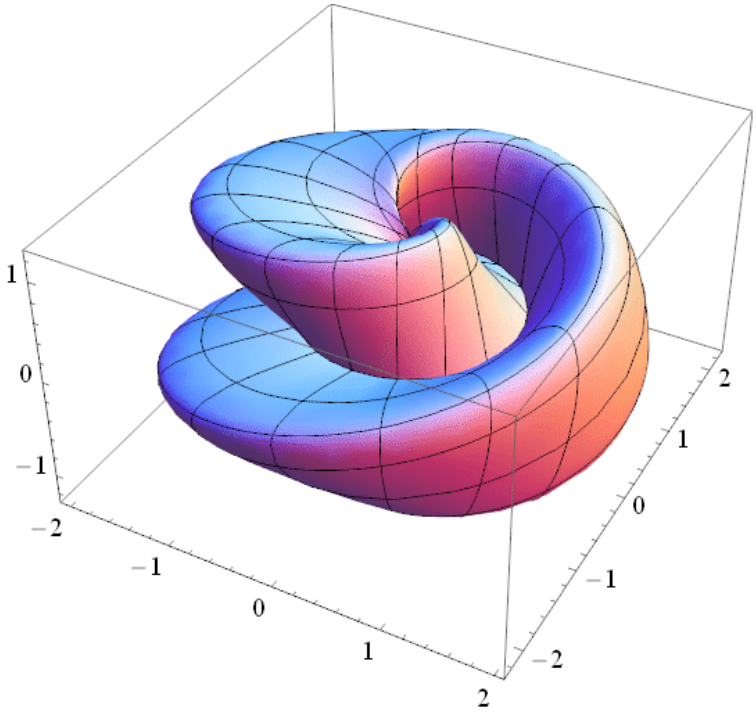


Figure 4.13. The graph of a Klein bottle.

At v=0, the u-coordinate curve is given by

$$\begin{cases} x = a\cos u, \\ y = a\sin u, \\ z = 0, \end{cases}$$

a circle of radius a in the xy-plane.

(* u-coordinate curve at v=0 *)

X[a][u,0]

(* Plot the u-coordinate curve at v=0 *)

ParametricPlot3D[X[2][u,0]//Evaluate,{u,0,2π},{v,0,2π}]

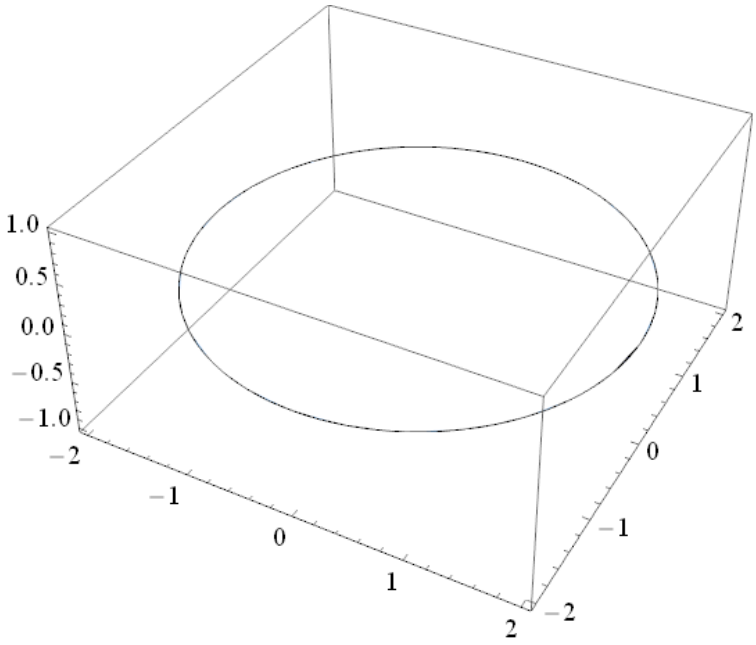


Figure 4.14. The u-coordinate curve at v=0 of a Klein bottle.

Next, we investigate the orientability of the Klein bottle. The standard normal vector along the u-coordinate curve at v=0 is given by

$$\mathbf{N}(u, v) \mid_{v=0} = (\mathbf{X}_u(u, v) \times \mathbf{X}_v(u, v)) \mid_{v=0}$$

.

So,

$$\mathbf{N}(u, 0) = \left(a \cos u \left(2 \cos \frac{u}{2} + \sin \frac{u}{2}\right),\right.$$
$$\left. a \sin u \left(2 \cos \frac{u}{2} + \sin \frac{u}{2}\right), -a \left(\cos \frac{u}{2} - 2 \sin \frac{u}{2}\right)\right)$$

(* Standard normal vector along the u-coordinate curve at v=0 *)

(* N(u,0) *)

Cross[D[X[a][u,v],u],D[X[a][u,v],v]]/.v->0//Simplify

X[a][0,0]==X[a][2π,0]

Mathematica verified that X(0,0)=X(2π,0). It means that if we travel along the Klein bottle via the u-coordinate circular path at v=0, the curve is continuous.

Now, we compare the normal vectors at these two points, that is, N(0,0) and N(2π,0).

(* Compute N(0,0) *)

Cross[D[X[a][u,v],u],D[X[a][u,v],v]]/.{u->0,v->0}

(* Compute N(2π,0) *)

```
Cross[D[X[a][u,v],u],D[X[a][u,v],v]]/.{u->2π,v->0}
```

From the results above, we get

X(0,0)=X(2π,0), but N(0,0)=-N(2π,0).

The sign reversal of the normal vector by traveling along a circular path shows that it is not continuously consistent. This implies that the surface is not orientable at all, it is one-sided. Therefore, the Klein bottle is a nonorientable surface.

Example 4.11. Let S be a funnel-shaped surface defined by

x^2+y^2=1 for 0<=z<=1,

and x^2+y^2=z^2 for 1<=z<=9.

(a) Plot the surface.

(b) Find the outward-pointing unit normal vectors to S.

(c) Find the flux of F through the surface S, where F=-y i+x j+z k.

Solution:

(a)

```
plot1=ContourPlot3D[x^2+y^2==1//Evaluate,{x,-2,2},{y,-2,2},
{z,0,1},Mesh->None,ContourStyle->Opacity[0.5],AxesLabel->
{"x","y","z"}];
```

```
plot2=ContourPlot3D[x^2+y^2==z^2//Evaluate,{x,-2,2},{y,-2,2},
{z,1,9},Mesh->None,ContourStyle->Opacity[0.5]];

Show[plot1,plot2,PlotRange->All]
```

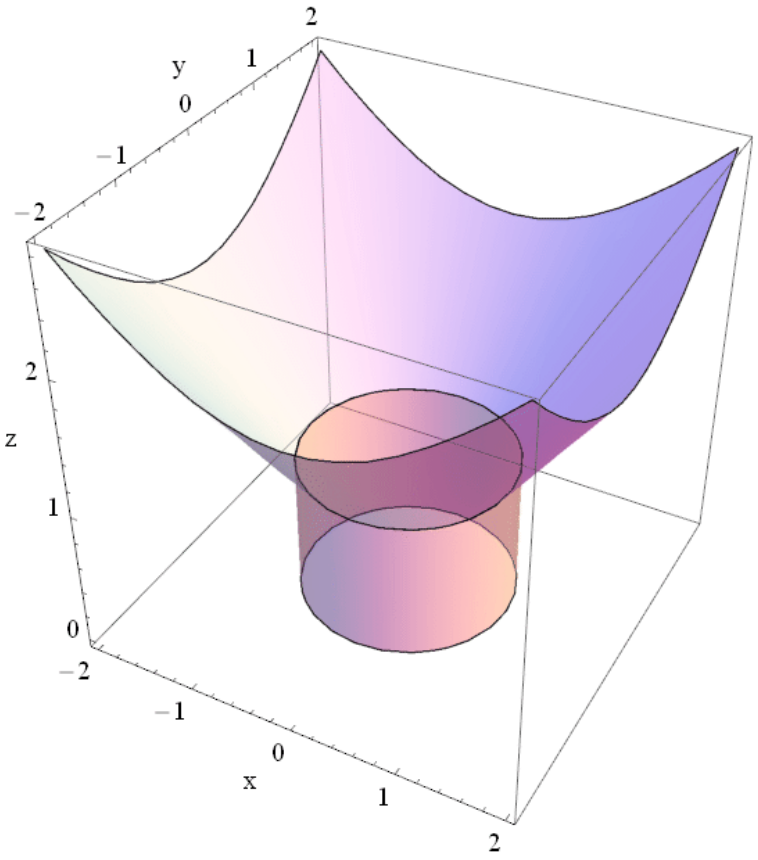


Figure 4.15. The graph of a funnel-shaped surface.

(b)

The funnel-shaped surface S consists of two piecewise smooth surfaces:

1. S_1 : x^2+y^2=1 for 0<=z<=1.

The parametric equation for this part of the surface:

$\mathbf{X}_1(u, v) = (\cos u, \sin u, v)$

, 0<=u<=2π, 0<=v<=1.

The normal vector is given by

$\mathbf{N}_1 = (\mathbf{X}_1)_u \times (\mathbf{X}_1)_v$,

$\mathbf{N}_1 = (\cos u, \sin u, 0)$.

Note that this vector is already pointing outwardly. Thus, the outward-pointing unit normal vector becomes

$\mathbf{u}_1 = (\cos u, \sin u, 0)$.

```
Clear[X1,u,v]

X1[u_,v_]:={Cos[u],Sin[u],v}
```

```
(* The normal vector of X1: N1 *)

Cross[D[X1[u,v],u],D[X1[u,v],v]]//Simplify
```

```
(* The unit normal vector of X1: u1 *)

Cross[D[X1[u,v],u],D[X1[u,v],v]]/Sqrt[Cross[D[X1[u,v],u],D[X1[u,v],v]].Cross[D[X1[u,v],u],D[X1[u,v],v]]]//Simplify
```

2. S_2 : x^2+y^2=z^2 for 1<=z<=9.

The parametric equation for this part of the surface:

$\mathbf{X}_2(u, v) = (u \cos v, u \sin v, u)$

, 1<=u<=9, 0<=v<=2π.

The normal vector is given by

$$\mathbf{N}_2 = (\mathbf{X}_2)_u \times (\mathbf{X}_2)_v ,$$

$$\mathbf{N}_2 = (-u\cos v, -u\sin v, u)$$

.

Since this vector is pointing inwardly, we multiply it with -1 to achieve the outward-pointing normal vector. Thus,

$$-\mathbf{N}_2 = (u\cos v, u\sin v, -u)$$

.

Hence, the outward-pointing unit normal vector is

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}(\cos v, \sin v, -1)$$

.

```
Clear[X2,u,v]

X2[u_,v_]:={u Cos[v],u Sin[v],u}


(* The normal vector of X2: N2 *)

Cross[D[X2[u,v],u],D[X2[u,v],v]]//Simplify


(* The outward-pointing unit normal vector of X2: u2 *)
```

-
(Cross[D[X2[u,v],u],D[X2[u,v],v]]/Sqrt[Cross[D[X2[u,v],u],D[X2[u,v],v]].
Cross[D[X2[u,v],u],D[X2[u,v],v]]])//Simplify//PowerExpand

(c)

Clear[F,x,y,z]

F[x_,y_,z_]:={-y,x,z}

We use VectorPlot3D command to generate a three-dimensional vector plot of F.

plot3=VectorPlot3D[F[x,y,z]//Evaluate,{x,-2,2},{y,-2,2},
{z,0,3},VectorScale->{Small, 0.3, Automatic}]

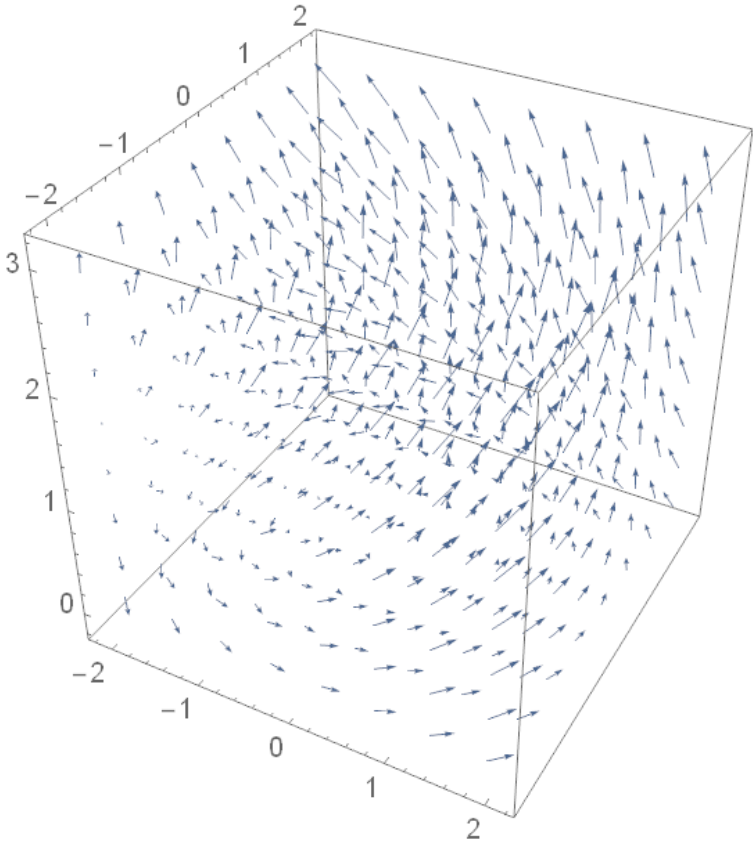


Figure 4.16. A 3D vector plot of the vector field F=-y i+x j+z k.

Show[plot1,plot2,plot3,PlotRange->All]

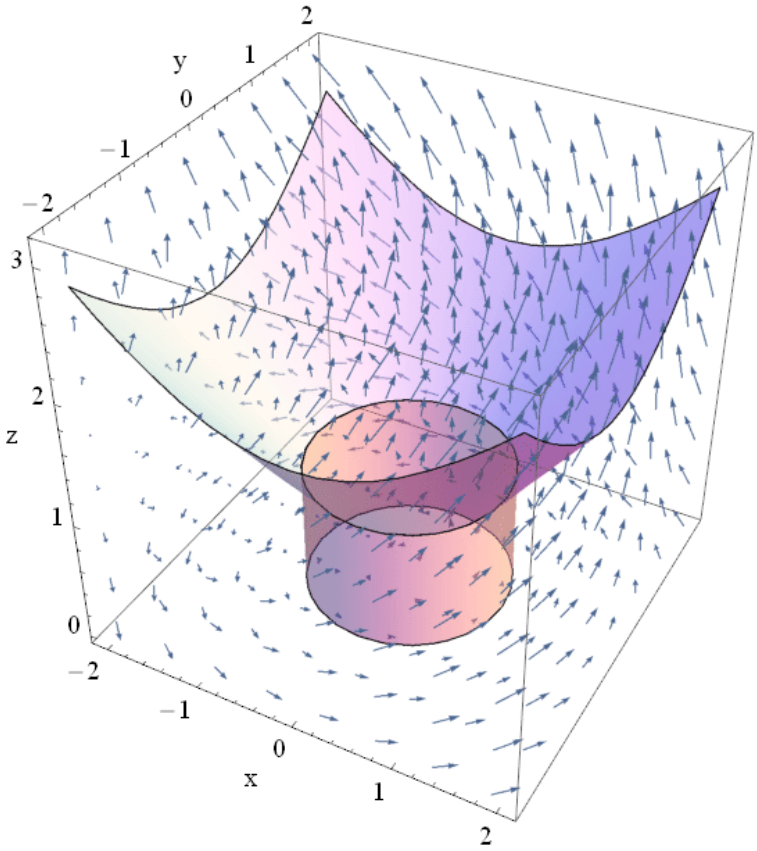


Figure 4.17. Vector plot of the vector field that is coming in and out of the surface.

To find the flux of F through the surface S, we compute the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where F=-y i+x j+z k.

The integral is divided into two parts according to two piecewise smooth surfaces:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_1} \mathbf{F} \cdot d\mathbf{S}_1 + \iint_{\mathbf{X}_2} \mathbf{F} \cdot d\mathbf{S}_1$$

$$= \iint_{D_1} \mathbf{F}(\mathbf{X}_1) \cdot \mathbf{N}_1(u, v)\, du\, dv + \iint_{D_2} \mathbf{F}(\mathbf{X}_2) \cdot (-1)\, \mathbf{N}_2(u, v)\, du\, dv$$

$$= \int_{u=0}^{2\pi} \int_{v=0}^{1} (-\sin u, \cos u, v) \cdot (\cos u, \sin u, 0)\, du\, dv$$

$$+ \int_{u=1}^{9} \int_{v=0}^{2\pi} (-u \sin v, u \cos v, u) \cdot (u \cos v, u \sin v, -u)\, du\, dv$$

$$= -\frac{1456\,\pi}{3}$$

(* Compute the vector surface integral *)

Integrate[{-Sin[u],Cos[u],v}.{Cos[u],Sin[u],0},{u,0,2π},
{v,0,1}]+Integrate[{-u Sin[v],u Cos[v],u}.{u Cos[v],u Sin[v],-u},{u,1,9},
{v,0,2π}]

If F represents the velocity vector field of a fluid, then the flux may be considered as the rate of fluid transported across S per unit time. Since it is negative value, there is more fluid flowing in than flowing out of S.

Let us compute the vector surface integral using the vectorSurfIntegral module and at the same time verifying our result:

vectorSurfIntegral[X1,F,{0,2π},{0,1}]+vectorSurfIntegral[X2,F,{1,9},
{0,2π},-1]

Notice that we have assigned -1 to the sign argument in the vectorSurfIntegral module above for $\mathbf{X}_2$ so that to reverse the normal vector to point away from the surface.

Example 4.12. The electric field due to a point charge located in $\mathbb{R}^3$ is $\mathbf{E} = k\frac{\mathbf{e}_r}{r^2}$, where k is constant and r^2=x^2+y^2+z^2. Find the flux of E through the surface S, a disk of radius 2 parallel to the xy-plane with center at (0,0,3).

Solution:

The parametric equation of the surface S (Figure 4.18) can be given in cylindrical coordinates by

X(u,v)=(u cos v,u sin v,3),

where 0<=u<=2, 0<=v<=2π.

```
Clear[X,u,v]

X[u_,v_]:={u Cos[v],u Sin[v],3}

plot1=ParametricPlot3D[X[u,v]//Evaluate,{u,0,2},{v,0,2π}]
```

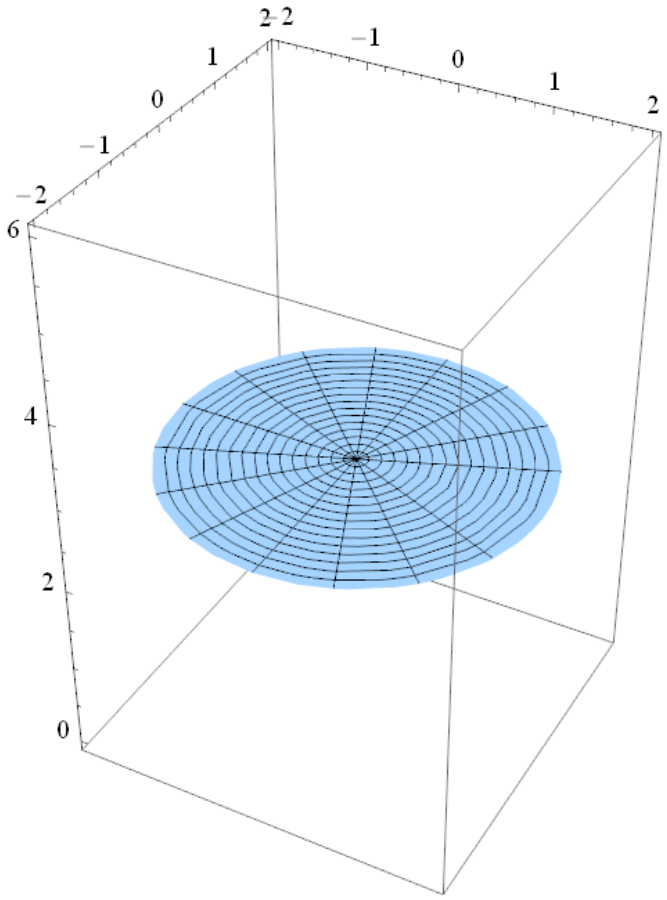


Figure 4.18. The graph of a disk of radius 2 parallel to the xy-plane with center (0,0,3) is parametrized as X(u,v)=(u cos v,u sin v,3), where 0<=u<=2, 0<=v<=2π.

The electric field $\mathbf{E} = k\,\frac{\mathbf{e}_r}{r^2}$ can be written as

$$\mathbf{E} = k\,\frac{\mathbf{e}_r}{r^2} = k\,\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}},$$

where $\mathbf{e}_r = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$.

In cylindrical coordinates and observing that z=3, the electric field becomes

$$\mathbf{E} = k\,\frac{u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + 3\,\mathbf{k}}{\left[(u\cos v)^2 + (u\sin v)^2 + 9\right]^{3/2}} = k\,\frac{u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + 3\,\mathbf{k}}{\left(9 + u^2\right)^{3/2}}$$

.

The electric field can be depicted using VectorPlot3D command (see Figure 4.19).

```
Clear[Efield,k,x,y,z]

Efield[x_,y_,z_]:=k/(x^2+y^2+z^2)^(3/2) {x,y,z}


plot2=VectorPlot3D[Efield[x,y,z]/.k->1//Evaluate,{x,-2,2},{y,-2,2},
{z,0,6},VectorScale->{Small, 0.3, Automatic}]
```

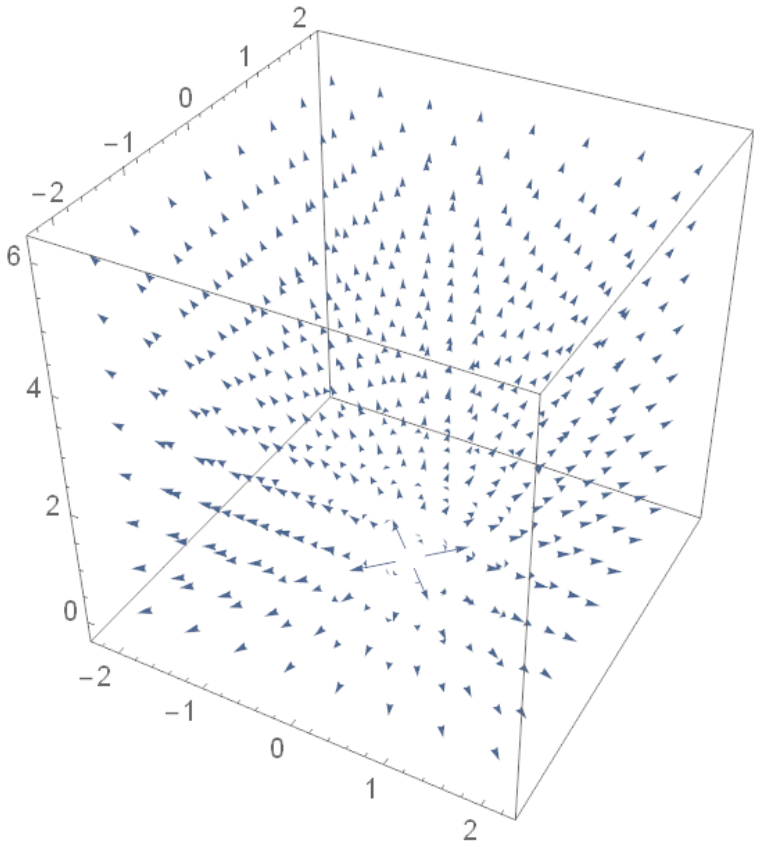


Figure 4.19. A 3D vector plot of the electric field E=(x i+y j+z k)/(x^2 + y^2 + z^2)^(3/2).

Show[plot1,plot2]

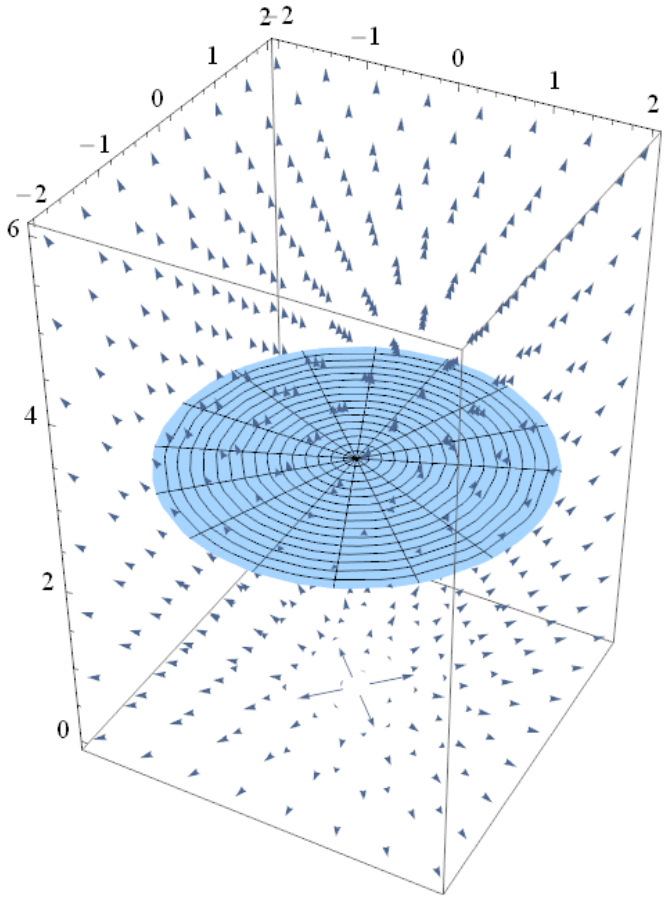


Figure 4.20. The electric field E=(x i+y j+z k)/(x^2 + y^2 + z^2)^(3/2) across the disk of radius 2 centered at (0,0,3).

We find the flux of E that passes through the disk:

X(u,v)=(u cos v,u sin v,3), 0<=u<=2, 0<=v<=2π.

by using equation (4.6):

$$\iint_X \mathbf{E} \cdot d\mathbf{S} = \iint_D \mathbf{E}(\mathbf{X}) \cdot \mathbf{N}(u, v)\, du\, dv$$

,

where the region D is a disk of radius 2 on the xy-plane with center at the origin.

$$\mathbf{X}_u = \partial_u \mathbf{X} = (\cos v,\ \sin v,\ 0)$$

,

$$\mathbf{X}_v = \partial_v \mathbf{X} = (-u \sin v,\ u \cos v,\ 0)$$

.

$$\mathbf{N}(u, v) = \mathbf{X}_u \times \mathbf{X}_v = (0, 0, u)$$

.

Notice that the normal vector is pointing away from the surface, that is, pointing upward in positive z-direction.

Cross[{Cos[v],Sin[v],0},{-u Sin[v],u Cos[v],0}]//Simplify

$$\iint_{\mathbf{X}} \mathbf{E} \cdot d\mathbf{S} = k \int_{u=0}^{2} \int_{v=0}^{2\pi} \frac{(u \cos v,\ u \sin v,\ 3)}{\left(9 + u^2\right)^{3/2}} \cdot (0, 0, u)\, du\, dv$$

$$= 3k \int_{u=0}^{2} \int_{v=0}^{2\pi} \frac{u}{\left(9 + u^2\right)^{3/2}}\, du\, dv$$

$$= \left(2 - \frac{6}{\sqrt{13}}\right) \pi k$$

3 k Integrate[u/(9+u^2)^(3/2),{u,0,2},{v,0,2π}]

Let us compute the flux of E using the vectorSurfIntegral module:

vectorSurfIntegral[X,Efield,{0,2},{0,2π}]

4.3 Curvatures of Surfaces

In this section we will investigate curvatures of surfaces. We may think of the curvatures of a surface as the analogs of the curvature of a curve. However, since surfaces are parametrically two-dimensional, there are various forms of surface curvature as well as increased complexity in dealing with the curvatures of a surface than the curvature of a curve. All of these various forms of curvatures arise from differentiating the normal vector function defined at each point on the surface. Since a surface is parametrically two-dimensional, we can differentiate the normal vector in any direction specified by a tangent vector. This leads us to the concept of shape operator.

Definition 4.3

Let $\mathbf{X}:(u, v)\to\mathbb{R}^3$ be a regular surface, and let $\boldsymbol{\alpha}(t)\equiv\mathbf{X}(\mathbf{r}(t))$ be a curve on the surface having T as tangent vector at point $P_0=\mathbf{X}(u_0, v_0)$. For a surface normal N to S at point P_0 , the shape operator is defined as the minus of the derivative of N in the direction of T:

(4.7)

$$S_{P_0}(\mathbf{T})=-D_{\mathbf{T}}\,\mathbf{N}=-\frac{d}{dt}\,\mathbf{N}(\boldsymbol{\alpha}(t))_{t=0}$$

.

We choose a curve parametrized by t on the surface:

$\boldsymbol{\alpha}(t)\equiv\mathbf{X}(\mathbf{r}(t))$,

such that the initial conditions of the curve satisfies

$$\mathbf{r}(0) = (u_0, v_0) \text{ , } \boldsymbol{\alpha}(0) = \mathbf{X}(\mathbf{r}(0)) = P_0 \text{ , } \boldsymbol{\alpha}'(0) = \mathbf{T} \text{ .}$$

S_{P_0} is an operator follows from the fact that it converts one tangent vector into another tangent vector. This can be shown by taking the derivative of $\mathbf{N} \cdot \mathbf{N}$ at point P_0 .

Since $(\mathbf{N} \cdot \mathbf{N})_{P_0} = \text{constant}$,

$$\frac{d}{dt}(\mathbf{N} \cdot \mathbf{N})_{P_0} = 0 \text{ ,}$$

$$2\,\mathbf{N}(P_0) \cdot \frac{d}{dt}\mathbf{N}(\boldsymbol{\alpha}(t))_{t=0} = 0$$

,

$$\mathbf{N}(P_0) \cdot S_{P_0}(\mathbf{T}) = 0 \text{ .}$$

Since $S_{P_0}(\mathbf{T})$ is perpendicular to $\mathbf{N}(P_0)$, it lies in the tangent plane at P_0 just as T does. Hence, the operator S_{P_0} takes one tangent vector into another tangent vector.

If the derivative of N is in the direction of tangent vector $\mathbf{X}_u$, we obtain the shape operator in this case

$$S(\mathbf{X}_u) = -D_{\mathbf{X}_u}\mathbf{N} = -\frac{\partial}{\partial u}\mathbf{N}(\mathbf{X}(u, v_0))_{P_0} = -\mathbf{N}_u$$

,

where $\mathbf{N}_u$ is the derivative of the normal vector N(u,v) along any u-parameter curve.

Similarly,

$$S(\mathbf{X}_v) = -D_{\mathbf{X}_v}\,\mathbf{N} = -\frac{\partial}{\partial v}\,\mathbf{N}(\mathbf{X}(u_0,\,v))_{P_0} = -\mathbf{N}_v$$

,

where $\mathbf{N}_v$ is the derivative of the normal vector N(u,v) along any v-parameter curve..

Furthermore, the following lemma shows that the shape operator is linear.

Lemma 4.4. At each point P of a regular surface $\mathbf{X} : (u,\,v) \to \mathbb{R}^3$, the shape operator is linear. That is, it satisfies

(4.8)

$$S_P(a\,\mathbf{T}_1 + b\,\mathbf{T}_2) = a\,S_P(\mathbf{T}_1) + b\,S_P(\mathbf{T}_2)$$

, $a,\,b \in \mathbb{R}$.

Proof.

That S_P is linear follows from Definition 4.3.

Since

$$S_{P_0}(\mathbf{T}) = -D_{\mathbf{T}}\,\mathbf{N} = -\frac{d}{dt}\,\mathbf{N}(\boldsymbol{\alpha}(t))_{t=0}$$

.

$$\begin{aligned} S_{P_0}(a\,\mathbf{T}_1 + b\,\mathbf{T}_2) &= -a\,\frac{d}{dt}\,\mathbf{N}_1(\boldsymbol{\alpha}(t))_{t=0} - b\,\frac{d}{dt}\,\mathbf{N}_2(\boldsymbol{\alpha}(t))_{t=0} \\ &= a\,S_{P_0}(\mathbf{T}_1) + b\,S_{P_0}(\mathbf{T}_2), \end{aligned}$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are two tangent vectors that lie on the tangent plane at the point P_0 ; and $\mathbf{N}_1$ and $\mathbf{N}_2$ are their respective normal vectors.

Since the point P_0 is arbitrary, for any point P on the surface we write

$$S_P(a\,\mathbf{T}_1 + b\,\mathbf{T}_2) = a\,S_P(\mathbf{T}_1) + b\,S_P(\mathbf{T}_2)$$

, $a, b \in \mathbb{R}$.

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are linearly independent vectors that span the tangent plane, then from linear algebra we can write both $S_P(\mathbf{T}_1)$ and $S_P(\mathbf{T}_2)$ as a linear combination of $\mathbf{T}_1$ and $\mathbf{T}_2$:

$$S_P(\mathbf{T}_1) = \alpha\,\mathbf{T}_1 + \beta\,\mathbf{T}_2 ,$$

$$S_P(\mathbf{T}_2) = \gamma\,\mathbf{T}_1 + \delta\,\mathbf{T}_2 , \quad \text{where } \alpha, \beta, \gamma, \delta \in \mathbb{R} .$$

Hence, the matrix representation of the shape operator S is given by

$$\mathbf{S} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} .$$

Note here that we have dropped the P sign for convenience, however, it is to be understood that the shape operator is computed at points on the surface.

The following theorem stated without proof, provides means to compute the shape operator in matrix representation. Readers who are interested in the proof may consult books on differential geometry.

Theorem 4.5. Let $\mathbf{X} : (u, v) \to \mathbb{R}^3$ be a regular surface. Then the shape operator S of X with respect to the basis $\{\mathbf{X}_u, \mathbf{X}_v\}$ is given by

$$\mathbf{S} = \begin{pmatrix} \frac{eG - fF}{EG - F^2} & \frac{fG - gF}{EG - F^2} \\ \frac{fE - eF}{EG - F^2} & \frac{gE - fF}{EG - F^2} \end{pmatrix}, \tag{4.9}$$

where

$$\begin{cases} E = \mathbf{X}_u \cdot \mathbf{X}_u, \\ F = \mathbf{X}_u \cdot \mathbf{X}_v = \mathbf{X}_v \cdot \mathbf{X}_u \\ G = \mathbf{X}_v \cdot \mathbf{X}_v. \end{cases}$$

$$\begin{cases} e = -\mathbf{U}_u \cdot \mathbf{X}_u = \mathbf{U} \cdot \mathbf{X}_{uu}, \\ f = -\mathbf{U}_v \cdot \mathbf{X}_u = \mathbf{U} \cdot \mathbf{X}_{uv} = \mathbf{U} \cdot \mathbf{X}_{vu} = -\mathbf{U}_u \cdot \mathbf{X}_v, \\ g = -\mathbf{U}_v \cdot \mathbf{X}_v = \mathbf{U} \cdot \mathbf{X}_{vv}, \end{cases}$$

and U is the unit normal vector to X.

E, F, and G are called the coefficients of the first fundamental form of X, whereas e, f, and g are the coefficients of the second fundamental form of X.

From Theorem 4.5 we notice that

$$f = -\mathbf{U}_v \cdot \mathbf{X}_u = -\mathbf{U}_u \cdot \mathbf{X}_v .$$

At any point P on the surface, this can also be written as

$$-\mathbf{N}_v \cdot \mathbf{X}_u = -\mathbf{N}_u \cdot \mathbf{X}_v .$$

By Definition 4.3, this implies that

$$S(\mathbf{X}_v) \cdot \mathbf{X}_u = S(\mathbf{X}_u) \cdot \mathbf{X}_v , \tag{4.10}$$

for any two tangent vectors $\mathbf{X}_u$ and $\mathbf{X}_v$ at any point P on the surface.

By Lemma 4.4, $\mathbf{X}_u$ and $\mathbf{X}_v$ span the tangent plane and is linear, so we can always choose two tangent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that they are orthonormal to each other to form a new basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Thus, the matrix representation of S with respect to this choice of basis becomes

$$S(\mathbf{v}_1) = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 ,$$

$$S(\mathbf{v}_2) = \gamma \mathbf{v}_1 + \delta \mathbf{v}_2 , \quad \text{where } \alpha, \beta, \gamma, \delta \in \mathbb{R} .$$

It follows that

$$S(\mathbf{v}_1) \cdot \mathbf{v}_2 = \beta \quad \text{and} \quad S(\mathbf{v}_2) \cdot \mathbf{v}_1 = \gamma .$$

But by equation (4.10):

$$S(\mathbf{v}_2) \cdot \mathbf{v}_1 = S(\mathbf{v}_1) \cdot \mathbf{v}_2 .$$

Therefore, $\beta = \gamma$.

This implies that S with respect to this new basis is represented by a symmetric matrix:

$$\mathbf{S} = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix} .$$

Using the fact from linear algebra that any symmetric matrix is always diagonalizable, thus, given a symmetric matrix

$$\mathbf{S} = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix} ,$$

we can choose an orthogonal matrix, such as a rotation by an angle θ

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

so that

$$\mathbf{S}' \equiv \mathbf{ASA}^{-1} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

In the new basis $\{\mathbf{T}_1, \mathbf{T}_2\}$, the shape operator is a diagonal matrix. Thus

$$S(\mathbf{T}_1) = k_1 \mathbf{T}_1, \qquad S(\mathbf{T}_2) = k_2 \mathbf{T}_2.$$

k_1, and k_2 are called the eigenvalues of S, and $\mathbf{T}_1$ and $\mathbf{T}_2$ are the corresponding eigenvectors.

k_1, and k_2 are also called the principal curvatures of X at point P.

$\mathbf{T}_1$ and $\mathbf{T}_2$ are perpendicular and the directions are called the principal vectors. The corresponding directions are called principal directions. A principal curve on X is a curve whose trace lies on X and whose tangent vector is everywhere principal.

Using Eigenvalues command from Mathematica, it can be shown that for a symmetric matrix:

$$\mathbf{S} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

the eigenvalues of S are given by

$$k_1,\, k_2 = \frac{1}{2}\left[\alpha + \gamma \pm \sqrt{(\alpha-\gamma)^2 + 4\,\beta^2}\right]$$

.

```
Clear[S,α,β,γ]

S={{α,β},{β,γ}};

Eigenvalues[S]
```

Definition 4.6. Let T be a unit tangent vector to a regular surface $\mathbf{X}:(u,\,v)\to\mathbb{R}^3$ at point P. Then the normal curvature of X in the direction T is given by

(4.11) $$k(\mathbf{T}) = S(\mathbf{T})\cdot\mathbf{T}\ .$$

More generally, if v is any nonzero tangent vector to X at P, we have

(4.12) $$k(\mathbf{v}) = \frac{S(\mathbf{v})\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\ .$$

We call k(v) the normal curvature of X in the direction v.

Consider again the orthonormal basis $\{\mathbf{T}_1,\,\mathbf{T}_2\}$ that spans the tangent space at point P. Any unit tangent vector T can be expressed in terms of $\mathbf{T}_1$ and $\mathbf{T}_2$ as

$$\mathbf{T} = a\,\mathbf{T}_1 + b\,\mathbf{T}_2\ .$$

Since T is unit tangent vector

$$|| \mathbf{T} ||^2 = 1 = a^2 + b^2 .$$

Then by Definition 4.6 the normal curvature of X in the direction T is given by

$$k(\mathbf{T}) = S(\mathbf{T}) \cdot \mathbf{T} .$$

$$\begin{aligned} k(\mathbf{T}) &= S(a\,\mathbf{T}_1 + b\,\mathbf{T}_2) \cdot (a\,\mathbf{T}_1 + b\,\mathbf{T}_2) \\ &= [a\,S(\mathbf{T}_1) + b\,S(\mathbf{T}_2)] \cdot (a\,\mathbf{T}_1 + b\,\mathbf{T}_2) \\ &= (a\,k_1\,\mathbf{T}_1 + b\,k_2\,\mathbf{T}_2) \cdot (a\,\mathbf{T}_1 + b\,\mathbf{T}_2) \end{aligned}$$

Therefore, $k(\mathbf{T}) = a^2\,k_1 + b^2\,k_2$.

Since the normal curvature of X in any direction T depends on the values of k_1 and k_2 , this implies if $k_1 \geq k_2$, the normal curvature is largest in the principal direction $\mathbf{T}_1$ and conversely if $k_1 \leq k_2$, the normal curvature is smallest in the principal direction $\mathbf{T}_1$. So, the maximum and minimum values of the normal curvature k of X at P are called the principal curvatures of X at P.

Recall from Section 2.1.2, the curvature of a unit-speed curve is

$$\kappa(s) \equiv || \mathbf{T}'(s) || ,$$

where T is the unit tangent vector field on the space curve r.

It turns out that there is a relation between the normal curvature of a regular surface with the curvature of a space curve which is stated in the following lemma:

Lemma 4.7. Let $\mathbf{X}:(u, v)\to\mathbb{R}^3$ be a regular surface and let v be a unit tangent vector to X at P. Let r be a unit-speed curve on X with r(0)=P and r'(0)=v. Then

(4.13) $$k(\mathbf{v}) = \kappa(0)\cos\theta\ ,$$

where κ(0) denotes the curvature of the curve r at 0, and θ is the angle between the normal vector N(0) of r and the surface normal vector U at P.

Proof.

By Definition 4.6 the normal curvature of X in the direction v is given by

$$k(\mathbf{v}) = S(\mathbf{v})\cdot\mathbf{v}\ .$$

Since r is a curve on X, then its tangent vector v is perpendicular to U, that is

$$\mathbf{v}\cdot\mathbf{U} = 0\ .$$

Differentiating, we get

$$\mathbf{v}'\cdot\mathbf{U} + \mathbf{v}\cdot\mathbf{U}' = 0\ .$$

By Definition 4.3:

$$\mathbf{U}' = -S(\mathbf{v})\ ,$$

so $$\mathbf{v}'\cdot\mathbf{U} = \mathbf{v}\cdot S(\mathbf{v})\ .$$

Thus,

$$k(\mathbf{v}) = \mathbf{v}'\cdot\mathbf{U}\ .$$

By Theorem 2.3, equation (2.14) we get

$$k(\mathbf{v}) = \kappa(0)\,\mathbf{N}(0) \cdot \mathbf{U} .$$

Hence,

$$k(\mathbf{v}) = \kappa(0)\cos\theta ,$$

where θ is the angle between the normal vector N(0) of r and the surface normal vector U at P.

Besides normal curvatures and principal curvatures, there are two more definitions of the curvatures of the surface. These are defined in Definition 4.8.

Definition 4.8. Let $\mathbf{X} : (u, v) \to \mathbb{R}^3$ be a regular surface. The Gaussian curvature K and the mean curvature H of X at point P are defined by

(4.14) $$K(P) = \det(S(P)) , \quad H(P) = \frac{1}{2}\operatorname{tr}(S(P)) .$$

Definition 4.8 implies that the Gaussian curvature K is the product of the principal curvatures k_1 and k_2 , and the mean curvature H is their average.

We say that X is a minimal surface if H vanishes and flat if K vanishes.

Theorem 4.9. Let $\mathbf{X} : (u, v) \to \mathbb{R}^3$ be a regular surface. The Gaussian curvature K and mean curvature H of X are given by the formulas

(4.15) $$K = \frac{eg - f^2}{EG - F^2} ,$$

(4.16) $$H = \frac{e\,G - 2\,f\,F + g\,E}{2\left(E\,G - F^2\right)}$$,

where e, f, g are the coefficients of the second fundamental form relative to X, and E, F, G are the coefficients of the first fundamental form.

Proof.

From equation (4.9), we have

$$\mathbf{S} = \begin{pmatrix} \frac{e\,G - f\,F}{E\,G - F^2} & \frac{f\,G - g\,F}{E\,G - F^2} \\ \frac{f\,E - e\,F}{E\,G - F^2} & \frac{g\,E - f\,F}{E\,G - F^2} \end{pmatrix}.$$

Using equation (4.14), we obtain

$$K = \frac{1}{\left(E\,G - F^2\right)^2}[(e\,G - f\,F)(g\,E - f\,F) - (f\,G - g\,F)(f\,E - e\,F)]$$

,

$$K = \frac{e\,g - f^2}{E\,G - F^2}.$$

and

$$H = \frac{1}{2\left(E\,G - F^2\right)}[e\,G - f\,F + g\,E - f\,F]$$

,

$$H = \frac{e\,G - 2\,f\,F + g\,E}{2\left(E\,G - F^2\right)}.$$

Of all definition of curvatures, the Gaussian curvature is the most important notion about the curvature of a surface. It also leads us to distinguish four

kinds of points on a surface.

Definition 4.10. Let $\mathbf{X}:(u, v) \rightarrow \mathbb{R}^3$ be a regular surface. Let K be the Gaussian curvature of X.

- Points are called elliptic if K>0 (equivalently, k_1 and k_2 have the same sign)

- Points are called hyperbolic if K<0 (equivalently, k_1 and k_2 have opposite signs)

- Points are called parabolic if K=0, but $S \neq 0$ (equivalently, exactly one of k_1 and k_2 is zero)

- Points are called planar if K=0 and S=0 (equivalently, $k_1 = k_2 = 0$)

Before we use Mathematica to calculate the shape operator of a surface. We shall need to define several modules in the following:

```
(* Define length of a vector *)

vectorLength[v_]:=Sqrt[Simplify[v.v]]


(* Define unit normal vector of a surface *)

unitnormal[X_][u_,v_]:=

Module[{U,V,Xu,Xv},

Xu=D[X[U,V],U];
```

```
Xv=D[X[U,V],V];

Nx=Cross[Xu,Xv]/.{U->u,V->v};

Simplify[Nx/vectorLength[Nx]]

]
```

The firstForm module compute the coefficients of the first fundamental form and arrange them in an array. Notice that we use the symbols ee, ff, and gg to denote E, F, and G, respectively. It is meant to avoid using the capital E which is a reserved symbol in Mathematica.

```
(* Compute the coefficients of the first fundamental form *)

firstForm[X_][u_,v_]:=

Module[{Xu,Xv,U,V},

Xu=D[X[U,V],U];

Xv=D[X[U,V],V];

ee=Xu.Xu/.{U->u,V->v}//Simplify;

ff=Xu.Xv/.{U->u,V->v}//Simplify;

gg=Xv.Xv/.{U->u,V->v}//Simplify;

{{ee,ff},{ff,gg}}

]
```

In secondForm module we compute the coefficients of the second fundamental form e, f, and g, and arrange the results in an array. In the module we use the symbols eee, fff, and ggg to denote e, f, and g, respectively.

```
(* Compute the coefficients of the second fundamental form *)

secondForm[X_][u_,v_]:=

Module[{Xuu,Xuv,Xvv,U,V},

Nx=unitnormal[X][u,v];

Xuu=D[X[U,V],U,U];

Xuv=D[X[U,V],U,V];

Xvv=D[X[U,V],V,V];

eee=Xuu.Nx/.{U->u,V->v}//Simplify;

fff=Xuv.Nx/.{U->u,V->v}//Simplify;

ggg=Xvv.Nx/.{U->u,V->v}//Simplify;

{{eee,fff}, {fff,ggg}}

]
```

Using Theorem 4.5 we define the shapeOperator module to compute the shape operator of the surface in matrix form:

```
(* Compute the shape operator of the surface *)
```

```
shapeOperator[X_][u_,v_]:=

Module[{},

Clear[ee,ff,gg,eee,fff,ggg];

secondForm[X][u,v];

(1/Det[firstForm[X][u,v]])*{{eee*gg-fff*ff,fff*gg-ggg*ff},{fff*ee-
eee*ff,ggg*ee-fff*ff}}//Simplify

]
```

Equation (4.15) and (4.16) in Theorem 4.9 are used to compute the Gaussian curvature and the mean curvature of a surface, respectively. The Gaussian curvature can be written as the division of the determinant of the first fundamental form matrix over the second fundamental form matrix which is implemented in gaussCurv module below:

```
(* Compute the Gaussian curvature *)

gaussCurv[X_][u_,v_]:=

Module[{f1,f2},

f1:=firstForm[X][u,v];

f2:= secondForm[X][u,v];

Simplify[Det[f2]/Det[f1]]

]
```

```
(* Compute the mean curvature *)

meanCurv[X_][u_,v_]:=

Module[{f1},

Clear[ee,ff,gg,eee,fff,ggg];

f1:=firstForm[X][u,v];

secondForm[X][u,v];

Simplify[(eee gg - 2 fff ff + ggg ee)/(2Det[f1])]

]
```

Using the fact that the principal curvatures are the eigenvalues of the shape operator, we define a function principalCurvs to compute the principal curvatures of a surface:

```
(* Compute the principle curvatures: Subscript[k, 1] and Subscript[k, 2] *)

principalCurvs[X_][u_,v_]:=Eigenvalues[shapeOperator[X][u,v]]//Simplify
```

Here are some examples:

Example 4.13. Consider a saddle surface defined by the function z=x y.

(a) Plot the surface.

(b) Find the shape operator.

(c) Compute the principal curvatures.

(d) Compute the Gaussian curvature and the mean curvature.

Solution:

(a)

The saddle surface can be parametrized as

X(u,v)=(u,v,u v),

where u=x, v=y.

```
Clear[X]

X[u_,v_]:={u,v,u v}

ParametricPlot3D[X[u,v]//Evaluate,{u,-2,2},{v,-2,2},AxesLabel->
{"x","y","z"},BoxRatios->{1,1,1}]
```

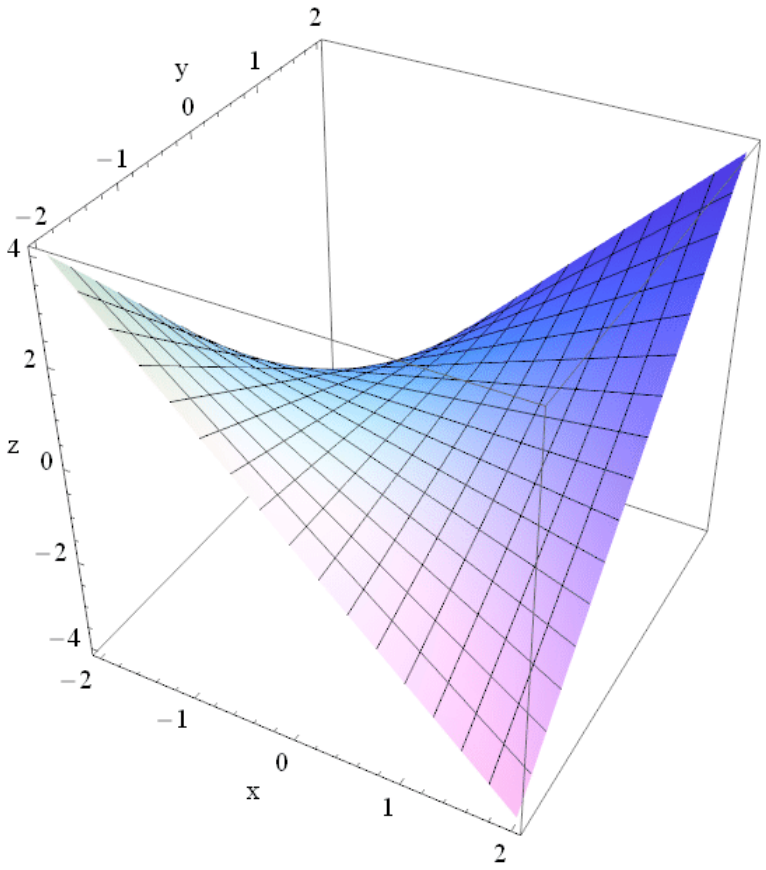


Figure 4.21. The graph of the saddle surface z=x y.

(b)

shapeOperator[X][u,v]//MatrixForm

The shape operator:

$$\mathbf{S} = \begin{pmatrix} -\frac{u\,v}{\left(1+u^2+v^2\right)^{3/2}} & \frac{1+u^2}{\left(1+u^2+v^2\right)^{3/2}} \\ \frac{1+v^2}{\left(1+u^2+v^2\right)^{3/2}} & -\frac{u\,v}{\left(1+u^2+v^2\right)^{3/2}} \end{pmatrix}$$

Note that the shape operator matrix is not symmetric. Because in general $\mathbf{X}_u$ is not perpendicular to $\mathbf{X}_v$.

(c)

principalCurvs[X][u,v]

The principal curvatures:

$$k_1 = \frac{-u\,v - \sqrt{\left(1+u^2\right)\left(1+v^2\right)}}{\left(1+u^2+v^2\right)^{3/2}}, \quad k_2 = \frac{-u\,v + \sqrt{\left(1+u^2\right)\left(1+v^2\right)}}{\left(1+u^2+v^2\right)^{3/2}}.$$

(d)

gaussCurv[X][u,v]

meanCurv[X][u,v]

The mean curvature:

$$H = -\frac{u\,v}{\left(1+u^2+v^2\right)^{3/2}}.$$

As can be seen from the formula, the Gaussian curvature of the saddle is always negative:

$$K = -\frac{1}{\left(1+u^2+v^2\right)^2} < 0.$$

Therefore, points on the surface are hyperbolic by Definition 4.10.

By transforming u->-u and v->-v, the Gaussian curvature is unchanged. So, it has reflection symmetry with respect to the origin. By transforming to polar coordinates, it is also invariant with respect to rotations about the z-axis.

Example 4.14. The parametric form of a torus is given by

torus[a,b,c](u,v)=((a+b cos v)cos u,(a+b cos v)sin u,c sin v),

where $u, v \in [0, 2\pi)$.

(a) Plot the surface.

(b) Find the shape operator.

(c) Compute the principal curvatures.

(d) Compute the Gaussian curvature and the mean curvature.

(e) Show parts of the torus where the Gaussian curvature is negative, zero, and positive.

Solution:

(a)

```
Clear[torus,a,b,c]

torus[a_,b_,c_][u_,v_]:={(a+b Cos[v])Cos[u],(a+b Cos[v])Sin[u],c Sin[v]}
```

We plot a torus with a=2, b=1, c=1:

```
ParametricPlot3D[torus[2,1,1][u,v]//Evaluate,{u,0,2π},{v,0,2π}]
```

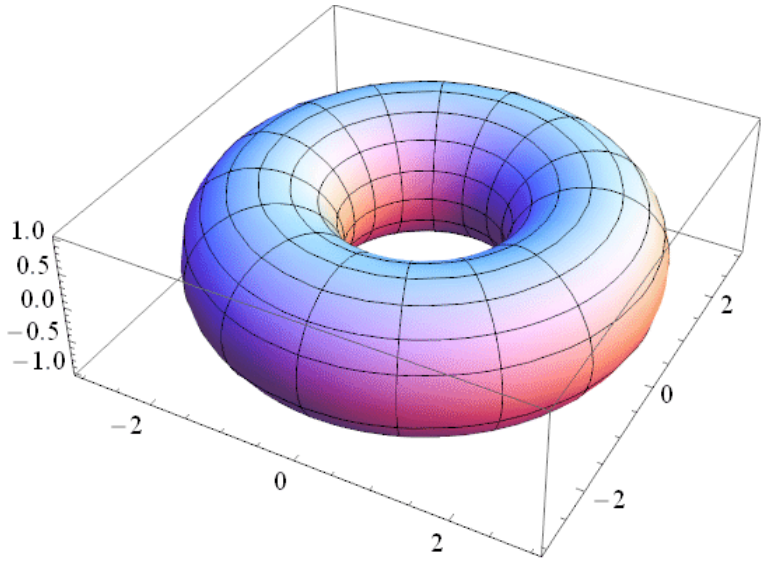


Figure 4.22. The graph of a torus[2,1,1].

(b)

shapeOperator[torus[a,b,c]][u,v]//MatrixForm

The shape operator of the torus:

$$\mathbf{S} = \begin{pmatrix} -\dfrac{\sqrt{2}\, c \cos v}{\sqrt{(a+b\cos v)^2\left[b^2+c^2+\left(-b^2+c^2\right)\cos 2v\right]}} & 0 \\ 0 & -\dfrac{2\sqrt{2}\, b c\,(a+b\cos v)^3}{\left[-(a+b\cos v)^2\left\{-b^2-c^2+\left(b^2-c^2\right)\cos 2v\right\}\right]^{3/2}} \end{pmatrix}$$

(c)

principalCurvs[torus[a,b,c]][u,v]

The principal curvatures can be read off from the diagonalized shape operator:

$$k_1 = -\frac{2\sqrt{2}\, b c\,(a+b\cos v)^3}{\left[(a+b\cos v)^2\left(b^2+c^2+\left(-b^2+c^2\right)\cos 2v\right)\right]^{3/2}}$$

,

$$k_2 = -\frac{\sqrt{2}\, c \cos v}{\sqrt{(a+b\cos v)^2\left(b^2+c^2+\left(-b^2+c^2\right)\cos 2v\right)}}$$

.

(d)

gaussCurv[torus[a,b,c]][u,v]

For specific values of a, b, and c; the sign of the Gaussian curvature is dependent on the v-parameter only:

$$K = \frac{4\,b\,c^2\cos v}{(a + b\cos v)\left[b^2 + c^2 + \left(-b^2 + c^2\right)\cos 2v\right]^2}$$

.

meanCurv[torus[a,b,c]][u,v]

The mean curvature of the torus:

$$H = -\left(\left(c\,(a + b\cos v)^2\left[4\,a\,b + \left(5\,b^2 + 3\,c^2\right)\cos v + \left(-b^2 + c^2\right)\cos 3v\right]\right)\Big/ \left(2\sqrt{2}\left[(a + b\cos v)^2\left(b^2 + c^2 + \left(-b^2 + c^2\right)\cos 2v\right)\right]^{3/2}\right)\right)$$

(e)

By fixing the v-parameter, the following graph shows several u-coordinate curves on the torus[2,1,1]:

ParametricPlot3D[Table[torus[2,1,1][u,j],{j,0,2π,1/2}]//Evaluate, {u,0,2π},PlotStyle->{{Blue,AbsoluteThickness[2]}}]

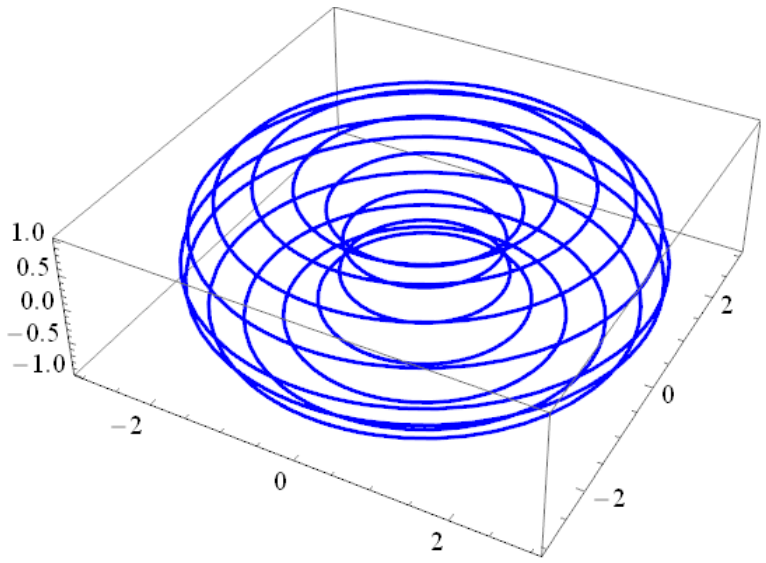


Figure 4.23. Several u-coordinate curves on the torus[2,1,1]

As an example, consider the torus with a=2, b=1, c=1, where $u, v \in [0, 2\pi)$. The Gaussian curvature for this torus becomes,

K=(cos v)/(2 + cos v).

gaussCurv[torus[2,1,1]][u,v]

Plotting the Gaussian curvature with respect to v, we obtain

Plot[gaussCurv[torus[2,1,1]][u,v]//Evaluate,{v,0,2π},Ticks->{{0,π/2,π,3/2π,2π},Automatic},AxesLabel->{"v","K"}]

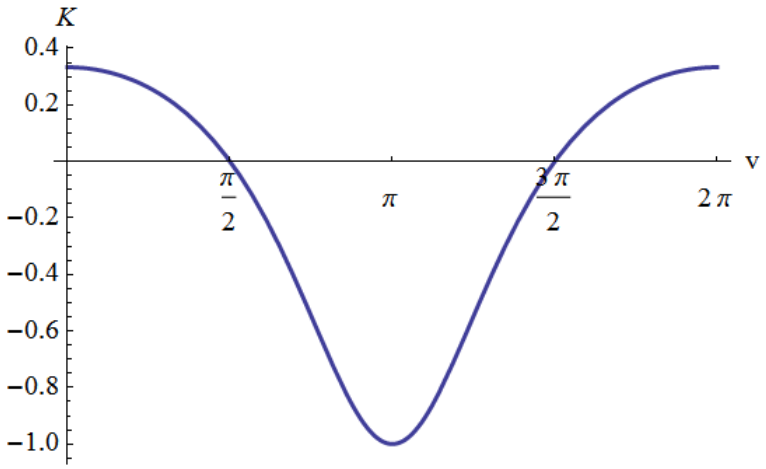


Figure 4.24. The graph of the Gaussian curvature of the torus[2,1,1].

Points on the surface is called elliptic if the Gaussian curvature is positive, that is, if

(cos v)/(2 + cos v)>0.

This occurs when

0<=v<π/2 and (3π)/2<=v<2π.

```
plot1=ParametricPlot3D[Table[torus[2,1,1][u,j],{j,0,π/2,1/2}]//Evaluate,
{u,0,2π},PlotStyle->{{Blue,AbsoluteThickness[2]}}]

plot2=ParametricPlot3D[Table[torus[2,1,1][u,j],{j,
(3π)/2,2π,1/2}]//Evaluate,{u,0,2π},PlotStyle->
{{Blue,AbsoluteThickness[2]}}]
```

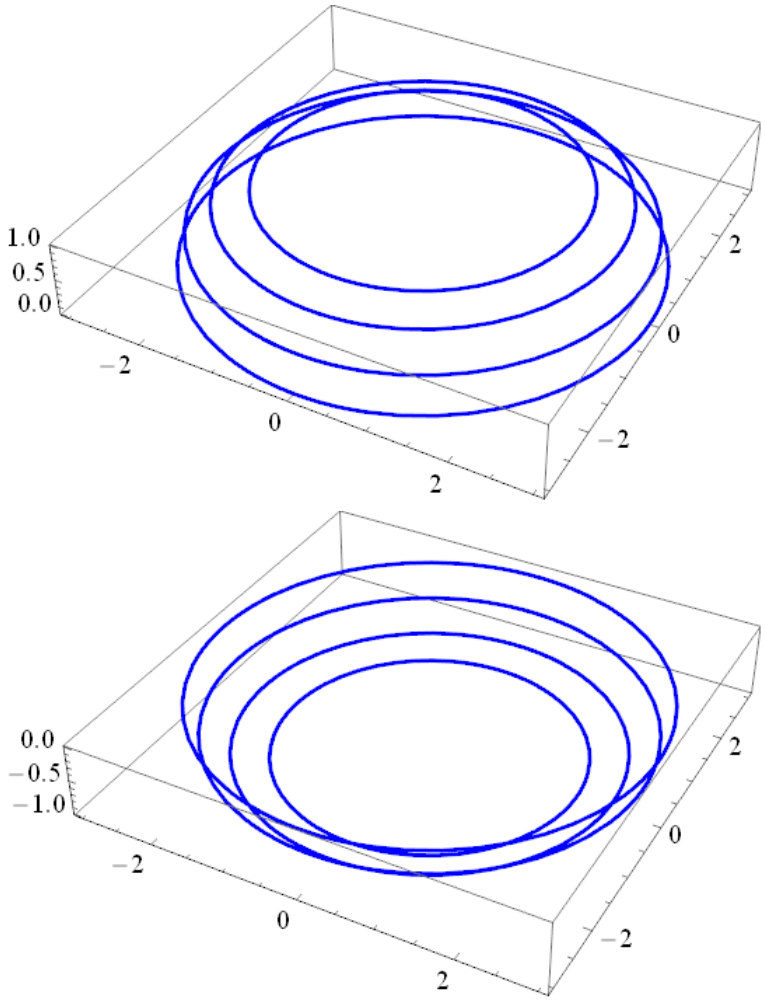


Figure 4.25. Some u-coordinate curves where v={0,1/2,1,3/2} (top) and v={(3π)/2,(1+3π)/2,(2+3π)/2,(3+3π)/2} (bottom)

So, combining the last two graphs we have elliptic points lie on these curves:

Show[plot1,plot2,PlotRange->All]

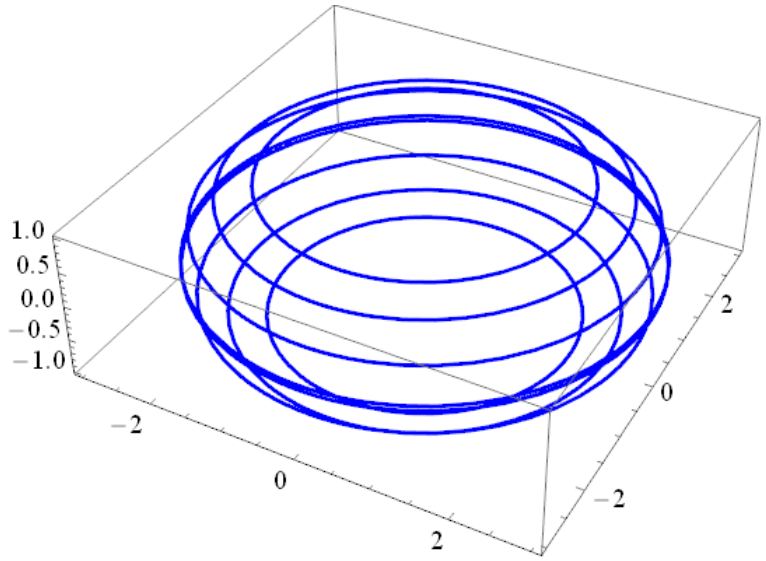


Figure 4.26. The graph of u-coordinate curves where points are elliptic by fixing v, where v={0,1/2,1,3/2,(3π)/2,(1+3π)/2,(2+3π)/2,(3+3π))/2}

Points on the surface is called hyperbolic if the Gaussian curvature is negative, that is, if

(cos v)/(2+cos v)<0.

This occurs when

π/2

The following graph shows hyperbolic points that lie on the curves.

plot3=ParametricPlot3D[Table[torus[2,1,1][u,j],{j,π/2,(3π)/2}]//Evaluate, {u,0,2π},PlotStyle->{{Green,AbsoluteThickness[2]}}]

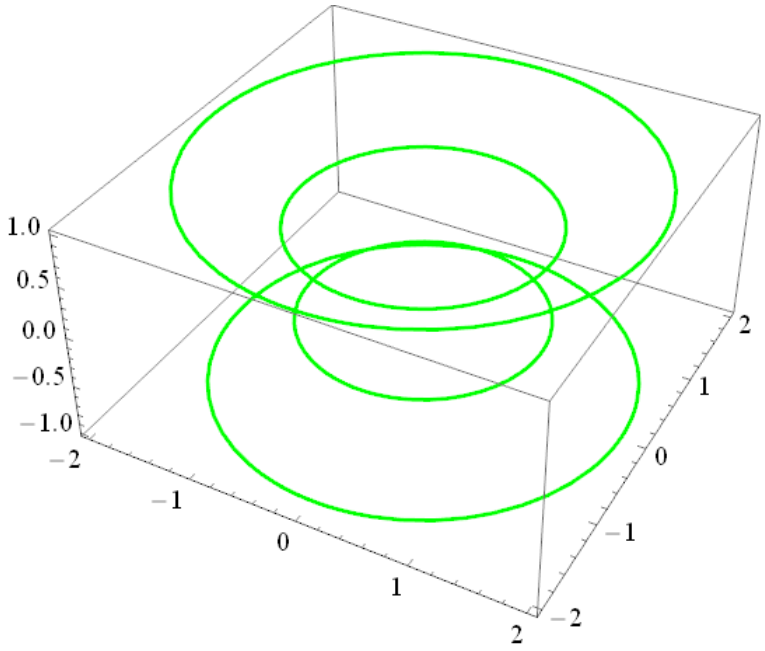


Figure 4.27. The graph of u-coordinate curves where points are hyperbolic.

The shape operator of the torus[2,1,1] can be computed and found to be

$$\mathbf{S} = \begin{pmatrix} -\frac{\cos v}{2+\cos v} & 0 \\ 0 & -1 \end{pmatrix}.$$

shapeOperator[torus[2,1,1]][u,v]//Simplify//PowerExpand//MatrixForm

Points on the surface is called parabolic if the Gaussian curvature vanishes and the shape operator is nonzero. This occurs when v=π/2 and v=(3π)/2.

There are only two curves that lie on the torus[2,1,1] which are parabolic as is shown in the following graph:

```
plot4=ParametricPlot3D[Table[torus[2,1,1][u,j],{j,{π/2,(3π)/2}}]//Evaluate,
{u,0,2π},PlotStyle->{{Red,AbsoluteThickness[2]}}]
```

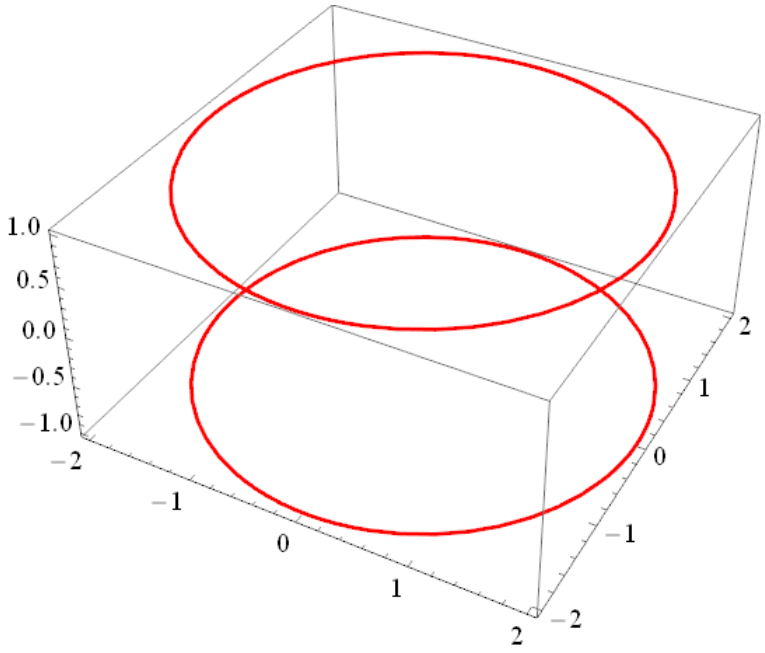


Figure 4.28. The graph of u-coordinate curves where points are parabolic.

Finally the figure below shows curves on the torus[2,1,1] where points are elliptic (blue lines), hyperbolic (green lines), and parabolic (red lines).

```
Show[plot1,plot2,plot3,plot4,PlotRange->All]
```

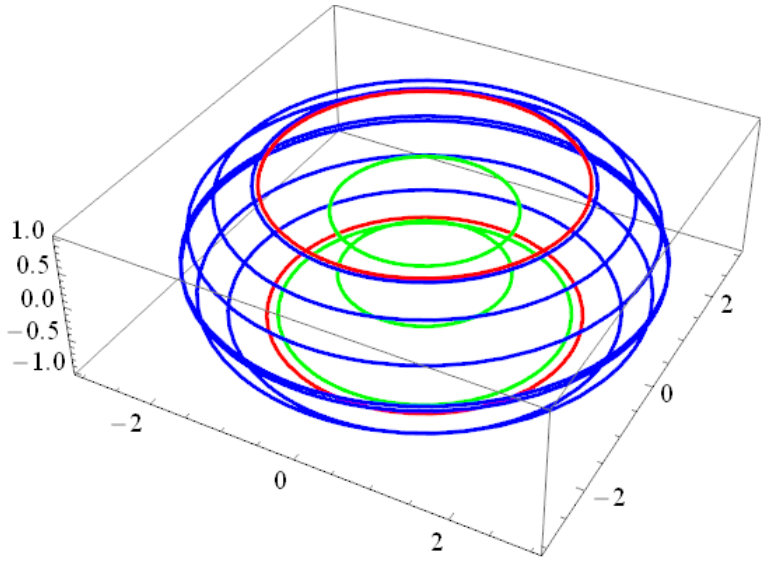


Figure 4.29. The graph of u-coordinate curves by fixing v, where points are elliptic (blue lines), hyperbolic (green lines), and parabolic (red lines).

4.4 Some Classifications of Surfaces

4.4.1 Monge Patch

A Monge patch on surface X is defined by the parametrization of the form

$X(u,v)=(u,v,\varphi(u,v))$,

provided φ is continuous.

Thus, in general a regular surface defined by $z=\varphi(x,y)$ is a Monge patch. It is a graph of functions of two variables.

The surface defined in Example 4.13 is an example of a Monge patch.

The following example gives another famous Monge patch called the monkey saddle.

Example 4.15. The monkey saddle is a surface defined by the function $z=x^3-3x\ y^2$.

(a) Plot the surface.

(b) Find the shape operator.

(c) Compute the principal curvatures.

(d) Compute the Gaussian curvature and the mean curvature.

Solution:

(a)

```
(* The parametric equation of a monkey saddle *)

Clear[monkeySaddle]

monkeySaddle[u_,v_]:={u,v,u^3-3u v^2}


ParametricPlot3D[monkeySaddle[u,v]//Evaluate,{u,-2,2},
{v,-2,2},AxesLabel->{"x","y","z"},BoxRatios->{1,1,1}]
```

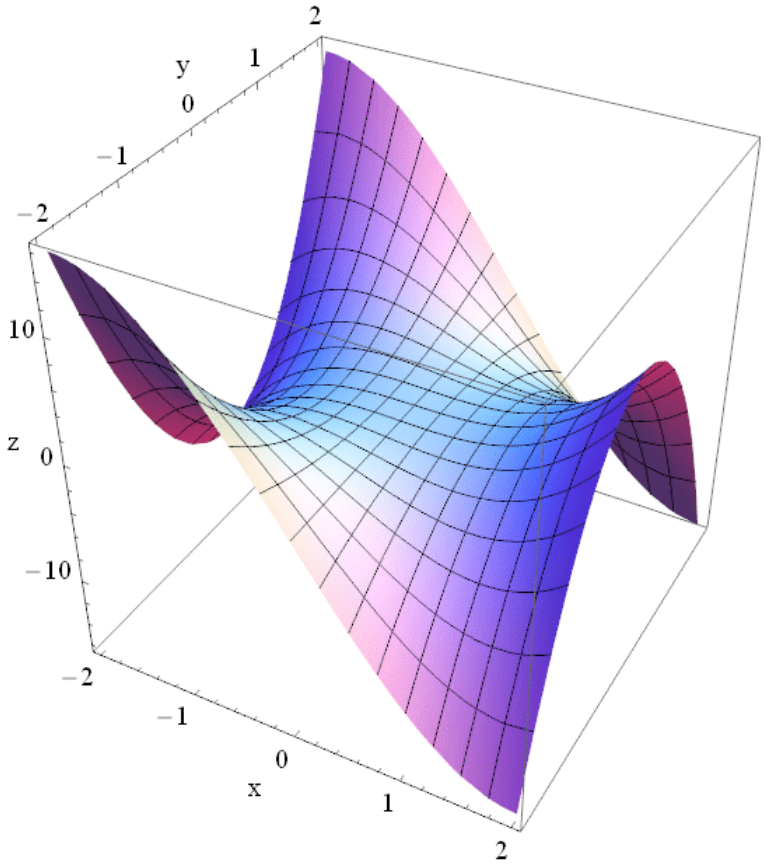


Figure 4.30. The graph of the monkey saddle z=x^3-3x y^2.

(b)

shapeOperator[monkeySaddle][u,v]//MatrixForm

The shape operator of the monkey saddle:

$$\mathbf{S} = \begin{pmatrix} \frac{6\left(u + 18\,u^3\,v^2 + 18\,u\,v^4\right)}{\left(1 + 9\,u^4 + 18\,u^2\,v^2 + 9\,v^4\right)^{3/2}} & -\frac{6\left(v + 18\,u^4\,v + 18\,u^2\,v^3\right)}{\left(1 + 9\,u^4 + 18\,u^2\,v^2 + 9\,v^4\right)^{3/2}} \\ -\frac{6\,v\left(1 - 9\,u^4 + 9\,v^4\right)}{\left(1 + 9\,u^4 + 18\,u^2\,v^2 + 9\,v^4\right)^{3/2}} & -\frac{6\left(u + 9\,u^5 - 9\,u\,v^4\right)}{\left(1 + 9\,u^4 + 18\,u^2\,v^2 + 9\,v^4\right)^{3/2}} \end{pmatrix}$$

.

(c)

principalCurvs[monkeySaddle][u,v]

(d)

gaussCurv[monkeySaddle][u,v]

meanCurv[monkeySaddle][u,v]

The principal curvatures of the monkey saddle looks complicated. However, the Gaussian curvature and mean curvature of the monkey saddle are rather simple, given by

$$K = -\frac{36\left(u^2 + v^2\right)}{\left(1 + 9u^4 + 18u^2v^2 + 9v^4\right)^2}$$

,

$$H = -\frac{27\left(u^5 - 2u^3v^2 - 3uv^4\right)}{\left(1 + 9u^4 + 18u^2v^2 + 9v^4\right)^{3/2}}$$

.

Clearly, points on the monkey saddle are hyperbolic.

Note that there are symmetries of the Gaussian curvature for the monkey saddle:

1. The Gaussian curvature is preserved under interchanging of parameters, that is u->v, v->u.

2. The Gaussian curvature is preserved under reflection, that is, transformation such that u->-u and v->-v.

3. We may rewrite the Gaussian curvature in polar coordinates as

$$K = -\frac{36r^2}{\left(1 + 9r^4\right)^2}$$.

Thus, the Gaussian curvature is invariant with respect to rotations about the z-axis.

gaussCurv[monkeySaddle][u,v]/.{u->r Cos[θ],v->r Sin[θ]}//Simplify

The first and the second fundamental form of the Monge patch, as well as the Gaussian curvature and mean curvatures can be computed straight forwardly and the results are stated in the following lemma:

Lemma 4.11. For a Monge patch X:(u,v)->(u,v,φ(u,v)), we have

(4.17) $$E = 1 + \varphi_u^2 , \qquad F = \varphi_u \varphi_v , \qquad G = 1 + \varphi_v^2 ,$$

(4.18) $$e = \frac{\varphi_{u\,u}}{\left(1 + \varphi_u^2 + \varphi_v^2\right)^{1/2}} , \quad f = \frac{\varphi_{u\,v}}{\left(1 + \varphi_u^2 + \varphi_v^2\right)^{1/2}} , \quad g = \frac{\varphi_{v\,v}}{\left(1 + \varphi_u^2 + \varphi_v^2\right)^{1/2}} ,$$

(4.19) $$K = \frac{\varphi_{u\,u}\,\varphi_{v\,v} - \varphi_{u\,v}^2}{\left(1 + \varphi_u^2 + \varphi_v^2\right)^2} ,$$

$$H = \frac{\left(1 + \varphi_v^2\right)\varphi_{u\,u} - 2\,\varphi_u\,\varphi_v\,\varphi_{u\,v} + \left(1 + \varphi_u^2\right)\varphi_{v\,v}}{2\left(1 + \varphi_u^2 + \varphi_v^2\right)^{3/2}}$$

,

where subscripts denote partial derivatives.

Proof.

It is quite tedious to prove these formulas by hands, so we employ Mathematica to do the tasks.

```
Clear[X,φ]

X[u_,v_]:={u,v,φ[u,v]}
```

```
firstForm[X][u,v]//MatrixForm
```

φ is a general function of u and v, so Mathematica use superscripts to denote partial derivatives. For example: $\varphi^{(1,0)}[u, v]$ denotes partial derivative of φ with respect to the first parameter which in this case is u. Likewise, $\varphi^{(0,1)}[u, v]$ denotes partial derivative of φ with respect to the second parameter which in this case is v.

This proves equation (4.17):

(4.17) $$E = 1 + \varphi_u{}^2 \,, \qquad F = \varphi_u \, \varphi_v \,, \qquad G = 1 + \varphi_v{}^2 \,.$$

```
secondForm[X][u,v]//MatrixForm
```

This proves equation (4.18):

(4.18) $$e = \frac{\varphi_{u\,u}}{\left(1 + \varphi_u{}^2 + \varphi_v{}^2\right)^{1/2}} \,, \quad f = \frac{\varphi_{u\,v}}{\left(1 + \varphi_u{}^2 + \varphi_v{}^2\right)^{1/2}} \,, \quad g = \frac{\varphi_{v\,v}}{\left(1 + \varphi_u{}^2 + \varphi_v{}^2\right)^{1/2}} \,,$$

where $\varphi^{(2,\,0)}[u, v]$, $\varphi^{(1,\,1)}[u, v]$, and $\varphi^{(0,\,2)}[u, v]$ denote second partial derivatives for $\varphi_{u\,u} \equiv \frac{\partial^2 \varphi}{\partial u^2}$, $\varphi_{u\,v} \equiv \frac{\partial^2 \varphi}{\partial v \, \partial u}$, and $\varphi_{v\,v} \equiv \frac{\partial^2 \varphi}{\partial v^2}$, respectively.

gaussCurv[X][u,v]

meanCurv[X][u,v]

The last two Mathematica commands prove equation (4.19) for the Gaussian curvature:

$$K = \frac{\varphi_{u\,u}\,\varphi_{v\,v} - \varphi_{u\,v}^{\,2}}{\left(1 + \varphi_u^{\,2} + \varphi_v^{\,2}\right)^2}$$

,

and the mean curvature:

$$H = \frac{\left(1 + \varphi_v^{\,2}\right)\varphi_{u\,u} - 2\,\varphi_u\,\varphi_v\,\varphi_{u\,v} + \left(1 + \varphi_u^{\,2}\right)\varphi_{v\,v}}{2\left(1 + \varphi_u^{\,2} + \varphi_v^{\,2}\right)^{3/2}}$$

,

respectively.

4.4.2 Quadric Surface

A quadric surface is a surface determined by a quadratic equation in x, y, and z.

The unit sphere centered at the origin that satisfies the quadratic equation:

x^2+y^2+z^2=1,

is an example of a quadric surface.

Example 4.16. The parametric equation of a unit sphere centered at the origin is given by

X(u,v)=(cos u sin v, sin u sin v, cos v),

$u \in [0, 2\pi]$, $v \in [0, \pi]$,

where u is called the longitude, v the colatitude.

(a) Show that the parametric equation satisfies the quadratic equation x^2+y^2+z^2=1.

(b) Plot the surface.

(c) Find the shape operator.

(d) Compute the principal curvatures.

(e) Compute the Gaussian curvature and the mean curvature.

Solution:

(a)

```
Clear[x,y,z,X]

x=Cos[u] Sin[v];

y=Sin[u] Sin[v];

z=Cos[v];

X[u_,v_]:={Cos[u] Sin[v],Sin[u] Sin[v],Cos[v]}
```

x^2+y^2+z^2==1//Simplify

Mathematica verified that the parametric equation satisfies the equation x^2+y^2+z^2=1.

(b)

ParametricPlot3D[X[u,v],{u,0,2π},{v,0,π}]

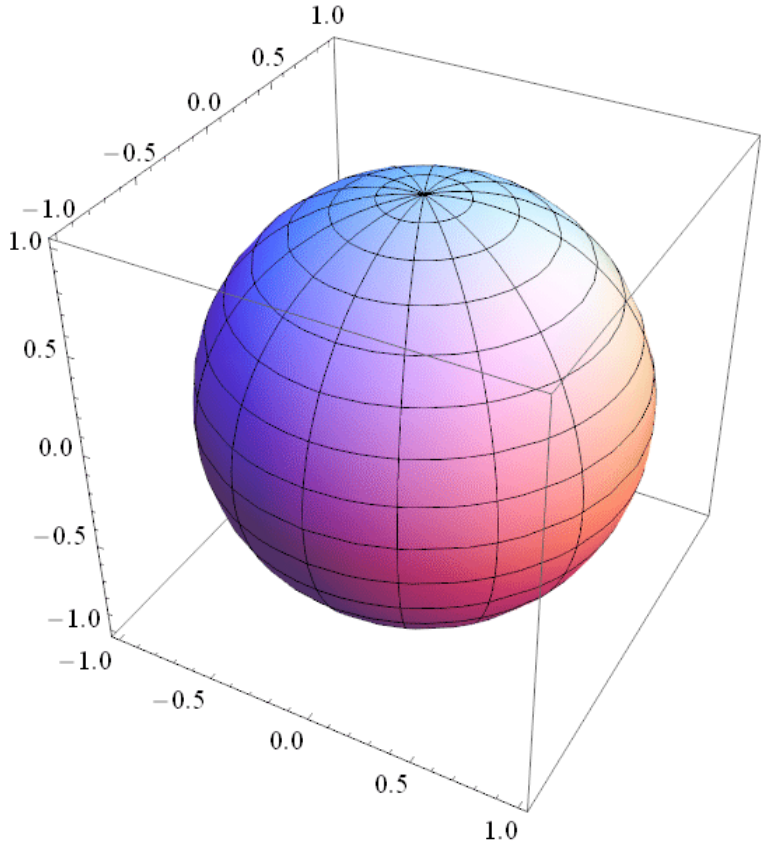


Figure 4.31. The graph of the unit sphere centered at origin.

(c)

shapeOperator[X][u,v]//PowerExpand//MatrixForm

It is not surprising that the shape operator of the unit sphere is an identity matix:

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(d)

principalCurvs[X][u,v]//PowerExpand

Since the shape operator is diagonalized, the principal curvatures can be read off from the matrix:

$k_1 = 1$ and $k_2 = 1$.

(e)

gaussCurv[X][u,v]

The Gaussian curvature of the unit sphere is everywhere constant, K=1.

meanCurv[X][u,v]//PowerExpand

The mean curvature equals the Gaussian curvature of the unit sphere:

H=1.

Example 4.17. A quadric surface is defined by the equation x y+x z=1.

(a) Find the parametric equation for the surface.

(b) Plot the surface.

(c) Find the shape operator.

(d) Compute the principal curvatures.

(e) Compute the Gaussian curvature and the mean curvature.

Solution:

(a)

Write u=y and v=z, then the parametric equation may be written as

X(u,v)=(1/(u + v),u,v).

(b)

```
Clear[X,u,v]

X[u_,v_]:={1/(u+v),u,v}

ParametricPlot3D[X[u,v]//Evaluate,{u,-5,5},{v,-5,5},BoxRatios->{1, 1, 1}]
```

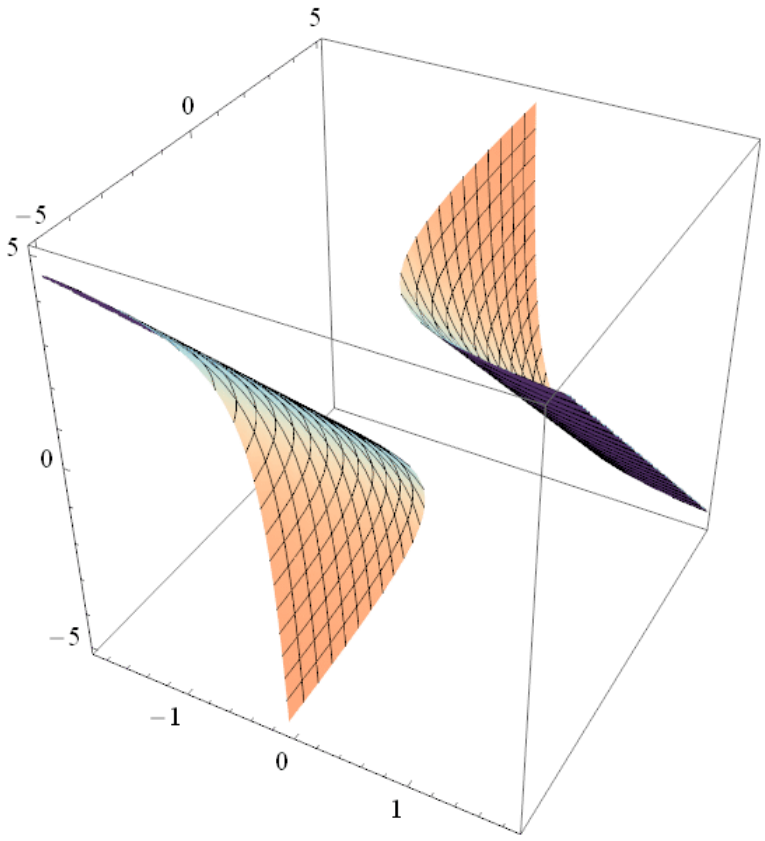


Figure 4.32. The graph of the quadric surface given by x y+x z=1.

(c)

shapeOperator[X][u,v]//MatrixForm

The shape operator is given by

$$\mathbf{S} = \frac{2\,(u+v)}{\left(2+u^4+4\,u^3\,v+6\,u^2\,v^2+4\,u\,v^3+v^4\right)\sqrt{1+\frac{2}{(u+v)^4}}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

.

It is interesting to note that all elements of the matrix are the same. However, the shape operator is undefined at u=v=0.

(d)

principalCurvs[X][u,v]//PowerExpand

$$k_1 = \frac{4\,(u+v)^3}{\left(2+u^4+4\,u^3\,v+6\,u^2\,v^2+4\,u\,v^3+v^4\right)^{3/2}}$$

, $k_2 = 0$.

One of the principal curvature is zero, so we expect the Gaussian curvature to vanish.

(e)

gaussCurv[X][u,v]

Since K=0 and $S \neq 0$ everywhere on the surface except the singular point at u=v=0, so points on this surface are parabolic except at u=v=0.

meanCurv[X][u,v]

The mean curvature is given by

$$H = \frac{2(u+v)}{\left(2+u^4+4u^3v+6u^2v^2+4uv^3+v^4\right)\sqrt{1+\frac{2}{(u+v)^4}}}$$

.

4.4.3 Cylindrical Surface

A surface is called cylindrical surface if it can be parametrized in the form:

X(u,v)=r(u) + v p.

where r(u) is a plane curve and p is a fixed direction, perpendicular to the plane of r.

As an example, consider a parabola curve in the xy-plane and the fixed (0,0,1) direction:

r(u)=(u, u^2,0), p=(0,0,1).

Thus,

X(u,v)=(u, u^2,v), $u \in [-2, 2]$, $v \in [-1, 1]$.

Example 4.18. A parabolic cylindrical surface is given by

X(u,v)=(u, u^2,v), $u \in [-2, 2]$, $v \in [-1, 1]$.

(a) Plot the surface.

(b) Find the shape operator.

(c) Compute the principal curvatures.

(d) Compute the Gaussian curvature and the mean curvature.

Solution:

(a)

Clear[X]

X[u_,v_]:={u,u^2,v}

ParametricPlot3D[X[u,v]//Evaluate,{u,-2,2},{v,-1,1}]

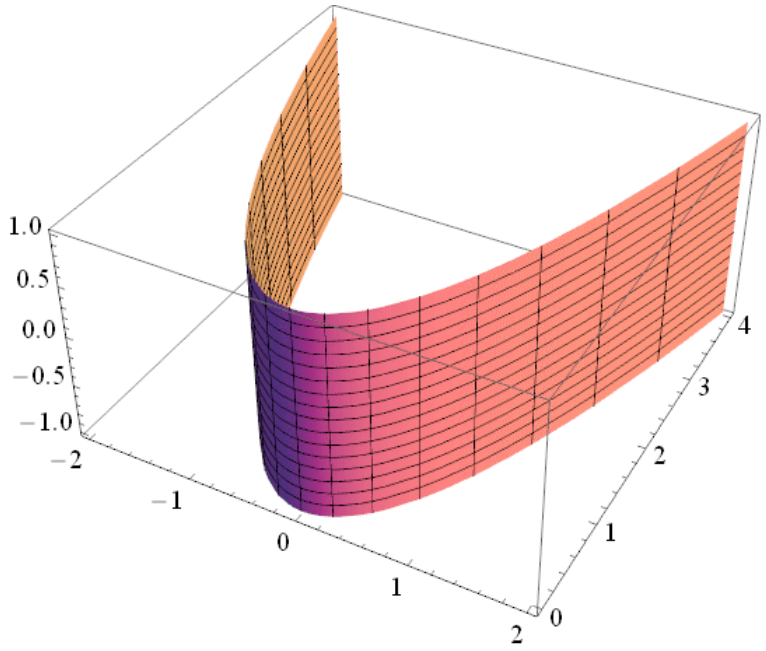


Figure 4.33. The graph of the cylindrical surface X(u,v)=(u, u^2,v).

(b)

shapeOperator[X][u,v]//MatrixForm

The shape operator is given by

$$\mathbf{S} = \begin{pmatrix} -\frac{2}{\left(1+4u^2\right)^{3/2}} & 0 \\ 0 & 0 \end{pmatrix}.$$

(c)

principalCurvs[X][u,v]

The principal curvatures are given by

$$k_1 = -\frac{2}{\left(1+4u^2\right)^{3/2}}, \qquad k_2 = 0 .$$

(d)

gaussCurv[X][u,v]

It is not surprising that points on the surface is everywhere paraboloid, as confirmed by the fact that K=0 and $S \neq 0$.

meanCurv[X][u,v]

The mean curvature is given by

$$H = -\frac{1}{\left(1+4u^2\right)^{3/2}} .$$

4.4.4 Ruled Surface

A ruled surface is a surface that can be swept out by moving a line in space. It therefore has a parameterization of the form:

$X(u,v)=r(u) + v\, p(u),$

where r is called a base curve which need not be planar and p is the director curve.

Clearly, a cylindrical surface is a special case of a ruled surface in which the direction is fixed and perpendicular to the base curve.

As an example, consider a base circle on the xy-plane:

$r(u)=(\cos u, \sin u,0),$

and the direction of a line:

$p(u)=(0,0,u).$

Thus,

$X(u,v)=r(u) + v\, p(u),$

$X(u,v)=(\cos u, \sin u,0)+ v\, (0,0,u)$

$=(\cos u, \sin u,v\, u).$

Example 4.19. A parametric equation of a ruled surface is given by

$X(u,v)=(\cos u, \sin u,v\, u).$

(a) Plot the surface.

(b) Find the shape operator.

(c) Compute the principal curvatures.

(d) Compute the Gaussian curvature and the mean curvature.

Solution:

(a)

```
Clear[X]

X[u_,v_]:={Cos[u],Sin[u],v u}

ParametricPlot3D[X[u,v],{u,0,2π},{v,-2,2},BoxRatios->{1, 1, 1}]
```

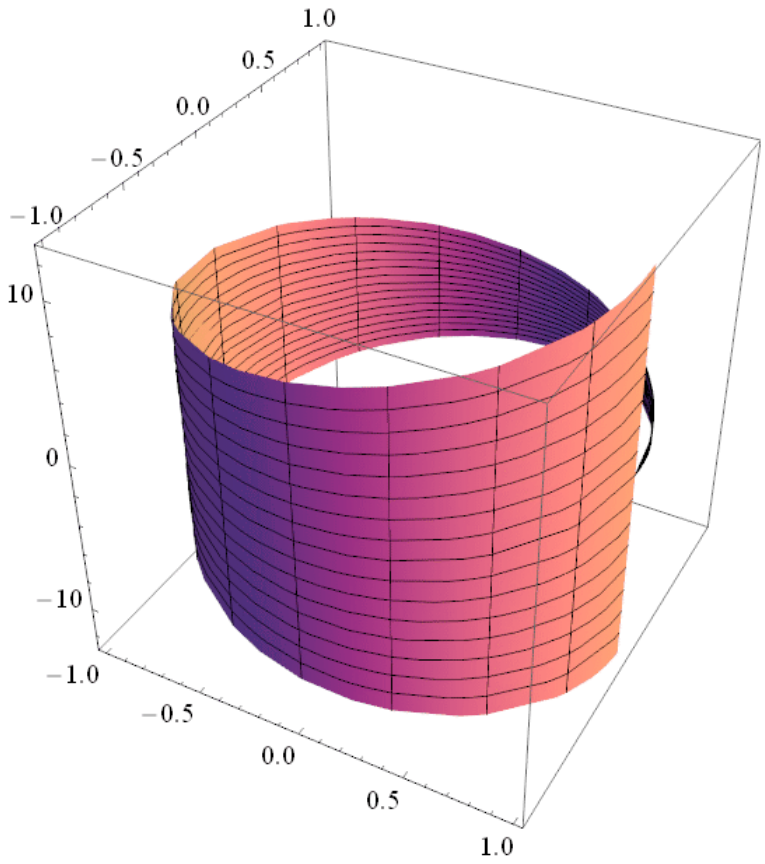


Figure 4.34. The graph of the ruled surface X(u,v)=(cos u, sin u,v u)

(b)

shapeOperator[X][u,v]//PowerExpand//MatrixForm

The shape operator of the ruled surface is given by

$$\mathbf{S} = \begin{pmatrix} -1 & 0 \\ v/u & 0 \end{pmatrix}.$$

(c)

principalCurvs[X][u,v]//PowerExpand

The principal curvatures of the ruled surface:

$k_1 = 0$ and $k_2 = -1$.

(d)

gaussCurv[X][u,v]

Since K=0 and $S \neq 0$, the points on the surface are parabolic.

meanCurv[X][u,v]//PowerExpand

The mean curvature of the ruled surface:

H=-(1/2).

4.4.5 Surface of Revolution

A surface of revolution is obtained by revolving a plane curve about an axis that does not meet the curve.

As an example, consider the plane curve to lie in the yz-plane parametrized by (0,g(u),h(u)). The general rotation matrix around the z-axis is given by

$$\begin{pmatrix} \cos v & \sin v & 0 \\ -\sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where v is the angle of rotation.

Since the plane curve lies in the yz-plane, the parametric equation of the surface is given by

$$\mathbf{X}(u, v) = \begin{pmatrix} \sin v & \cos v & 0 \\ -\cos v & \sin v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ g(u) \\ h(u) \end{pmatrix} = \begin{pmatrix} g(u) \cos v \\ g(u) \sin v \\ h(u) \end{pmatrix}$$

.

Example 4.20. An inverted bell surface is given by the parametric equation

X(u,v)=(u cos v,u sin v,1-E^-u^2), 0<=u<=3, 0<=v<=2π.

(a) Plot the surface.

(b) Find the shape operator.

(c) Compute the principal curvatures.

(d) Compute the Gaussian curvature and the mean curvature.

Solution:

(a)

```
Clear[X]

X[u_,v_]:={u Cos[v],u Sin[v],1-E^-u^2}


ParametricPlot3D[X[u,v],{u,0,3},{v,0,2π},AxesLabel->
{"x","y","z"},BoxRatios->{1, 1, 1}]
```

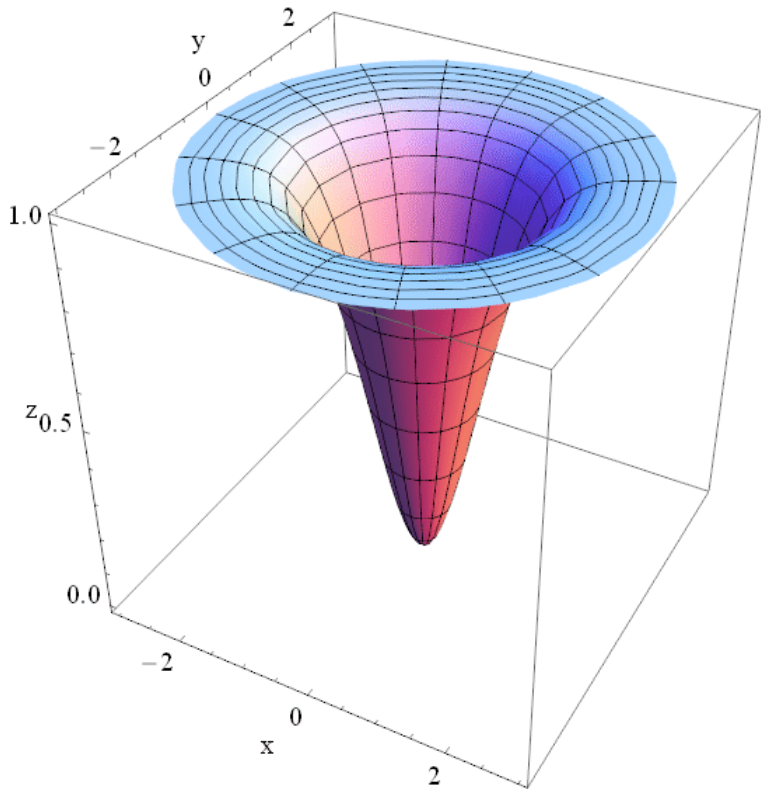


Figure 4.35. The graph of the inverted bell surface X(u,v)=(u cos v,u sin v,1-E^-u^2).

(b)

shapeOperator[X][u,v]//MatrixForm

The shape operator of the inverted bell surface is given by

$$\mathbf{S} = \begin{pmatrix} \dfrac{e^{-u^2} u^2 \left(2u - 4u^3\right)}{\left(u^2 + 4e^{-2u^2} u^4\right)^{3/2}} & 0 \\ 0 & \dfrac{2e^{-u^2} u}{\sqrt{u^2 + 4e^{-2u^2} u^4}} \end{pmatrix}$$

.

(c)

principalCurvs[X][u,v]

Since the shape operator in matrix form is diagonal, we can read off from the matrix the principal curvatures:

$$k_1 = \frac{e^{-u^2} u^2 \left(2u - 4u^3\right)}{\left(u^2 + 4e^{-2u^2} u^4\right)^{3/2}}, \quad k_2 = \frac{2e^{-u^2} u}{\sqrt{u^2 + 4e^{-2u^2} u^4}}.$$

(d)

gaussCurv[X][u,v]

The Gaussian curvature of the inverted bell surface is given by

$$K = \frac{4 e^{2 u^2} \left(1 - 2 u^2\right)}{\left(e^{2 u^2} + 4 u^2\right)^2}$$.

Note that the Gaussian curvature depends only on the parameter u.

meanCurv[X][u,v]

The mean curvature of the inverted bell surface is given by

$$H = \frac{e^{-u^2} \left(4 u^3 - 2 e^{2 u^2} u \left(-1 + u^2\right)\right)}{\left(e^{2 u^2} + 4 u^2\right) \sqrt{u^2 + 4 e^{-2 u^2} u^4}}$$

.

Now, let us investigate points on the surface by plotting the Gaussian curvature:

Plot[gaussCurv[X][u,v]//Evaluate,{u,-3,3},PlotRange->All,AxesLabel->{"u","K"}]

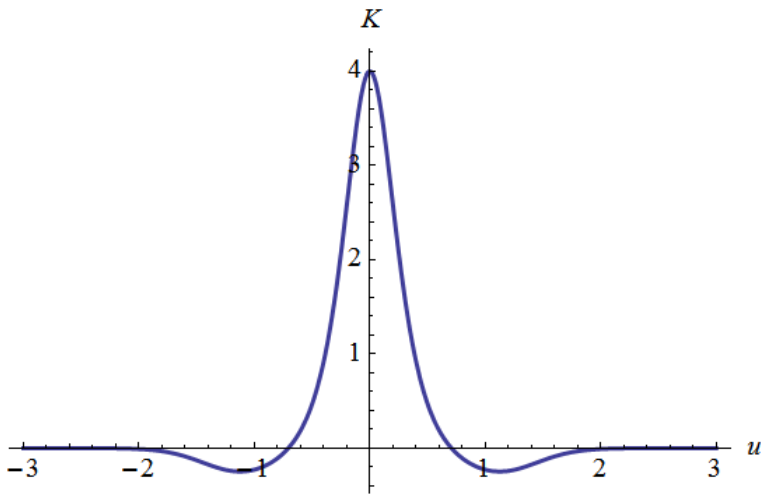


Figure 4.36. The graph of the Gaussian curvature of the inverted bell surface.

Note that the Gaussian curvature is symmetric about the u-axis.

Since the z-coordinate is defined as z=1-E^-u^2, the surface is also symmetric about the z-axis, which is not surprising as can be seen from the graph of the surface.

(* K > 0 *)

Reduce[gaussCurv[X][u,v]>0,u]

K>0, the points are elliptic for region -(1/Sqrt[2])

(* K < 0 *)

Reduce[gaussCurv[X][u,v]<0,u]

K<0, the points are hyperbolic for region u<-(1/Sqrt[2]) or u>1/Sqrt[2].

(* K = 0 *)

Reduce[gaussCurv[X][u,v]==0,u]

K=0, and since $S \neq 0$ the points are parabolic when u=-(1/Sqrt[2]) or u=1/Sqrt[2].

Let us find the extreme values of K by using FindMinimum command and FindMaximum command for local minimum and local maximum, respectively. From Figure 4.36 we may estimate the extrema for these commands to start the search.

(* Find the local minimum of K near u=-1 *)

FindMinimum[gaussCurv[X][u,v]//Evaluate,{u,-1}]

(* Find the local minimum of K near u=1 *)

FindMinimum[gaussCurv[X][u,v]//Evaluate,{u,1}]

(* Find the local maximum of K near u=0.01 *)

FindMaximum[gaussCurv[X][u,v]//Evaluate,{u,0.01}]

K have local minima at u=-1.1252 and u=1.1252. The minimum value of K is -0.247643.

K has a local maximum at u=0. The maximum value of K is 4.

Chapter 5. Optimization in Several Variables

In this chapter, we will study the techniques of finding the extrema of functions of several variables. The process is called optimization. Recall in single variable case, if f(a) is a local extreme value, then a is called a critical point. If the tangent line exists, it is horizontal at x=a.

A local extremum (either maximum or minimum) always occurs at a critical point, that is, at a point where the derivative f '(a)=0. However, the reverse is not always true. A critical point may not be a local extremum. To find the nature of the critical points we deploy the second derivative test. If f '(a)=0 and f "(a)>0, then f has a local minimum at x=a; and if f '(a)=0 and f "(a)<0, then f has a local maximum at x=a.

Similarly for a function of two variables, if f(a,b) is a local extreme value, then the point (a,b) is a critical point of z=f(x,y). However, in this case, it is the tangent plane (if it exists) that must be horizontal. By equation (3.5), the tangent plane to z=f(x,y) at (a,b) is given by

$$h(x, y) = f(a, b) + \frac{\partial f(a, b)}{\partial x}(x - a) + \frac{\partial f(a, b)}{\partial y}(y - b)$$

,

in which we have assumed that f(x,y) is differentiable.

If f(x,y) has a local extremum at a=(a,b), then we require that the derivative of f(x,y) vanishes at a. By equation (3.2):

$$D f_{\mathbf{a}} = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right)_{\mathbf{a}} = (0 \;\; 0)$$

.

Note that this is equivalent to setting the gradient of f(x,y) (equation 3.1) equal to zero.

Or written in component form:

$\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$.

This implies that the equation of the tangent plane is

z=h(x,y)=f(a,b).

Thus, the tangent plane is horizontal if $D f_{\mathbf{a}} = \mathbf{0}$.

Initialization

```
Clear["Global`*"]

SetOptions[Plot3D,PlotTheme->"Classic"];

SetOptions[ContourPlot3D,PlotTheme->"Classic"];

SetOptions[Plot,PlotTheme->"Classic",PlotStyle->AbsoluteThickness[2]];

SetOptions[ParametricPlot3D,PlotTheme->"Classic",PlotStyle-
>AbsoluteThickness[2]];
```

Definition 5.1

We say that f(x,y) has a local maximum at the point a=(a,b) if f(a,b)>=f(x,y) for all (x,y) close to (a,b).

Similarly, we say that f(x,y) has a local minimum at the point a=(a,b) if f(a,b)<=f(x,y) for all (x,y) close to (a,b).

Let us consider some examples:

Example 5.1. (a) Find the critical points of the function:

f(x,y)=x^2-2y^2+2x+3.

(b) Plot the graph and determine the nature of the critical point.

Solution:

To find the critical points of f(x,y), we set the derivative of f(x,y) to zero:

$$Df(x, y) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = (0 \quad 0)$$

,

and solve the system of equations

$$\begin{cases} 2x + 2 = 0, \\ -4y = 0. \end{cases}$$

There is a critical point at (x,y)=(-1,0).

Clear[f]

f[x_,y_]:=x^2-2y^2+2x+3

Solve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y}]

Notice that we have used the Thread command to make sure that the partial derivatives with respect to x and y are assigned to zero respectively.

(b)

```
Module[{plot,a,b},

a=-1;

b=0;

plot[1]=Plot3D[f[x,y]//Evaluate,{x,-2,0},{y,-1,1},AxesLabel->
{"x","y","z"}];

plot[2]=Graphics3D[{PointSize[0.03],Red,Point[{a,b,f[a,b]}]}];

Show[Array[plot,2]]

]
```

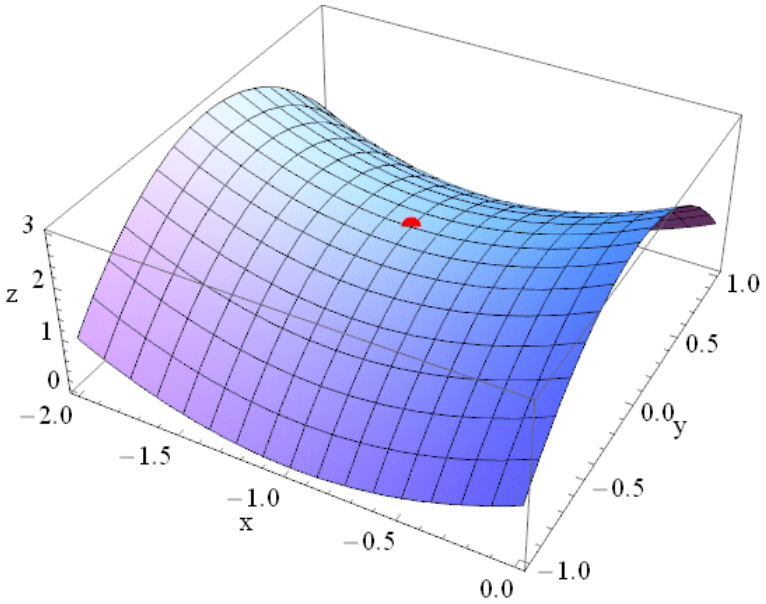


Figure 5.1. The graph of f(x,y)=x^2-2y^2+2x+3 and the critical point.

It can be seen from Figure 5.1 the site where the critical point (-1,0) occurs is neither a maximum nor minimum. As a matter of fact, it is a saddle point.

Example 5.2. (a) Find the critical points of the function:

f(x,y)=(x + y)/(x^2 + 4y^2 + 3).

(b) Plot the graph and determine the nature of the critical point.

Solution:

(a) We solve the equation

$$Df(x, y) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = (0 \quad 0)$$

to find the critical points.

$$\begin{cases} \dfrac{3 - x^2 - 2xy + 4y^2}{\left(3 + x^2 + 4y^2\right)^2} = 0, \\ \dfrac{3 + x^2 - 8xy - 4y^2}{\left(3 + x^2 + 4y^2\right)^2} = 0. \end{cases}$$

Clear[f]

f[x_,y_]:=(x+y)/(x^2+4y^2+3)

```
Grad[f[x,y],{x,y},"Cartesian"]//Together
```

This system of equation is difficult to solve in close form, so we resort to numerical solutions using NSolve command.

```
NSolve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y}]
```

There are two critical points found at

(1.54919,0.387298) and (-1.54919,-0.387298).

(b)

```
Module[{plot,x1,x2,y1,y2},

{x1,y1}={1.54919,0.387298};

{x2,y2}={-1.54919,-0.387298};

plot[1]=Plot3D[f[x,y]//Evaluate,{x,-8,8},{y,-8,8},PlotPoints->
{20,20},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{PointSize[0.03],Red,Point[{{x1,y1,f[x1,y1]},
{x2,y2,f[x2,y2]}}]}];

Show[Array[plot,2],ViewPoint->{10,-12,15}]

]
```

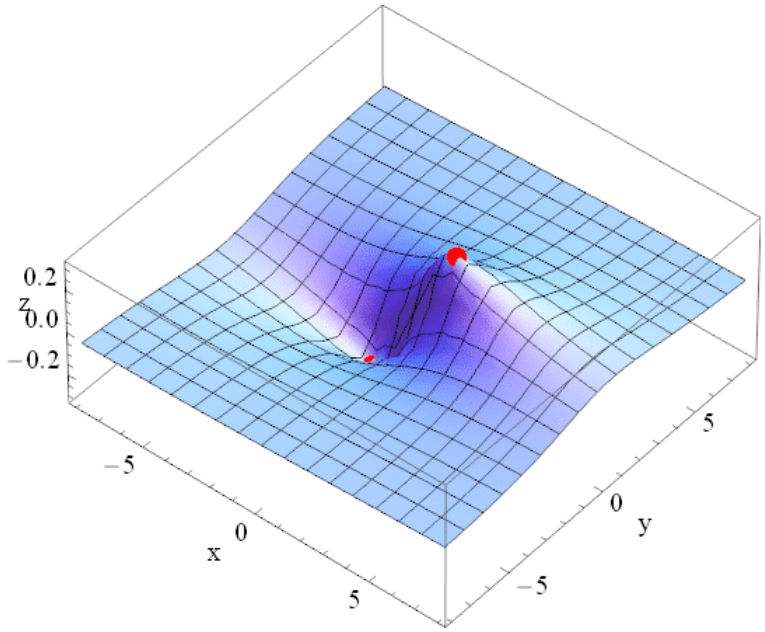


Figure 5.2. The graph of f(x,y)=(x + y)/(x^2 + 4y^2 + 3) and the two extrema.

From Figure 5.2 we identity the point (1.54919,0.387298) as the local maximum and the point (-1.54919,-0.387298) as the local minimum.

As can be seen from the two examples above, graphs provide a powerful tool to reveal information about the nature of critical points. However, it is not always the case. As we will see later that sometimes symbolic or numerical computations are more revealing (see Example 5.18, Figure 5.25)

5.1 Second Derivative Test

As in the single variable case, there is a second derivative test for determining the nature of the critical points of a function of two variables. It depends on the Hessian matrix defined by

$$Hf(x, y) \equiv \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\, \partial x} \\ \frac{\partial^2 f}{\partial x\, \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$

For a smooth function, that is, function which has continuous second partial derivatives, we have $\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$. Hence, the Hessian matrix is symmetric.

The test relies on the sign of the determinant of the Hessian matrix. Below is the theorem of the second derivative test for function of two variables which is stated without proof. We invite readers who are interested in the proof to consult textbooks on multivariable calculus.

Theorem 5.2. Second Derivative Test for local extrema (function of two variables)

Let a=(a,b) be the critical point of a function f(x,y) which has continuous second partial derivatives. The Hessian matrix of f(x,y) evaluated at a is given by

$$Hf(\mathbf{a}) \equiv \begin{pmatrix} \frac{\partial^2 f(\mathbf{a})}{\partial x^2} & \frac{\partial^2 f(\mathbf{a})}{\partial y\, \partial x} \\ \frac{\partial^2 f(\mathbf{a})}{\partial x\, \partial y} & \frac{\partial^2 f(\mathbf{a})}{\partial y^2} \end{pmatrix}. \tag{5.1}$$

Define d as the determinant of the Hessian matrix H f(a), that is, d=det H f(a). Then:

(i) If d>0 and $\frac{\partial^2 f(\mathbf{a})}{\partial x^2} > 0$, then f(a,b) is a local minimum.

(ii) If d>0 and $\frac{\partial^2 f(\mathbf{a})}{\partial x^2} < 0$, then f(a,b) is a local maximum.

(iii) If d<0, then f has a saddle point at (a,b).

(iv) If d=0, the test is inconclusive.

Example 5.3. Find the critical points of the function:

f(x,y)=x^2+2y^2-4y+6x

and analyze them using Theorem 5.2.

Solution:

Step 1. Find the critical points.

$\frac{\partial f}{\partial x} = 2x + 6$, $\frac{\partial f}{\partial y} = 4y - 4$.

Setting the derivative of f equals to zero:

$$Df(x, y) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = (0 \quad 0)$$

,

we solve the system of equations

$$\begin{cases} 2x+6=0, \\ 4y-4=0. \end{cases}$$

The solution is (x,y)=(-3,1).

So, there is only one critical point at (x,y)=(-3,1).

Clear[f]

f[x_,y_]:=x^2+2y^2-4y+6x

Solve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y}]

Step 2. Compute the second partial derivatives.

$$\frac{\partial^2 f}{\partial x^2}=2, \qquad \frac{\partial^2 f}{\partial y\,\partial x}=0,$$

$$\frac{\partial^2 f}{\partial x\,\partial y}=0, \qquad \frac{\partial^2 f}{\partial y^2}=4.$$

Step 3. Apply Theorem 5.2.

The Hessian matrix is

$$Hf(-3,\,1)=\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

The determinant of H f(-3,1):

d=8.

Since d>0 and $\frac{\partial^2 f(-3,1)}{\partial x^2} > 0$, then f(-3,1) is a local minimum.

```
Module[{plot,a,b},

a=-3;

b=1;

plot[1]=Plot3D[f[x,y]//Evaluate,{x,-6,0},{y,-2,4},BoxRatios->{1, 1,
1},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{PointSize[0.03],Blue,Point[{a,b,f[a,b]}]}];

Show[Array[plot,2],ViewPoint->{-4,-2,0}]

]
```

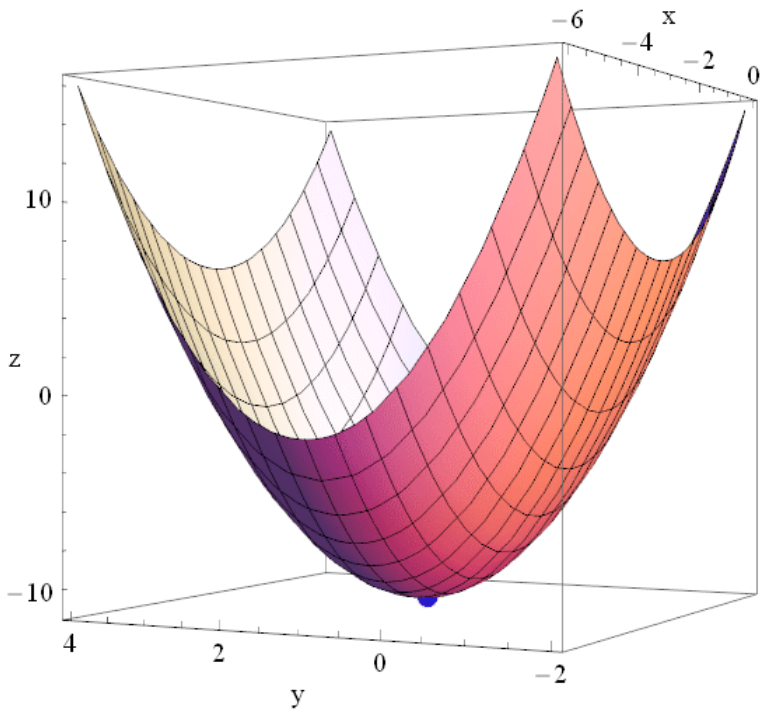


Figure 5.3. The graph of f(x,y)=x^2+2y^2-4y+6x with the critical point (-3,1,-11).

Example 5.4. (a) Find the critical points of the function:

f(x,y)=y^4-2x y^2+x^3-x.

(b) Determine the nature of the critical points in part (a).

Solution:

(a)

Step 1. Find the critical points.

$$\frac{\partial f}{\partial x} = -2y^2 + 3x^2 - 1, \quad \frac{\partial f}{\partial y} = 4y^3 - 4xy.$$

Setting the derivative of f equals to zero

$$Df(x, y) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = (0 \quad 0)$$

,

we solve the system of equations

$$\begin{cases} -2y^2 + 3x^2 - 1 = 0, \\ 4y^3 - 4xy = 0. \end{cases}$$

Clear[f]

f[x_,y_]:=y^4-2x y^2+x^3-x

D[f[x,y],x]

D[f[x,y],y]

Solve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y}]

There are six critical points as solutions to the system of equation:

- four real critical points at: (1,-1), (1,1), (-(1/Sqrt[3]),0), (1/Sqrt[3],0), and

- two complex critical points at: (-(1/3),-(I/Sqrt[3])) and (-(1/3),I/Sqrt[3]).

However, only the real critical points are applicable.

We can specify the domain of the solutions in the Solve command so that it returns only real values:

Solve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y},Reals]

(b)

Step 2. Compute the second partial derivatives.

$\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial y \partial x} = -4y$,

$$\frac{\partial^2 f}{\partial x\,\partial y} = -4y \quad , \quad \frac{\partial^2 f}{\partial y^2} = -4x + 12y^2 \quad .$$

(* The Hessian matrix *)

{{D[f[x,y],{x,2}],D[f[x,y],x,y]},{D[f[x,y],y,x],D[f[x,y],
{y,2}]}}//MatrixForm

Step 3. Apply the Second Derivative Test

The Hessian matrix is

$$Hf(x, y) = \begin{pmatrix} 6x & -4y \\ -4y & -4x + 12y^2 \end{pmatrix}$$

The determinant of H f(x,y):

d=-24 x^2-16y^2+72x y^2.

(* Define the determinant of the Hessian matrix to test critical points *)

d[x_,y_]=6 x (-4x+12y^2)-(-4y)(-4y)//Simplify

Critical point at (1,-1) is a local minimum with value f(1,-1)=-1.

d[1,-1]

D[f[x,y],{x,2}]/.{x->1,y->-1}

f[1,-1]

Critical point at (1,1) is also a local minimum with value f(1,1)=-1.

d[1,1]

D[f[x,y],{x,2}]/.{x->1,y->1}

f[1,1]

Critical point at (-(1/Sqrt[3]),0) is a saddle point with value f(-(1/Sqrt[3]),0)=0.3849.

d[-(1/Sqrt[3]),0]

D[f[x,y],{x,2}]/.{x->-(1/Sqrt[3]),y->0}

f[-(1/Sqrt[3]),0]//N

Critical point at (1/Sqrt[3],0) is also a saddle point with value f(1/Sqrt[3],0)=-0.3849.

d[1/Sqrt[3],0]

D[f[x,y],{x,2}]/.{x->1/Sqrt[3],y->0}

f[1/Sqrt[3],0]//N

```
Module[{plot,x1,x2,x3,x4,y1,y2,y3,y4},

{x1,y1}={1,1};

{x2,y2}={1,-1};

{x3,y3}={-(1/Sqrt[3]),0};

{x4,y4}={1/Sqrt[3],0};

plot[1]=Plot3D[f[x,y]//Evaluate,{x,-2,2},{y,-2,2},AxesLabel->
{"x","y","z"},PlotRange->All];

plot[2]=Graphics3D[{{PointSize[0.03],Red,Point[{{x1,y1,f[x1,y1]},
{x2,y2,f[x2,y2]}}]},{PointSize[0.03],Blue,Point[{{x3,y3,f[x3,y3]},
{x4,y4,f[x4,y4]}}]}}];

Show[Array[plot,2]]

]
```

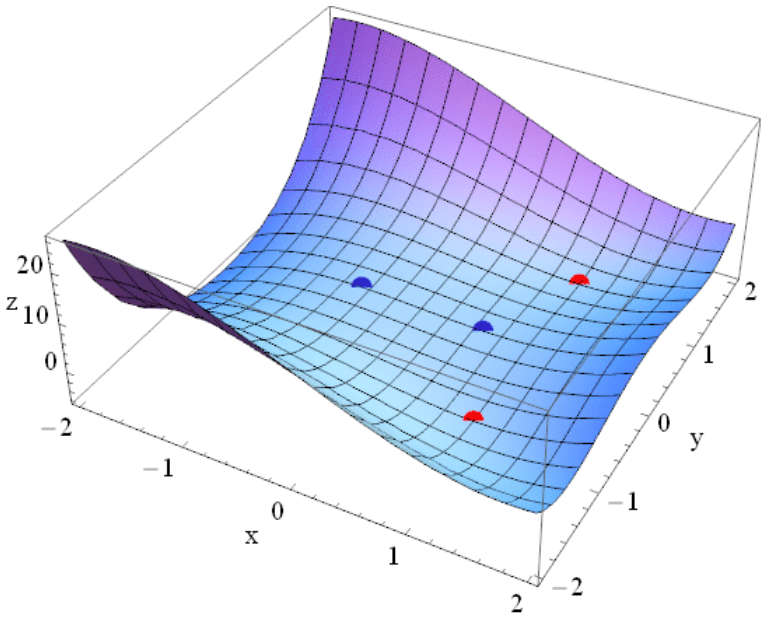


Figure 5.4. The graph of f(x,y)=y^4-2x y^2+x^3-x with local minimum (red) and saddle point (blue).

Example 5.5. (a) Find the critical points of the function:

f(x,y)=x^2-2x y+y^2.

(b) Determine the nature of the critical points in part (a).

Solution:

(a)

Step 1. Find the critical points.

$$\frac{\partial f}{\partial x} = 2x - 2y \quad , \qquad \frac{\partial f}{\partial y} = -2x + 2y \quad .$$

Setting the derivative of f equals to zero

$$Df(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = (0 \;\; 0)$$

,

we solve the system of equations

$$\begin{cases} 2x - 2y = 0, \\ -2x + 2y = 0. \end{cases}$$

However, the first and the second equation is equivalent. So, we do not have a definite answer for critical point.

```
Clear[f]

f[x_,y_]:=x^2-2x y+y^2


D[f[x,y],x]

D[f[x,y],y]


Solve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y}]
```

There are infinite number of critical points as solutions. As a matter of fact the line y=x is the solution for this system of equations.

(b)

Step 2. Compute the second partial derivatives.

$\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y\, \partial x} = -2$,

$\frac{\partial^2 f}{\partial x\, \partial y} = -2$, $\frac{\partial^2 f}{\partial y^2} = 2$.

```
(* The Hessian matrix *)

{{D[f[x,y],{x,2}],D[f[x,y],x,y]},{D[f[x,y],y,x],D[f[x,y],
{y,2}]}}//MatrixForm
```

Step 3. Apply the Second Derivative Test

The Hessian matrix:

$$Hf(x, y) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

The determinant of H f(x,y):

d=0.

By Theorem 5.2 the test is inconclusive, we need to use other method to determine the nature of the extremum.

Note that the function f(x,y)=x^2-2x y+y^2 can be written as

f(x,y)=x^2-2x y+y^2=(x-y)^2.

Clearly, it is non-negative and the minimum value for this function is zero. Thus, all points on the line y=x are the minima.

Let us plot the graph:

```
Module[{plot},

plot[1]=Plot3D[f[x,y]//Evaluate,{x,-2,2},{y,-2,2},AxesLabel->
{"x","y","z"}];

plot[2]=Graphics3D[{AbsoluteThickness[2],Red,Dashed,Line[{{-2,-2,f[-2,-
2]},{2,2,f[2,2]}}]}];

Show[Array[plot,2]]

]
```

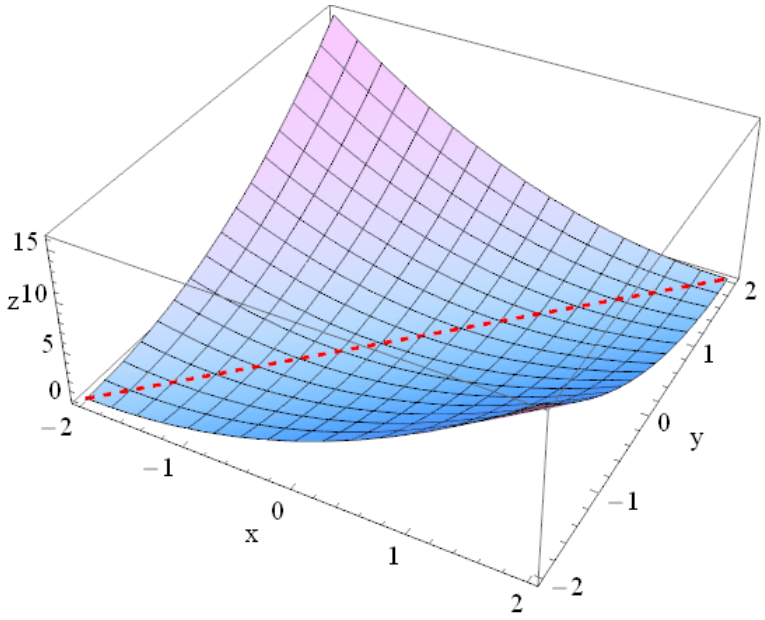


Figure 5.5. The graph of f(x,y)=x^2-2x y+y^2 and minimum points occur on the line y=x.

As can be seen from examples above, Theorem 5.2 is applicable to determine the nature of critical points of functions of two variables. However, there are cases when we need to handle problems of functions of three variables or even more. Then, we may need to extend Theorem 5.2 so that it can be used to test critical points of functions of n variable for instance. Fortunately, there is such tool available (see for example [Colley] page 268) which we state here omitting the proof.

Theorem 5.3. Second Derivative Test for local extrema (function of n variables)

Let $\mathbf{a} \equiv (a_1, a_2, \ldots, a_n)$ be the critical point of a function $f(x_1, x_2, \ldots, x_n)$ which has continuous second partial derivatives. The Hessian matrix of $f(x_1, x_2, \ldots, x_n)$ evaluated at a is given by

(5.2)

$$H f(\mathbf{a}) \equiv \begin{pmatrix} \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_1} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{a})}{\partial x_n \partial x_1} \\ \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{a})}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_n} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(\mathbf{a})}{\partial x_n \partial x_n} \end{pmatrix}$$

.

Define d_k as the determinant of the principal minors of H f(a), that is, $d_k \equiv \det H_k f(\mathbf{a})$, where $H_k f(\mathbf{a})$ is the upper leftmost $k \times k$ submatrix of H f(a) and k=1,2,3,...,n. That is,

$$d_1 = \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_1} ,$$

$$d_2 = \begin{vmatrix} \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_1} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_2} \end{vmatrix} ,$$

$$d_3 = \begin{vmatrix} \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_1} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{a})}{\partial x_3 \partial x_1} \\ \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_2} & \frac{\partial^2 f(\mathbf{a})}{\partial x_3 \partial x_2} \\ \frac{\partial^2 f(\mathbf{a})}{\partial x_1 \partial x_3} & \frac{\partial^2 f(\mathbf{a})}{\partial x_2 \partial x_3} & \frac{\partial^2 f(\mathbf{a})}{\partial x_3 \partial x_3} \end{vmatrix}$$

,

...

$$d_n = |H f(\mathbf{a})| .$$

Then:

(i) If $d_k > 0$ for k=1,2,...,n, then $f(a_1, a_2, \ldots, a_n)$ is a local minimum.

(ii) If $d_k < 0$ for k odd and $d_k > 0$ for k even, then $f(a_1, a_2, \ldots, a_n)$ is a local maximum.

(iii) If neither case (i) nor case (ii) holds, then f has a saddle point at $(a_1, a_2, \ldots, a_n)$.

(iv) If $d_n = 0$, the test is inconclusive. The critical point a is said to be degenerate.

Note that for smooth functions the Hessian matrix is symmetric.

Example 5.6. (a) Find the critical points of the function:

f(x,y,z)=2-x^4 y^4-z^4.

(b) Determine the nature of the critical points in part (a).

Solution:

(a)

Step 1. Find the critical points.

$$\frac{\partial f}{\partial x} = -4x^3y^4, \quad \frac{\partial f}{\partial y} = -4x^4y^3, \quad \frac{\partial f}{\partial z} = -4z^3.$$

To find the critical points we set the derivative of f equals to zero and solve for x, y and z

$$Df(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} = (0 \ 0 \ 0)$$

.

In component form:

$$\begin{cases} -4x^3y^4 = 0, \\ -4x^4y^3 = 0, \\ -4z^3 = 0. \end{cases}$$

There is only one critical point, that is, at the origin (0,0,0).

(b)

Step 2. Compute the second partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = -12x^2y^4, \quad \frac{\partial^2 f}{\partial y\,\partial x} = -16x^3y^3, \quad \frac{\partial^2 f}{\partial z\,\partial x} = 0,$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -16x^3y^3, \quad \frac{\partial^2 f}{\partial y^2} = -12x^4y^2, \quad \frac{\partial^2 f}{\partial z\,\partial y} = 0,$$

$$\frac{\partial^2 f}{\partial x\,\partial z} = 0, \quad \frac{\partial^2 f}{\partial y\,\partial z} = 0, \quad \frac{\partial^2 f}{\partial z^2} = -12z^2.$$

```
Clear[f]

f[x_,y_,z_]:=2-x^4 y^4-z^4


(* The Hessian matrix *)

{{D[f[x,y,z],{x,2}],D[f[x,y,z],x,y],D[f[x,y,z],x,z]},
{D[f[x,y,z],y,x],D[f[x,y,z],{y,2}],D[f[x,y,z],y,z]},
{D[f[x,y,z],z,x],D[f[x,y,z],z,y],D[f[x,y,z],{z,2}]}}//MatrixForm
```

Step 3. Apply Theorem 5.3. Second Derivative Test for local extrema

The Hessian matrix:

$$Hf(x, y, z) = \begin{pmatrix} -12x^2y^4 & -16x^3y^3 & 0 \\ -16x^3y^3 & -12x^4y^2 & 0 \\ 0 & 0 & -12z^2 \end{pmatrix}$$

.

$$d_1 = -12x^2y^4 ,$$

$$d_2 = \begin{vmatrix} -12x^2y^4 & -16x^3y^3 \\ -16x^3y^3 & -12x^4y^2 \end{vmatrix}$$

,

$$d_3 = \begin{vmatrix} -12x^2y^4 & -16x^3y^3 & 0 \\ -16x^3y^3 & -12x^4y^2 & 0 \\ 0 & 0 & -12z^2 \end{vmatrix}$$

.

At the origin (x,y,z)=(0,0,0):

$$d_1 = d_2 = d_3 = 0 .$$

By Theorem 5.3 (iv) the test is inconclusive.

We have to use other method to determine the nature of the critical point at the origin.

Observing that the value of f(x,y,z) at (0,0,0) is 2 and f(x,y,z)=2-x^4 y^4-z^4<2 for other points near the origin, we conclude that the origin is a local

maximum.

Example 5.7. (a) Find the critical points of the function:

f(x,y,z,w)=-x^2-2z^2+w^2-x y z+y w.

(b) Determine the nature of the critical points in part (a).

Solution:

(a)

Step 1. Find the critical points.

$$\frac{\partial f}{\partial x} = -2x - yz, \quad \frac{\partial f}{\partial y} = -xz + w,$$

$$\frac{\partial f}{\partial z} = -4z - xy, \quad \frac{\partial f}{\partial w} = 2w + y.$$

.

Setting the derivative of f equals to zero:

$$Df(x, y, z, w) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \quad \frac{\partial f}{\partial w} \right)$$
$$= (0 \ 0 \ 0 \ 0)$$

we solve the system of equations

$$\begin{cases} -2x - yz = 0, \\ -xz + w = 0, \\ -4z - xy = 0, \\ 2w + y = 0. \end{cases}$$

Clear[f]

f[x_,y_,z_,w_]:=-x^2-2z^2+w^2-x y z+y w

D[f[x,y,z,w],x]

D[f[x,y,z,w],y]

D[f[x,y,z,w],z]

D[f[x,y,z,w],w]

Solve[Grad[f[x,y,z,w],{x,y,z,w},"Cartesian"]=={0,0,0,0}//Thread, {x,y,z,w}]

The critical points are found at

(x,y,z,w)=(0,0,0,0),

(x,y,z,w)=(-Sqrt[2],-2 Sqrt[2],-1,Sqrt[2]),

(x,y,z,w)=(-Sqrt[2],2Sqrt[2],1,-Sqrt[2]),

(x,y,z,w)=(Sqrt[2],-2 Sqrt[2],1,Sqrt[2]),

(x,y,z,w)=(Sqrt[2],2Sqrt[2],-1,-Sqrt[2]).

(b)

Step 2. Compute the second partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial y\, \partial x} = -z, \quad \frac{\partial^2 f}{\partial z\, \partial x} = -y, \quad \frac{\partial^2 f}{\partial w\, \partial x} = 0,$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = -z, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial z\, \partial y} = -x, \quad \frac{\partial^2 f}{\partial w\, \partial y} = 1,$$

$$\frac{\partial^2 f}{\partial x\, \partial z} = -y, \quad \frac{\partial^2 f}{\partial y\, \partial z} = -x, \quad \frac{\partial^2 f}{\partial z^2} = -4, \quad \frac{\partial^2 f}{\partial w\, \partial z} = 0,$$

$$\frac{\partial^2 f}{\partial x\, \partial w} = 0, \quad \frac{\partial^2 f}{\partial y\, \partial w} = 1, \quad \frac{\partial^2 f}{\partial z\, \partial w} = 0, \quad \frac{\partial^2 f}{\partial w^2} = 2.$$

(* The Hessian matrix *)

```
{{D[f[x,y,z,w],{x,2}],D[f[x,y,z,w],x,y],D[f[x,y,z,w],x,z],D[f[x,y,z,w],x,w]},
{D[f[x,y,z,w],y,x],D[f[x,y,z,w],{y,2}],D[f[x,y,z,w],y,z],D[f[x,y,z,w],y,w]},
{D[f[x,y,z,w],z,x],D[f[x,y,z,w],z,y],D[f[x,y,z,w],{z,2}],D[f[x,y,z,w],z,w]},
{D[f[x,y,z,w],w,x],D[f[x,y,z,w],w,y],D[f[x,y,z,w],w,z],D[f[x,y,z,w],
{w,2}]}}//MatrixForm
```

Step 3. Apply Theorem 5.3. Second Derivative Test for local extrema

The Hessian matrix:

$$Hf(x, y, z, w) = \begin{pmatrix} -2 & -z & -y & 0 \\ -z & 0 & -x & 1 \\ -y & -x & -4 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

.

$$d_1 = -2\,,$$

$$d_2 = \begin{vmatrix} -2 & -z \\ -z & 0 \end{vmatrix} = -z^2\,,$$

$$d_3 = \begin{vmatrix} -2 & -z & -y \\ -z & 0 & -x \\ -y & -x & -4 \end{vmatrix} = 2x^2 - 2xyz + 4z^2$$

,

$$d_4 = \begin{vmatrix} -2 & -z & -y & 0 \\ -z & 0 & -x & 1 \\ -y & -x & -4 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix} = -8 + 4x^2 + y^2 - 4xyz + 8z^2$$

.

d2=Det[{{-2,-z},{-z,0}}]

d3=Det[{{-2,-z,-y},{-z,0,-x},{-y,-x,-4}}]

d4=Det[{{-2,-z,-y,0},{-z,0,-x,1},{-y,-x,-4,0},{0,1,0,2}}]

Critical point 1: (x,y,z,w)=(0,0,0,0):

$d_1 = -2, \quad d_2 = 0, \quad d_3 = 0, \quad d_4 = -8$

.

f has a saddle point at (x,y,z,w)=(0,0,0,0).

d2/.{x->0,y->0,z->0,w->0}

d3/.{x->0,y->0,z->0,w->0}

d4/.{x->0,y->0,z->0,w->0}

Critical point 2: (x,y,z,w)=(-Sqrt[2],-2 Sqrt[2],-1,Sqrt[2]):

$d_1 = -2, \quad d_2 = -1, \quad d_3 = 16, \quad d_4 = 32$

.

f has a saddle point at (x,y,z,w)=(-Sqrt[2],-2 Sqrt[2],-1,Sqrt[2]).

d2/.{x->-Sqrt[2],y->-2 Sqrt[2],z->-1,w->Sqrt[2]}

d3/.{x->-Sqrt[2],y->-2 Sqrt[2],z->-1,w->Sqrt[2]}

d4/.{x->-Sqrt[2],y->-2 Sqrt[2],z->-1,w->Sqrt[2]}

Critical point 3: (x,y,z,w)=(-Sqrt[2],2Sqrt[2],1,-Sqrt[2]):

$d_1 = -2, \quad d_2 = -1, \quad d_3 = 16, \quad d_4 = 32$

.

f has a saddle point at (x,y,z,w)=(-Sqrt[2],2Sqrt[2],1,-Sqrt[2]).

d2/.{x->-Sqrt[2],y->2Sqrt[2],z->1,w->-Sqrt[2]}

d3/.{x->-Sqrt[2],y->2Sqrt[2],z->1,w->-Sqrt[2]}

d4/.{x->-Sqrt[2],y->2Sqrt[2],z->1,w->-Sqrt[2]}

Critical point 4: (x,y,z,w)=(Sqrt[2],-2 Sqrt[2],1,Sqrt[2]):

$d_1 = -2, \quad d_2 = -1, \quad d_3 = 16, \quad d_4 = 32$

.

f has a saddle point at (x,y,z,w)=(Sqrt[2],-2 Sqrt[2],1,Sqrt[2]).

d2/.{x->Sqrt[2],y->-2 Sqrt[2],z->1,w->Sqrt[2]}

d3/.{x->Sqrt[2],y->-2 Sqrt[2],z->1,w->Sqrt[2]}

d4/.{x->Sqrt[2],y->-2 Sqrt[2],z->1,w->Sqrt[2]}

Critical point 5: (x,y,z,w)=(Sqrt[2],2Sqrt[2],-1,-Sqrt[2]):

$d_1 = -2, \quad d_2 = -1, \quad d_3 = 16, \quad d_4 = 32$

.

f has a saddle point at (x,y,z,w)=(Sqrt[2],2Sqrt[2],-1,-Sqrt[2]).

d2/.{x->Sqrt[2],y->2Sqrt[2],z->-1,w->-Sqrt[2]}

d3/.{x->Sqrt[2],y->2Sqrt[2],z->-1,w->-Sqrt[2]}

d4/.{x->Sqrt[2],y->2Sqrt[2],z->-1,w->-Sqrt[2]}

So, all the critical points of the function f(x,y,z,w)=-x^2-2z^2+w^2-x y z+y w are saddle points. Note that there is no available tool to plot a five-dimensional hypersurface.

5.2 Global Extrema

Thus far, we have considered only local extrema. These are, local maximum, local minimum and saddle point. However, sometimes we are also interested in finding the maximum and minimum value of a function on a certain type of domain $\mathcal{D}$. These are called global or absolute extreme values of the function. The type of domain we consider is said to be compact, which is defined below:

Definition 5.4. Compact set in $\mathbb{R}^3$

A domain $\mathcal{D} \subseteq \mathbb{R}^3$ is bounded if there is a positive number $\mathcal{M}$ such that $\|\mathbf{x}\| < \mathcal{M}$ for all $\mathbf{x} \in \mathcal{D}$. The domain $\mathcal{D}$ is said to be closed if it contains all its boundary points.

The domain $\mathcal{D} \subseteq \mathbb{R}^3$ is said to be compact if it is both closed and bounded.

Definition 5.4 can be generalized to n-dimension, where n can be 1, 2 or higher numbers. In this book we will consider only n<=3.

Example 5.8. Compact domain

Figure 5.6 shows a cuboid domain. It is bounded (contained in a sphere of radius 2 centered at the origin) and closed (contains all boundary points including the edges). So, the cuboid domain is compact in the interval:

-1<=x<=1, -1<=y<=1, -1<=z<=1.

```
Module[{plot},

plot[1]=Graphics3D[{EdgeForm[{AbsoluteThickness[2],Blue}],FaceForm[
Yellow],Cuboid[{-1,-1,-1},{1,1,1}]}];

plot[2]=Graphics3D[{Opacity[0.5],Sphere[{0,0,0},2]}];

Show[Array[plot,2],Axes->True,AxesLabel->{"x","y","z"},PlotRange->
{{-2,2},{-2,2},{-2,2}}]

]
```

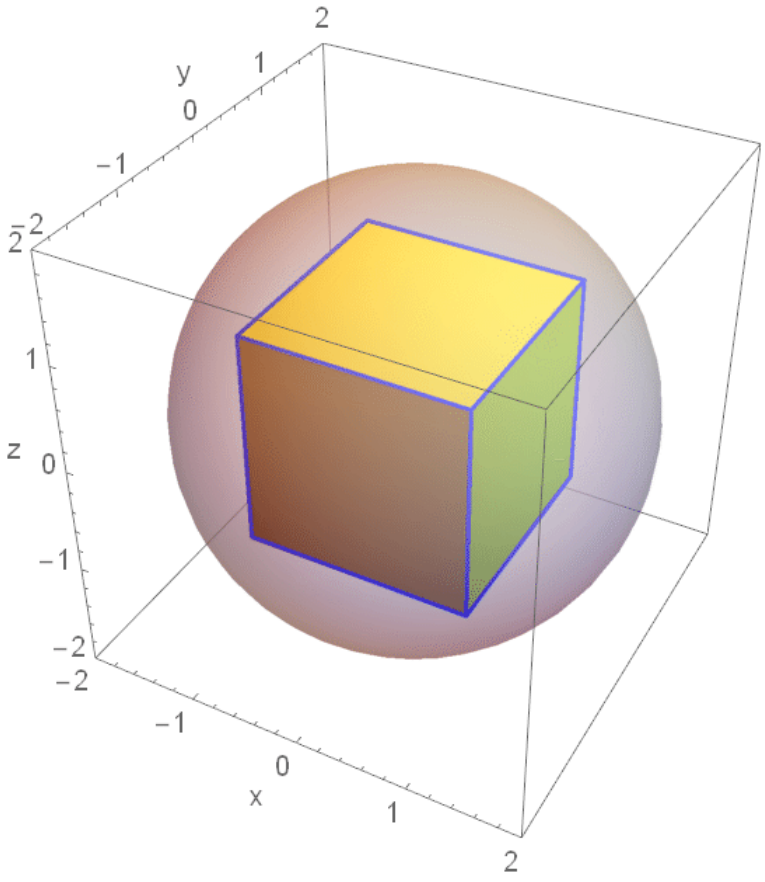


Figure 5.6. The cuboid domain is compact in the interval: -1<=x<=1, -1<=y<=1, -1<=z<=1.

Now, we are ready to state our next theorem without proof.

Theorem 5.5. Extreme Value Theorem

If $\mathcal{D} \subseteq \mathbb{R}^3$ is compact and let f(x,y,z) be a continuous function on $\mathcal{D}$. Then:

(i) f(x,y,z) must have both a global maximum and a global minimum value on $\mathcal{D}$.

(ii) The extreme values of f(x,y,z) occur either at critical points in the interior of $\mathcal{D}$ or at points on the boundary of $\mathcal{D}$.

Hence, we must compare the values of f(x,y,z) at any interior critical points with those at the boundary of $\mathcal{D}$ to determine which is largest and smallest.

Although Theorem 5.5 is stated for function of three variables, it is applicable to function of two variables as well as it is given in the following example.

Example 5.9. Find the global extrema of f(x,y)=x^2+x y+y^2-6y on the rectangle { (x,y)|-3<=x<=3, 0<=y<=5 }.

Solution:

Clearly, the rectangle

$$\mathcal{D} \equiv \{(x, y) \mid -3 \le x \le 3,\ 0 \le y \le 5\}$$

(Figure 5.7) is closed because it contains all the points that make up its boundary. It is also bounded because there is some (open or closed) disk that contains it. Hence, the domain $\mathcal{D}$ is compact.

By Theorem 5.5, the extremum occurs either at a critical point or on the boundary of the rectangle (Figure 5.7).

```
Module[{domain},

domain=Graphics[{EdgeForm[{AbsoluteThickness[2],Blue}],FaceForm[Ye
llow],Rectangle[{-3,0},{3,5}]}];

Show[domain,Axes->True,PlotRange->{{-4,4},{-1,6}},Ticks->{{-3,3},
{0,5}},AxesLabel->{"x","y"}]

]
```

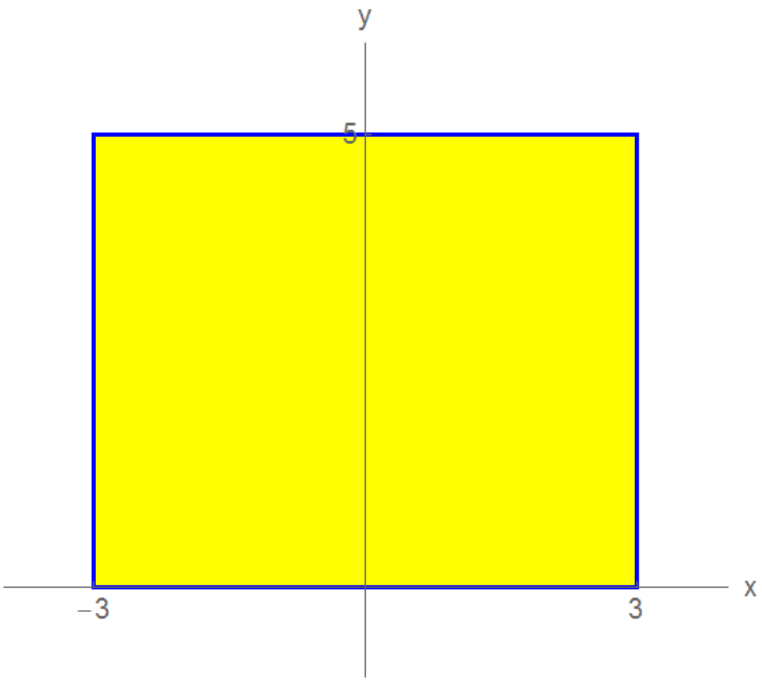


Figure 5.7. The domain

$$\mathcal{D} = \{(x, y) \mid -3 \le x \le 3,\ 0 \le y \le 5\}$$

.

Step 1. Find the critical points.

We compute the partial derivatives

$$\frac{\partial f}{\partial x} = 2x + y, \qquad \frac{\partial f}{\partial y} = x + 2y - 6,$$

and set the derivative of f equals to zero

$$Df(x, y) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = (0 \quad 0)$$

.

Solving the system of equations

$$\begin{cases} 2x + y = 0, \\ x + 2y - 6 = 0. \end{cases}$$

we found that there is a critical point at (x,y)=(-2,4) and f(-2,4)=-12.

```
Clear[f]

f[x_,y_]:=x^2+x y +y^2-6 y

Solve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y}]

f[-2,4]
```

Step 2. Check the boundary.

Side 1. The bottom edge of $\mathcal{D}$ is the set

$$E_1 = \{(x, y) \mid y = 0, -3 \le x \le 3\}$$

.

On this edge, f is given by

$$f_1(x) \equiv f(x, 0) = x^2$$.

as $f_1'(x) = 2x$, the function f_1 has a critical point at x=0.

The possible extrema are: (0,0), (-3,0), (3,0).

The smallest value of $f_1(x)$ is 0. The largest value of $f_1(x)$ is 9.

Side 2. The top edge of $\mathcal{D}$ is the set

$$E_2 = \{(x, y) \mid y = 5, -3 \le x \le 3\}$$

.

Consequently, we define f_2 by

$$f_2(x) \equiv f(x, 5) = x^2 + 5x + 25 - 30 = x^2 + 5x - 5$$

.

Thus, f_2 is the restriction of f(x,y) to E_2 .

We calculate $f_2'(x) = 2x + 5$, which implies that x=-(5/2) is a critical point of f_2 .

Hence, we consider (-(5/2),5), (-3,5), (3,5) for possible extrema.

Side 3. The left edge of $\mathcal{D}$ is the set

$$E_3 = \{(x, y) \mid x = -3,\ 0 \le y \le 5\}$$

.

Thus, the restriction of f(x,y) to E_3 is given by

$$f_3(y) \equiv f(-3, y) = 9 - 3y + y^2 - 6y = y^2 - 9y + 9$$

.

We have $f_3'(y) = 2y - 9$, and so y=9/2 is the critical point of f_3 .

The possible extrema are: (-3,9/2), (-3,0), (-3,5). But the last two has already been considered in previous analysis.

So, there is only one critical point of f_3 to be considered here, that is at (-3,9/2).

Side 4. The right edge of $\mathcal{D}$ is the set

$$E_4 = \{(x, y) \mid x = 3,\ 0 \le y \le 5\}$$

.

On E_4 , the function f(x,y) is given by

$$f_4(y) \equiv f(3, y) = 9 + 3y + y^2 - 6y = y^2 - 3y + 9$$

.

So, we have $f_4'(y) = 2y - 3$, and so y=3/2 is the critical point of f_4 .

The possible extrema are: (3,3/2), (3,0), (3,5). But the last two has already been considered in previous analysis.

So, there is only one critical point at (3,3/2) to be considered as possible extrema.

As a summary, there are nine possible locations for global extrema. These are,

	Interior of $\mathcal{D}$	**Boundary** of $\mathcal{D}$
Critical points :	$(-2, 4)$	$(0, 0), (-3, 0), (3, 0), \left(-\frac{5}{2}, 5\right), (-3, 5), (3, 5), \left(-3, \frac{9}{2}\right), \left(3, \frac{3}{2}\right).$

Module[{domain,point,points},

point[1]={-2,4};

point[2]={0,0};

point[3]={-3,0};

point[4]={3,0};

point[5]={-(5/2),5};

point[6]={-3,5};

point[7]={3,5};

point[8]={-3,9/2};

point[9]={3,3/2};

```
points=Graphics[{Red,PointSize[0.03],Point[Array[point,9]]}];

domain=Graphics[{EdgeForm[{AbsoluteThickness[2],Blue}],FaceForm[Ye
llow],Rectangle[{-3,0},{3,5}]}];

Show[domain,points,Axes->True,PlotRange->{{-4,4},{-1,6}},Ticks->
{{-3,3},{0,5}},AxesLabel->{"x","y"}]

]
```

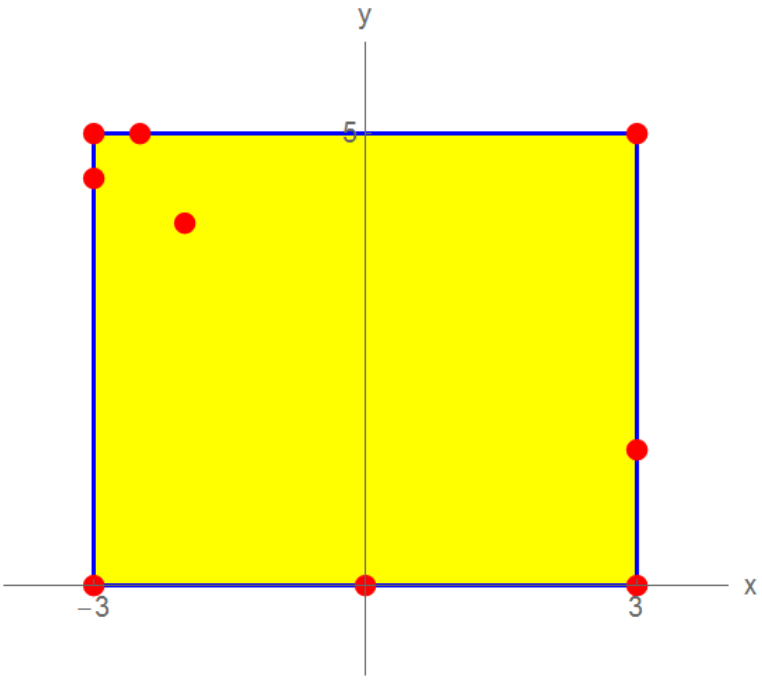


Figure 5.8. The possible extrema on the domain

$$\mathcal{D} = \{(x, y) \mid -3 \le x \le 3,\ 0 \le y \le 5\}$$

.

Step3. Compare the values of f.

Next, we compare the values of f(x,y) at these points:

{f[-2,4],f[0,0],f[-3,0],f[3,0],f[-(5/2),5],f[-3,5],f[3,5],f[-3,9/2],f[3,3/2]}

So, we obtain the global maximum of f(x,y) is 19 at the point (x,y)=(3,5).

The global minimum of f(x,y) is -12 at the point (x,y)=(-2,4).

```
Module[{plot,x1,x2,x3,x4,x5,x6,x7,x8,x9,y1,y2,y3,y4,y5,y6,y7,y8,y9},
{x1,y1}={-2,4};
{x2,y2}={0,0};
{x3,y3}={-3,0};
{x4,y4}={3,0};
{x5,y5}={-(5/2),5};
{x6,y6}={-3,5};
{x7,y7}={3,5};
{x8,y8}={-3,9/2};
{x9,y9}={3,3/2};
plot[1]=Plot3D[f[x,y]//Evaluate,{x,-3,3},{y,0,5},AxesLabel->
{"x","y","z"}];
plot[2]=Graphics3D[{PointSize[0.03],{Yellow,Point[{x1,y1,f[x1,y1]}]},
{Black,Point[{x7,y7,f[x7,y7]}]},{Red,Point[{{x2,y2,f[x2,y2]},
{x3,y3,f[x3,y3]},{x4,y4,f[x4,y4]},{x5,y5,f[x5,y5]},{x6,y6,f[x6,y6]},
{x8,y8,f[x8,y8]},{x9,y9,f[x9,y9]}}]}}];
```

Show[Array[plot,2]]

]

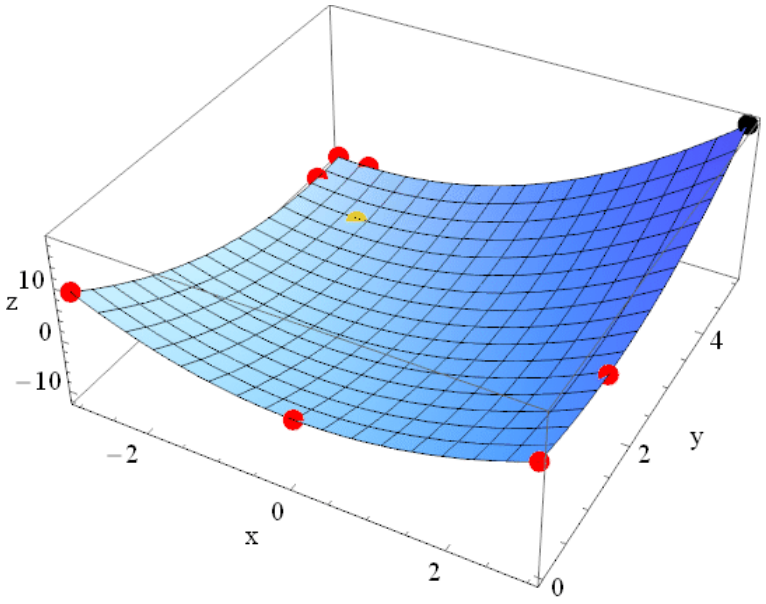


Figure 5.9. The graph of f(x,y)=x^2+x y+y^2-6y with the global maximum (black) and global minimum (yellow).

Example 5.10. Find the global extrema of f(x,y,z)=x^2+x z-y^2+2z^2+x y+5x on the cuboid domain { (x,y,z)|-5<=x<=0, 0<=y<=3,0<=z<=2 } (see Figure 5.10).

Solution:

By Definition 5.4, the cuboid

$$\mathcal{D} \equiv \{(x, y, z) \mid -5 \le x \le 0,\ 0 \le y \le 3,\ 0 \le z \le 2\}$$

(Figure 5.10) is compact.

By Theorem 5.5, the extremum occurs either at a critical point or on the boundary of the cuboid domain.

```
Module[{domain},

domain=Graphics3D[{EdgeForm[{AbsoluteThickness[2],Blue}],FaceForm
[{Opacity[0.1]}],Cuboid[{0,0,0},{-5,3,2}]}];

Show[domain,Axes->True,AxesLabel->{"x","y","z"}]

]
```

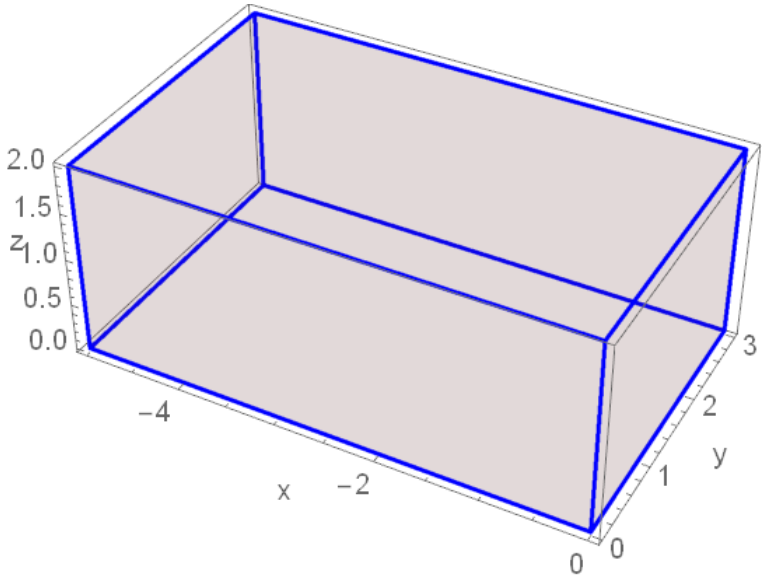


Figure 5.10. The cuboid domain

$$\mathcal{D} \equiv \{(x, y, z) \mid -5 \le x \le 0,\ 0 \le y \le 3,\ 0 \le z \le 2\}$$

.

Step 1. Find the critical points.

We compute the partial derivatives

$$\frac{\partial f}{\partial x} = 2x + y + z + 5, \quad \frac{\partial f}{\partial y} = x - 2y, \quad \frac{\partial f}{\partial z} = x + 4z$$

.

and set the derivative of f equals to zero

$$Df(x, y, z) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right) = (0 \quad 0 \quad 0)$$

.

This yields the system of equations to solve:

$$\begin{cases} 2x + y + z + 5 = 0, \\ x - 2y = 0, \\ x + 4z = 0. \end{cases}$$

There is only one critical point at (x,y,z)=(-(20/9),-(10/9),5/9) and f(-(20/9),-(10/9),5/9)=-(50/9).

However, the critical point is not in $\mathcal{D}$. And we still need to check the value of f along the boundary of the domain.

Clear[f]

f[x_,y_,z_]:=x^2+x z -y^2+2 z^2+x y+5x

D[f[x,y,z],x]

D[f[x,y,z],y]

D[f[x,y,z],z]

Solve[Grad[f[x,y,z],{x,y,z},"Cartesian"]=={0,0,0}//Thread,{x,y,z}]

f[-(20/9),-(10/9),5/9]

Step 2. Check the boundary.

Side 1. The front xz-plane of $\mathcal{D}$ is the set

$$F_1 = \{(x, y, z) \mid y = 0, -5 \le x \le 0, 0 \le z \le 2\}$$

.

On this side, f is given by

$$f_1(x, z) \equiv f(x, 0, z) = x^2 + x z - y^2 + 2 z^2 + x y + 5 x$$
$$= x^2 + x z + 2 z^2 + 5 x$$

.

f_1 is a function of two variables. To find the critical points, we use the same method for treating function of two variables, so

$$\frac{\partial f_1}{\partial x} = 2 x + z + 5, \quad \frac{\partial f_1}{\partial z} = 4 z + x$$

.

The system of equations to solve:

$$\begin{cases} 2 x + z + 5 = 0, \\ 4 z + x = 0. \end{cases}$$

The function f_1 has a critical point at (x,z)=(-(20/7),5/7).

Solve[{2x+z+5==0,4z+x==0},{x,z}]

Together with the boundary points, the possible extrema are

(-(20/7),0,5/7), (-5,0,0), (0,0,0),(-5,0,2),(0,0,2).

Side 2. The front yz-plane of $\mathcal{D}$ is the set

$$F_2 = \{(x, y, z) \mid x = 0,\ 0 \le y \le 3,\ 0 \le z \le 2\}$$

.

The restriction of f(x,y,z) to F_2 defines a new function f_2 given by

$$f_2(y, z) \equiv f(0, y, z) = x^2 + x z - y^2 + 2 z^2 + x y + 5 x$$
$$= -y^2 + 2 z^2$$

.

The partial derivatives:

$$\frac{\partial f_2}{\partial y} = -2 y, \quad \frac{\partial f_2}{\partial z} = 4 z$$.

Solving the system of equations:

$$\begin{cases} -2 y = 0, \\ 4 z = 0. \end{cases}$$

we obtain the critical point at (y,z)=(0,0).

Including boundary points, these are the candidates for possible extrema:

(0,0,0), (0,3,0), (0,3,2), (0,0,2).

Some points are already included in previous analysis, so there are only two points of f_2 for possible extrema:

(0,3,0) and (0,3,2).

Side 3. The back xz-plane of $\mathcal{D}$ is the set

$$F_3 = \{(x, y, z) \mid y = 3,\ -5 \le x \le 0,\ 0 \le z \le 2\}$$

.

On this side, the restriction of f(x,y,z) to F_3 defines a new function f_3 given by

$$\begin{aligned} f_3(x, z) \equiv f(x, 3, z) &= x^2 + x z - y^2 + 2 z^2 + x y + 5 x \\ &= x^2 + x z - 9 + 2 z^2 + 3 x + 5 x \\ &= x^2 + x z + 2 z^2 + 8 x - 9 \end{aligned}$$

So, the partial derivatives are

$$\frac{\partial f_3}{\partial x} = 2 x + z + 8, \quad \frac{\partial f_3}{\partial z} = 4 z + x$$

.

The system of equations to solve:

$$\begin{cases} 2 x + z + 8 = 0, \\ 4 z + x = 0. \end{cases}$$

The function f_3 has a critical point at (x,z)=(-(32/7),8/7).

Solve[{2x+z+8==0,4z+x==0},{x,z}]

These are the possible extrema:

(-(32/7),3,8/7), (-5,3,0), (-5,3,2), (0,3,0),(0,3,2).

Since some points are already included in previous analysis, we consider only these points:

(-(32/7),3,8/7), (-5,3,0), (-5,3,2).

Side 4. The back yz-plane of $\mathcal{D}$ is the set

$$F_4 = \{(x, y, z) \mid x = -5,\ 0 \le y \le 3,\ 0 \le z \le 2\}$$

.

On this side, f is given by

$$\begin{aligned} f_4(y, z) \equiv f(-5, y, z) &= x^2 + x z - y^2 + 2 z^2 + x y + 5 x \\ &= 25 - 5 z - y^2 + 2 z^2 - 5 y - 25 \\ &= -y^2 + 2 z^2 - 5 z - 5 y \end{aligned}$$

The partial derivatives:

$$\frac{\partial f_4}{\partial y} = -2 y - 5, \quad \frac{\partial f_4}{\partial z} = 4 z - 5$$

.

Solving the system of equations:

$$\begin{cases} -2 y - 5 = 0, \\ 4 z - 5 = 0, \end{cases}$$

we obtain the critical point (y,z)=(-(5/2),5/4).

The possible extrema are: (-5,-(5/2),5/4), (-5,3,0), (-5,3,2), (-5,0,0), (-5,0,2).

However, the point (-5,-(5/2),5/4) is not in our domain while others have been included above, so there is none for this side to consider.

Side 5. The bottom xy-plane of $\mathcal{D}$ is the set

$$F_5 = \{(x, y, z) \mid z = 0, -5 \le x \le 0, 0 \le y \le 3\}$$

.

On this side, the restriction of f(x,y,z) to F_5 defines a new function f_5 given by

$$f_5(x, y) \equiv f(x, y, 0) = x^2 + x z - y^2 + 2 z^2 + x y + 5 x$$
$$= x^2 - y^2 + x y + 5 x$$

The partial derivatives:

$$\frac{\partial f_5}{\partial x} = 2 x + y + 5, \quad \frac{\partial f_5}{\partial y} = -2 y + x$$

.

Solving the system of equations:

$$\begin{cases} 2 x + y + 5 = 0, \\ -2 y + x = 0, \end{cases}$$

we obtain the only critical point of f_5 at (x,y)=(-2,-1).

Solve[{2x+y+5==0,-2y+x==0},{x,y}]

The point (-2,-1,0) is not in our domain, and all vertices on this side have been included in previous analysis, so there is none to consider for this side.

Side 6. The top xy-plane of $\mathcal{D}$ is the set

$$F_6 = \{(x, y, z) \mid z = 2, -5 \le x \le 0, 0 \le y \le 3\}$$

.

The restriction of f(x,y,z) to F_6 defines a new function f_6 given by

$$\begin{aligned} f_6(x, y) \equiv f(x, y, 2) &= x^2 + x z - y^2 + 2 z^2 + x y + 5 x \\ &= x^2 + 2 x - y^2 + 8 + x y + 5 x \\ &= x^2 - y^2 + x y + 7 x + 8 \end{aligned}$$

The partial derivatives:

$$\frac{\partial f_6}{\partial x} = 2 x + y + 7, \quad \frac{\partial f_6}{\partial y} = -2 y + x$$

.

Solving the system of equations:

$$\begin{cases} 2 x + y + 7 = 0, \\ -2 y + x = 0, \end{cases}$$

the function f_6 has a critical point at (x,y)=(-(14/5),-(7/5)).

Solve[{2x+y+7==0,-2y+x==0},{x,y}]

Since all vertices are already accounted for in previous analysis and the point (-(14/5),-(7/5),2) is not in our domain, there is none to consider for this side.

In summary, there are fourteen possible locations for global extrema including those that are not defined in our domain (these are highlighted):

$$\mathcal{D} \equiv \{(x, y, z) \mid -5 \le x \le 0,\ 0 \le y \le 3,\ 0 \le z \le 2\}$$

.

	Interior of $\mathcal{D}$	**Boundary points** of $\mathcal{D}$
Critical points :	$\left(-\frac{20}{9}, -\frac{10}{9}, \frac{5}{9}\right)$	$\left(-\frac{20}{7}, 0, \frac{5}{7}\right)$, $(-5, 0, 0)$, $(0, 0, 0)$, $(-5, 0, 2)$, $(0, 0, 2)$, $(0, 3, 0)$, $(0, 3, 2)$, $\left(-\frac{32}{7}, 3, \frac{8}{7}\right)$, $(-5, 3, 0)$, $(-5, 3, 2)$, $\left(-5, -\frac{5}{2}, \frac{5}{4}\right)$, $(-2, -1, 0)$, $\left(-\frac{14}{5}, -\frac{7}{5}, 2\right)$.

Module[{domain,point,points},

point[1]={-(20/7),0,5/7};

point[2]={-5,0,0};

point[3]={0,0,0};

point[4]={-5,0,2};

point[5]={0,0,2};

```
point[6]={0,3,0};

point[7]={0,3,2};

point[8]={-(32/7),3,8/7};

point[9]={-5,3,0};

point[10]={-5,3,2};

points=Graphics3D[{Red,PointSize[0.03],Point[Array[point,10]]}];

domain=Graphics3D[{EdgeForm[{AbsoluteThickness[2],Blue}],FaceForm
[{Opacity[0.1]}],Cuboid[{0,0,0},{-5,3,2}]}];

Show[domain,points,Axes->True,AxesLabel->{"x","y"}]

]
```

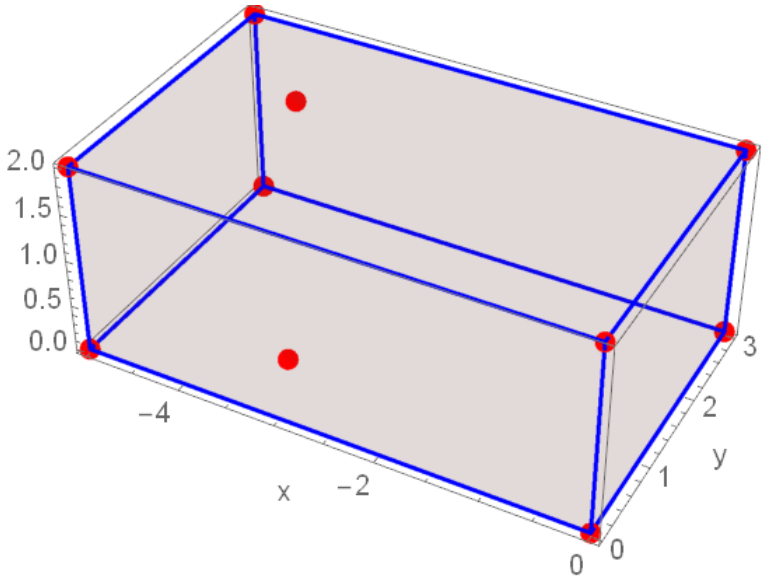


Figure 5.11. The possible extrema on the cuboid domain { (x,y,z)|-5<=x<=0, 0<=y<=3,0<=z<=2 }.

```
{f[-(20/7),0,5/7],f[-5,0,0],f[0,0,0],f[-5,0,2],f[0,0,2],f[0,3,0],f[0,3,2],f[-
(32/7),3,8/7],f[-5,3,0],f[-5,3,2]}//N
```

Comparing the results above, we conclude that

Global (absolute) maximum of f(x,y,z) is 8 at (0,0,2).

Global (absolute) minimum of f(x,y,z) is -27.2857 at (-(32/7),3,8/7).

Note that the function f(x,y,z) is four-dimensional hypersurface, so it is not possible to graph it in three-dimensional graphics. However, we still can illustrate by graph the global maximum and minimum of the function f confined on the domain $\mathcal{D}$ by reducing the dimension to three-dimensional surface by plotting the level surfaces of f(x,y,z) at these extreme values, and together with the corresponding cuboid domain as is shown in Figure 5.12. Notice that the two extrema on the domain touches the level surfaces of f at f(x,y,z)=8 and f(x,y,z)=-27.2857, respectively.

```
Module[{domain,plot,max,min},

max={0,0,2};

min={-(32/7),3,8/7};

domain=Graphics3D[{EdgeForm[{AbsoluteThickness[2],Red}],FaceForm[
{Opacity[0.1]}],Cuboid[{0,0,0},{-5,3,2}]}];

plot=ContourPlot3D[{f[x,y,z]==8,f[x,y,z]==-27.2857},{x,-10,5},{y,-3,6},
{z,-2,4},Mesh->None,ContourStyle->{{Yellow,Opacity[0.2]},
{Red,Opacity[0.4]}}];

points=Graphics3D[{{PointSize[0.02],Blue,Point[max]},
{PointSize[0.02],Red,Point[min]}}];
```

Show[domain,plot,points,Axes->True,AxesLabel->
{"x","y","z"},ImageSize->400]

]

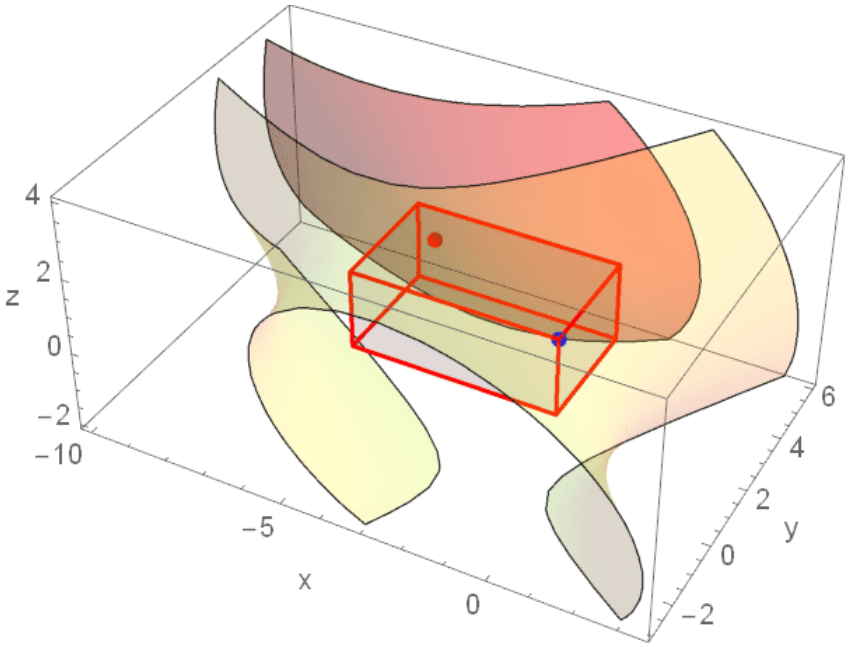


Figure 5.12. The global maximum and minimum of the function f(x,y,z)=x^2+x z-y^2+2z^2+x y+5x plotted as level surfaces of f(x,y,z)=8 and f(x,y,z)=-27.2857, respectively; confined on the cuboid domain

$$\mathcal{D} \equiv \{(x, y, z) \mid -5 \le x \le 0,\ 0 \le y \le 3,\ 0 \le z \le 2\}$$

.

5.3 Lagrange Multipliers: Constrained Extrema

Thus far, our discussion has been finding extrema of a function without constraints. Although we have considered functions confined on certain type of domains, these are not the kind of constraints we are going to discuss in this section. In some optimization problems we may need to find the extreme of a function subject to one or more constraint functions. For example, we want to find the extreme values of a function f(x,y) subject to a constraint g(x,y)=0. The method of solving this type of problems is called the Lagrange Multipliers method which is stated in the following theorem without proof:

Theorem 5.6. Lagrange Multipliers

Let $\mathcal{D} \subseteq \mathbb{R}^2$ and $f, g : (x, y) \in \mathcal{D} \to \mathbb{R}$ be differentiable. If f(x,y) has an extremum on the constraint g(x,y)=c at a point (a,b), and if $\nabla g(a, b) \neq \mathbf{0}$, then there is a scalar λ such that

(5.3) $$\nabla f(a, b) = \lambda \nabla g(a, b)$$.

Equation (5.3) is referred to as the Lagrange condition and the parameter λ is called the Lagrange multiplier.

Though Theorem 5.6 is stated for functions of two variables, it can be generalized for functions of three variables. As a matter of fact, the method of Lagrange multipliers is valid in any number of variables and for multiple constraints. In such cases, the Lagrange condition becomes

(5.4)

$$\nabla f(a_1, a_2, \ldots, a_n) = \lambda_1 \nabla g_1(a_1, a_2, \ldots, a_n) + \lambda_2 \nabla g_2(a_1, a_2, \ldots, a_n),$$
$$+ \ldots + \lambda_k \nabla g_k(a_1, a_2, \ldots, a_n),$$

where

(5.5)
$$\begin{cases} g_1(x_1, x_2, \ldots, x_n) = c_1, \\ g_2(x_1, x_2, \ldots, x_n) = c_2, \\ \cdots \\ g_k(x_1, x_2, \ldots, x_n) = c_k, \end{cases}$$ and k

Example 5.11. Use Lagrange multipliers method to find the extreme values of

f(x,y)=x^3+x y+y^3,

subject to the given constraint:

g(x,y)=x^3-x y+y^3=1.

Solution:

Step 1. Write out the Lagrange equations.

The constraint is

g(x,y)=1, where g(x,y)=x^3-x y+y^3.

So, the gradient of f and g are

$$\nabla f = \left(3x^2 + y, 3y^2 + x\right), \quad \nabla g = \left(3x^2 - y, 3y^2 - x\right).$$

The Lagrange conditions:

$$\nabla f(x, y) = \lambda \nabla g(x, y),$$

when written in component forms are the Lagrange equations:

$$\begin{cases} 3x^2 + y = \lambda(3x^2 - y), \\ 3y^2 + x = \lambda(3y^2 - x). \end{cases}$$

Step 2. Solve for λ in terms of x and y.

$$\lambda = \frac{3x^2 + y}{3x^2 - y}, \quad \lambda = \frac{3y^2 + x}{3y^2 - x}.$$

Step 3. Solve for x and y using the constraint.

Since the two expressions for λ must be equal, we have

(3x^2 + y)/(3x^2 - y)=(3y^2 + x)/(3y^2 - x).

Solving the above equation, we obtain (discarding complex solutions):

y=x.

Solve[(3x^2+y)/(3x^2-y)==(3y^2+x)/(3y^2-x),{x,y}]

Using the constraint:

g(x,y)=x^3-x y+y^3=1 and y=x,

we get

2x^3-x^2=1,

2x^3-x^2-1=0,

Solve[2x^3-x^2-1==0,x]

Taking only the real solution we obtain the extremum point at (x,y)=(1,1).

Step 4. Find the extreme value.

The extreme value of f(x,y) at (x,y)=(1,1) is 3.

```
Clear[f,g]

f[x_,y_]:=x^3+x y+y^3

g[x_,y_]:=x^3-x y+y^3
```

Let us plot some graphs:

```
Module[{plot},

plot[1]=Plot3D[f[x,y],{x,-3,3},{y,-3,3},AxesLabel->{"x","y","z"}];

plot[2]=ContourPlot[g[x,y],{x,-3,3},{y,-3,3},AxesLabel->
{"x","y","z"},Contours->{1}];

GraphicsGrid[{Table[plot[i],{i,1,2}]},ImageSize->500]

]
```

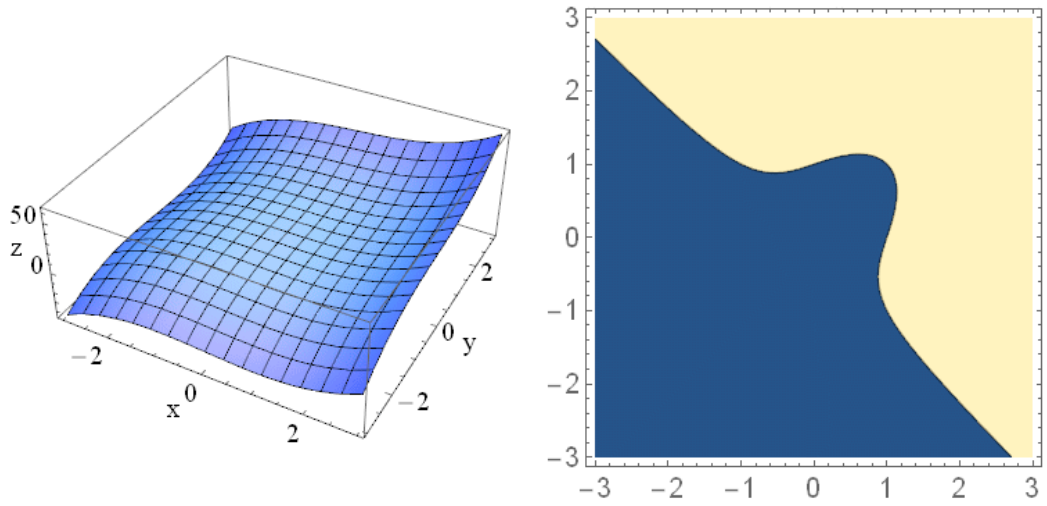


Figure 5.13. The graph of f(x,y)=x^3+x y+y^3 and the level curve of the constraint function g(x,y)=x^3-x y+y^3=1.

To analyze the nature of the critical point we use ContourPlot to plot the level curves of both functions:

```
Module[{plot,x,y},

{x,y}={1,1};

plot[1]=ContourPlot[f[x,y],{x,-3,3},{y,-3,3},Contours->
{-5,-3,-1,0,1,3,5},ContourShading->None,ContourLabels-
>True,ContourStyle->Green];

plot[2]=ContourPlot[g[x,y]==1,{x,-3,3},{y,-3,3},ContourShading-
>None,ContourLabels->True];

plot[3]=Graphics[{PointSize[0.02],Red,Point[{x,y}]}];

Show[Array[plot,3]]

]
```

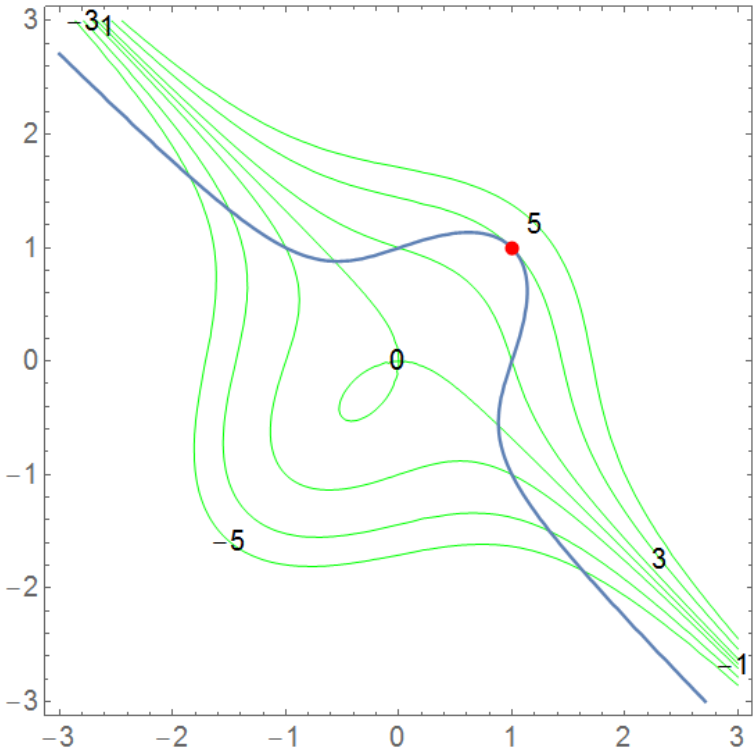


Figure 5.14. The level curves of the function to be optimized f(x,y)=x^3+x y+y^3 and the constraint g(x,y)=x^3-x y+y^3=1 meet at the extreme point (1,1).

Figure 5.14 shows the level curve of f(x,y)=3 is tangent to the constraint curve g(x,y)=1 at the extreme point (x,y)=(1,1). Since, all other level curves of f(x,y) that intersect the level curves of g(x,y)=1 have values lower than 3, we conclude that the extremum is a local maximum of f(x,y) subject to the constraint. Looking at the behavior of the level curves of f(x,y) it is likely that this is also the global maximum of f(x,y).

Example 5.12. Suppose a toy manufacturer is investing in x units of labor (at $75 per unit) and y units of capital (at $60 per unit) to produce P(x,y)=100x^0.25 y^0.75 toys. Find the maximum production level P on a budget of $250,000.

Solution:

The total cost of production is the constraint in this problem.

Thus, the constraint is given by

g(x,y)=75x+60y=250000.

Step 1. Write out the Lagrange equations.

Write

f(x,y)=P(x,y)=100x^0.25 y^0.75.

$\nabla f = \left(25\, x^{-0.75}\, y^{0.75},\ 75\, x^{0.25}\, y^{-0.25}\right)$

, $\nabla g = (75,\ 60)$.

Using the Lagrange condition:

$\nabla f(x,\ y) = \lambda \nabla g(x,\ y)$,

we get the Lagrange equations:

$$\begin{cases} 25\, x^{-0.75}\, y^{0.75} = 75\, \lambda, \\ 75\, x^{0.25}\, y^{-0.25} = 60\, \lambda. \end{cases}$$

Step 2. Solve for λ in terms of x and y.

λ =1/3 (y/x)^0.75, λ =5/4 (y/x)^-0.25.

The two expressions for λ must be equal, so

1/3 (y/x)^0.75=5/4 (y/x)^-0.25.

Step 3. Solve for x and y using the constraint.

Multiplying both sides with 3(y/x)^0.25, we get

y/x=15/4, y=3.75x.

Solve[{25x^-0.75 y^0.75==75λ,75x^0.25 y^-0.25==60λ}]

Substituting y=3.75x into the constraint

g(x,y)=75x+60y=250000.

we obtain

x=833.333 and y=3125.

solx=Solve[75x+60y==250000/.y->3.75x,x]

soly=3.75*x /.solx[[1]]

Step 4. Find the maximum production level

The maximum production level is obtained by substituting x=833.3333 and y=3125 into P(x,y) to yield 224565 toys with a cost per toy of 250000/224565, or about $1.11.

Note that we have assumed that the extremum is a maximum as stated by the problem. This is justifiable since exponential functions are increasing, so we

expect it to hit a point where it meets the constraint as the maximum point subject to this constraint. Actually, there is a Second Derivative Test which will be discussed in the following where we can use to determine the nature of the extremum precisely.

```
Clear[P]

P[x_,y_]:=100x^0.25 y^0.75



P[x,y]/.solx[[1]]/.y->soly



250000/224565//N



Module[{plot,x,y},

x=833.333;

y=3125;

plot[1]=ContourPlot[P[x,y],{x,0,1500},{y,2500,3500},Contours->
{150000,200000,224565,250000}];

plot[2]=ContourPlot[75x+60y==250000,{x,0,1500},
{y,2500,3500},ContourStyle->{AbsoluteThickness[2],Red}];

plot[3]=Graphics[{PointSize[0.03],Blue,Point[{x,y}]}];

Show[Array[plot,3]]

]
```

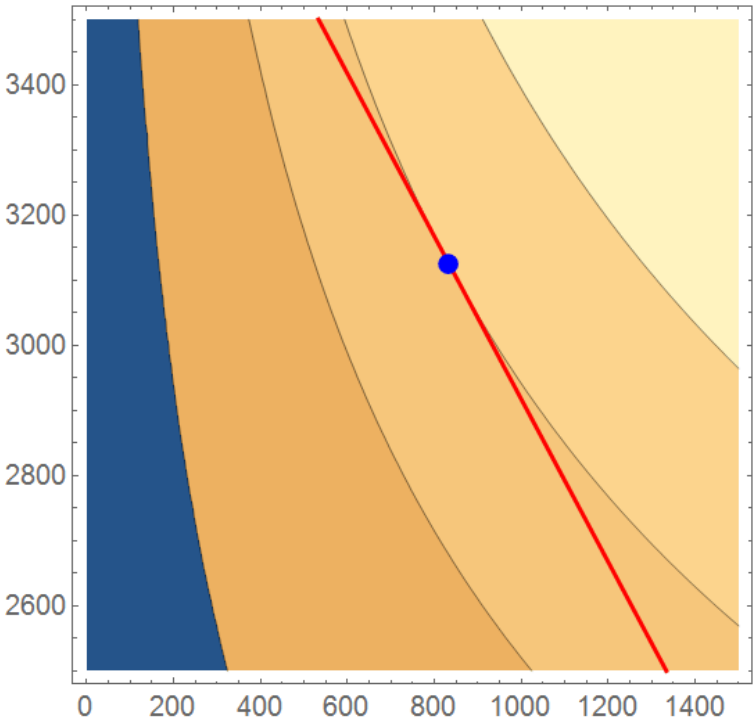


Figure 5.15. The level curves of production function P(x,y)=100x^0.25 y^0.75 subject to the budget line 75x+60y=250000. The production function is known as Cobb-Douglas production function and the level curves are called isoquants.

In Example 5.11 and 5.12 above we determined the nature of the extrema (maximum or minimum) by investigating the problem at hand. However, often it is not easy to do so when the variables are more than two and the function is subject to multiple constraints. So, we need a tool similar to the unconstrained case. Therefore, we present in the following a Second Derivative Test like the one in Theorem 5.3 for n variables with k constraints. Similar to the unconstrained case, however, this test only determines the local extremum of a critical point, not a global extremum.

Theorem 5.7. Second Derivative Test for constrained local extrema

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a constrained critical point of $f(x_1, x_2, \ldots, x_n)$ subject to the constraints

(5.6)
$$\begin{cases} g_1(x_1, x_2, \ldots, x_n) = c_1, \\ g_2(x_1, x_2, \ldots, x_n) = c_2, \\ \qquad\qquad \cdots \\ g_k(x_1, x_2, \ldots, x_n) = c_k, \end{cases}$$ where k

Construct a Lagrangian function $\mathscr{L}$ defined by

(5.7)

$$\begin{aligned} \mathcal{L}(\lambda; \mathbf{x}) &\equiv \mathcal{L}(\lambda_1, \lambda_2, \ldots, \lambda_k, x_1, x_2, \ldots, x_n) \\ &= f(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{k} \lambda_i \left[g_i(x_1, x_2, \ldots, x_n) - c_i \right] \end{aligned}$$

The Hessian of $\mathcal{L}$ is the matrix given by

(5.8)

$$H\,\mathcal{L}(\lambda; \mathbf{a}) = \begin{pmatrix} \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_1 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_2 \partial \lambda_1} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_k \partial \lambda_1} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_1 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_2 \partial \lambda_1} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_n \partial \lambda_1} \\ \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_2 \partial \lambda_2} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_k \partial \lambda_2} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_2 \partial \lambda_2} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_n \partial \lambda_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_1 \partial \lambda_k} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_2 \partial \lambda_k} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_k \partial \lambda_k} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_1 \partial \lambda_k} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_2 \partial \lambda_k} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_n \partial \lambda_k} \\ \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_2 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_k \partial x_1} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_n \partial x_1} \\ \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_1 \partial x_2} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_2 \partial x_2} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_k \partial x_2} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_1 \partial x_n} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_2 \partial x_n} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial \lambda_k \partial x_n} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_1 \partial x_n} & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 \mathcal{L}(\mathbf{a})}{\partial x_n \partial x_n} \end{pmatrix}$$

Note that $H\,\mathcal{L}(\lambda; \mathbf{a})$ is an $(n+k) \times (n+k)$ matrix.

Let $H_j \mathcal{L}(\lambda; \mathbf{a})$ be the upper leftmost $j \times j$ submatrix of $H \mathcal{L}(\lambda; \mathbf{a})$.

Write

$d_j \equiv \det H_j \mathcal{L}(\lambda; \mathbf{a})$.

For j=1,2,...,n+k, calculate the following sequence of n-k numbers:

(5.9) $\quad (-1)^k d_{2k+1}, \ (-1)^k d_{2k+2}, \ \ldots, \ (-1)^k d_{k+n}$.

Then:

(i) If the sequence in (5.9) consists entirely of positive numbers, then f has a local minimum at a subject to the constraints in (5.6).

(ii) If the first number in (5.9) is a negative number and thereafter alternates in sign, then f has a local maximum at a subject to the constraints in (5.6).

(iii) If neither case (i) nor case (ii) holds, then f has a constrained saddle point at $(a_1, a_2, \ldots, a_n)$.

(iv) If $d_{k+n} = \det H \mathcal{L}(\lambda; \mathbf{a}) = 0$, the test is inconclusive. The critical point a is said to be degenerate.

Example 5.13. Find the extreme values of f(x,y,z)=3x y-4y z+5x z subject to the constraints

3x+y+2 z=12 and 2x-3y+5z=0.

Solution:

Step 1. Write out the Lagrange equations.

There are two constraints in this problem

$$\begin{cases} g_1(x, y, z) \equiv 3x + y + 2z = 12, \\ g_2(x, y, z) \equiv 2x - 3y + 5z = 0. \end{cases}$$

Finding the gradient of f, g_1 and g_2 , we get

$$\nabla f(x, y, z) = (3y + 5z, 3x - 4z, 5x - 4y)$$

.

$$\begin{cases} \nabla g_1(x, y, z) = (3, 1, 2), \\ \nabla g_2(x, y, z) = (2, -3, 5). \end{cases}$$

By equation (5.4) the Lagrange condition for two constraints is

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z)$$

.

Thus, the Lagrange equations:

$$\begin{cases} 3y + 5z = 3\lambda_1 + 2\lambda_2, \\ 3x - 4z = \lambda_1 - 3\lambda_2, \\ 5x - 4y = 2\lambda_1 + 5\lambda_2. \end{cases}$$

```
Clear[f,g1,g2]

f[x_,y_,z_]:=3x y-4y z+5x z

g1[x_,y_,z_]:=3x+y+2 z

g2[x_,y_,z_]:=2x-3y+5 z


Grad[f[x,y,z],{x,y,z},"Cartesian"]


Grad[g1[x,y,z],{x,y,z},"Cartesian"]
```

Grad[g2[x,y,z],{x,y,z},"Cartesian"]

Step 2. Solve for λ_1 , λ_2 , x, y, z.

Solve[{Grad[f[x,y,z],{x,y,z},"Cartesian"]==λ1 Grad[g1[x,y,z],
{x,y,z},"Cartesian"]+λ2 Grad[g2[x,y,z],
{x,y,z},"Cartesian"],g1[x,y,z]==12,g2[x,y,z]==0}//Thread]

There is one critical point including Lagrange multipliers at

(λ_1 , λ_2 ,x,y,z)=(482/121,-(107/121),31/11,29/11,5/11).

The extreme value at this point is f(31/11,29/11,5/11)=23.9.

To find out if it is a local maximum, minimum or saddle point, we need the second derivative test, Theorem 5.7.

f[31/11,29/11,5/11]//N

Step 3. Second Derivative Test for constrained local extrema

We construct the Lagrangian function using equation (5.7):

$$\mathcal{L}(\lambda_1, \lambda_2, x, y, z) = f(x, y, z) - \lambda_1 [g_1(x, y, z) - c_1] - \lambda_2 [g_2(x, y, z) - c_2],$$

,

$$\mathcal{L}(\lambda_1, \lambda_2, x, y, z) = 3xy - 4yz + 5xz - \lambda_1(3x + y + 2z) + 12\lambda_1 - \lambda_2(2x - 3y + 5z).$$

By equation (5.8), the Hessian matrix of $\mathcal{L}$ is

$$H\mathcal{L}(\lambda_1, \lambda_2, x, y, z) = \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial y \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial z \partial \lambda_1} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial y \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial z \partial \lambda_2} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial x} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial x} & \frac{\partial^2 \mathcal{L}}{\partial x \partial x} & \frac{\partial^2 \mathcal{L}}{\partial y \partial x} & \frac{\partial^2 \mathcal{L}}{\partial z \partial x} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial y} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial y} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} & \frac{\partial^2 \mathcal{L}}{\partial y \partial y} & \frac{\partial^2 \mathcal{L}}{\partial z \partial y} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial z} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial z} & \frac{\partial^2 \mathcal{L}}{\partial x \partial z} & \frac{\partial^2 \mathcal{L}}{\partial y \partial z} & \frac{\partial^2 \mathcal{L}}{\partial z \partial z} \end{pmatrix}$$

.

Computing the second derivatives and enter into the matrix we get

$$H\mathcal{L}(\lambda_1, \lambda_2, x, y, z) = \begin{pmatrix} 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -2 & 3 & -5 \\ -3 & -2 & 0 & 3 & 5 \\ -1 & 3 & 3 & 0 & -4 \\ -2 & -5 & 5 & -4 & 0 \end{pmatrix}$$

.

```
Clear[c1,c2,lagrangian]

c1=12;

c2=0;

lagrangian[λ1_,λ2_,x_,y_,z_]:=f[x,y,z]-λ1*(g1[x,y,z]-c1)-λ2*(g2[x,y,z]-c2)
```

```
(* Find the Hessian matrix of the Lagrangian function *)

Module[{n,k},

Clear[λ1,λ2,x,y,z];

n=3;

k=2;

X[1]=λ1;

X[2]=λ2;

X[3]=x;

X[4]=y;

X[5]=z;

Table[D[lagrangian[X[1],X[2],X[3],X[4],X[5]],X[i],X[j]],{i,1,n+k},
{j,1,n+k}]//MatrixForm

]
```

Since the Hessian matrix is constant, we just need to find the sequence of numbers as required by Theorem 5.7.

There are three variables and two constraints, so n=3 and k=2, respectively. There is 1 (n-k=1) sequence of number by equation (5.9):

$$(-1)^2 d_5 = \begin{vmatrix} 0 & 0 & -3 & -1 & -2 \\ 0 & 0 & -2 & 3 & -5 \\ -3 & -2 & 0 & 3 & 5 \\ -1 & 3 & 3 & 0 & -4 \\ -2 & -5 & 5 & -4 & 0 \end{vmatrix} = -2904$$

.

(-1)^2*Det[{{0,0,-3,-1,-2},{0,0,-2,3,-5},{-3,-2,0,3,5},{-1,3,3,0,-4},
{-2,-5,5,-4,0}}]

By Theorem 5.7 (iii) the point at (x,y,z)=(31/11,29/11,5/11) is a constrained saddle point of f(x,y,z)=3x y-4y z+5x z subject to the constraints 3x+y+2 z=12 and 2x-3y+5z=0.

We select a point very close to our critical point and specify it in FindMinimum and FindMaximum to verify our results below:

(* The critical point *)

{31/11,29/11,5/11}//N

FindMinimum[{f[x,y,z],g1[x,y,z]==12,g2[x,y,z]==0},{{x,2.818181},
{y,2.636363},{z,0.45454545}}]

FindMaximum[{f[x,y,z],g1[x,y,z]==12,g2[x,y,z]==0},{{x,2.818181},
{y,2.636363},{z,0.45454545}}]

In one direction the critical point is minimum while in another direction a maximum. This suggests that the critical point is indeed a saddle point.

```
Module[{plot,x,y,z},

{x,y,z}={31/11,29/11,5/11};

plot[1]=ContourPlot3D[f[x,y,z],{x,0,6},{y,-1,6},{z,0,1},Contours->
{23.9},ContourStyle->Yellow];

plot[2]=ContourPlot3D[g1[x,y,z]==12,{x,0,6},{y,-1,6},{z,0,1},Mesh-
>None,ContourStyle->{Red,Opacity[0.2]}];

plot[3]=ContourPlot3D[g2[x,y,z]==0,{x,0,6},{y,-1,6},{z,0,1},Mesh-
>None,ContourStyle->{Blue,Opacity[0.2]}];

plot[4]=Graphics3D[{Blue,PointSize[0.04],Point[{x,y,z}]}];

Show[Array[plot,4],AxesLabel->{"x","y","z"},ViewPoint->{6,1,3}]

]
```

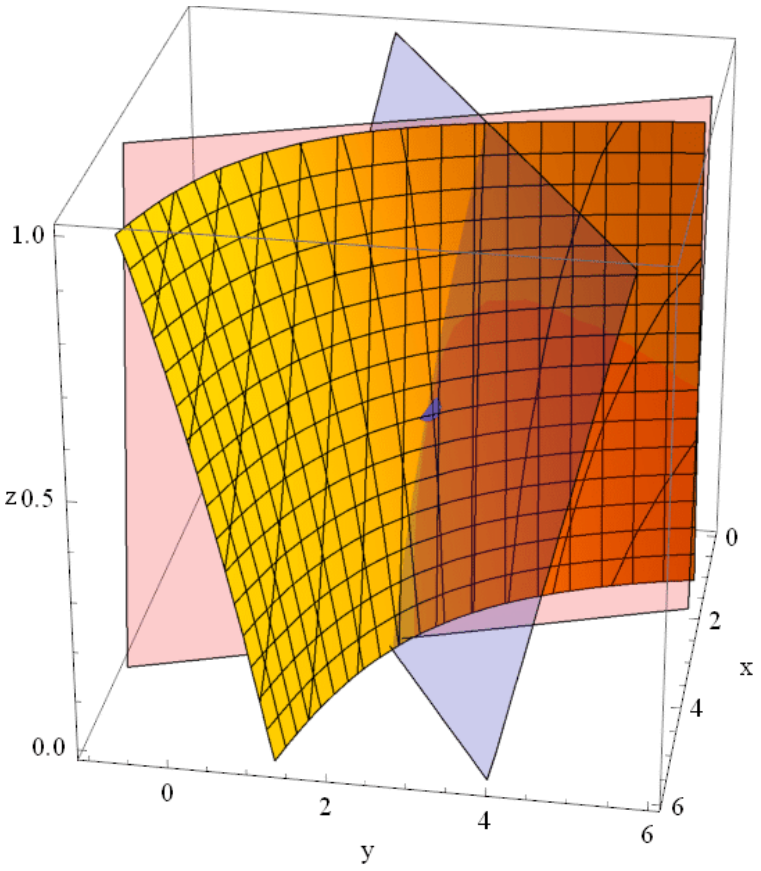


Figure 5.16. The extreme value of the function f(x,y,z)=3x y-4y z+5x z subject to the constraints 3x+y+2 z=12 and 2x-3y+5z=0, plotted as the level surface of f(x,y,z)=23.9 at the saddle point (x,y,z)=(31/11,29/11,5/11).

Example 5.14. Find the extreme values of f(x,y,z,w)=x^2+y^2+z^2+w^2 subject to the constraints

x^2+y^2=1, x+y+z+w=1, x-y+z-w=0.

Solution:

Step 1. Write out the Lagrange equations.

In this problem we have four variables (a five-dimensional hypersurface) and three constraints:

$$\begin{cases} g_1(x, y, z, w) \equiv x^2 + y^2 = 1, \\ g_2(x, y, z, w) \equiv x + y + z + w = 1, \\ g_3(x, y, z, w) \equiv x - y + z - w = 0. \end{cases}$$

The gradient of f, g_1 , g_2 , and g_3 :

$$\nabla f(x, y, z, w) = (2x, 2y, 2z, 2w)$$

,

$$\begin{cases} \nabla g_1(x, y, z, w) = (2x, 2y, 0, 0), \\ \nabla g_2(x, y, z, w) = (1, 1, 1, 1), \\ \nabla g_3(x, y, z, w) = (1, -1, 1, -1). \end{cases}$$

The Lagrange conditions becomes

$$\nabla f(x, y, z, w) = \lambda_1 g_1(x, y, z, w) + \lambda_2 g_2(x, y, z, w) + \lambda_3 g_3(x, y, z, w)$$

We obtain the Lagrange equations:

$$\begin{cases} 2x = 2x\lambda_1 + \lambda_2 + \lambda_3, \\ 2y = 2y\lambda_1 + \lambda_2 - \lambda_3, \\ 2z = \lambda_2 + \lambda_3, \\ 2w = \lambda_2 - \lambda_3. \end{cases}$$

to be solved with three constraints.

```
Clear[f,g1,g2,g3]

f[x_,y_,z_,w_]:=x^2+y^2+z^2+w^2

g1[x_,y_,z_,w_]:=x^2+y^2
```

g2[x_,y_,z_,w_]:=x+y+z+w

g3[x_,y_,z_,w_]:=x-y+z-w

Grad[f[x,y,z,w],{x,y,z,w},"Cartesian"]

Grad[g1[x,y,z,w],{x,y,z,w},"Cartesian"]

Grad[g2[x,y,z,w],{x,y,z,w},"Cartesian"]

Grad[g3[x,y,z,w],{x,y,z,w},"Cartesian"]

Step 2. Solve for λ_1 , λ_2 , λ_3 , x, y, z, w.

Solve[{Grad[f[x,y,z,w],{x,y,z,w},"Cartesian"]==λ1 Grad[g1[x,y,z,w],
{x,y,z,w},"Cartesian"]+λ2 Grad[g2[x,y,z,w],{x,y,z,w},"Cartesian"]+λ3
Grad[g3[x,y,z,w],
{x,y,z,w},"Cartesian"],g1[x,y,z,w]==1,g2[x,y,z,w]==1,g3[x,y,z,w]==0}//Thr
ead]

The solutions are two constrained critical points including the Lagrange multipliers at

$$\lambda_1 = 2 + \frac{1}{\sqrt{2}},\ \lambda_2 = 1 + \sqrt{2},\ \lambda_3 = 0,$$
$$x = -\frac{1}{\sqrt{2}},\ y = -\frac{1}{\sqrt{2}},\ z = \frac{1}{2}\left(1 + \sqrt{2}\right),\ w = \frac{1}{2}\left(1 + \sqrt{2}\right)$$

and

$$\lambda_1 = 2 - \frac{1}{\sqrt{2}},\ \lambda_2 = 1 - \sqrt{2},\ \lambda_3 = 0,$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}},\ z = \frac{1}{2}\left(1 - \sqrt{2}\right),\ w = \frac{1}{2}\left(1 - \sqrt{2}\right)$$

.

Step 3. Second Derivative Test for constrained local extrema

We construct the Lagrangian function using equation (5.7):

$$\mathcal{L}(\lambda_1, \lambda_2, x, y, z, w) = f(x, y, z, w) - \lambda_1 [g_1(x, y, z, w) - c_1] - \lambda_2 [g_2(x, y, z, w) - c_2] - \lambda_3 [g_3(x, y, z, w) - c_3],$$
$$\mathcal{L}(\lambda_1, \lambda_2, x, y, z, w) = x^2 + y^2 + z^2 + w^2 - \lambda_1 \left(x^2 + y^2\right) + \lambda_1 - \lambda_2 (x + y + z + w) + \lambda_2 - \lambda_3 (x - y + z - w).$$

```
Clear[c1,c2,c3,lagrangian]

c1=1;

c2=1;

c3=0;

lagrangian[λ1_,λ2_,λ3_,x_,y_,z_,w_]:=f[x,y,z,w]-λ1*(g1[x,y,z,w]-c1)-λ2*
(g2[x,y,z,w]-c2)-λ3*(g3[x,y,z,w]-c3)
```

By equation (5.8), the Hessian matrix of $\mathscr{L}$:

$$H\mathcal{L}(\lambda_1, \lambda_2, \lambda_3, x, y, z, w) =$$

$$\begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial y \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial z \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial w \partial \lambda_1} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial y \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial z \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial w \partial \lambda_2} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_3} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_3} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_3 \partial \lambda_3} & \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda_3} & \frac{\partial^2 \mathcal{L}}{\partial y \partial \lambda_3} & \frac{\partial^2 \mathcal{L}}{\partial z \partial \lambda_3} & \frac{\partial^2 \mathcal{L}}{\partial w \partial \lambda_3} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial x} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial x} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_3 \partial x} & \frac{\partial^2 \mathcal{L}}{\partial x \partial x} & \frac{\partial^2 \mathcal{L}}{\partial y \partial x} & \frac{\partial^2 \mathcal{L}}{\partial z \partial x} & \frac{\partial^2 \mathcal{L}}{\partial w \partial x} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial y} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial y} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_3 \partial y} & \frac{\partial^2 \mathcal{L}}{\partial x \partial y} & \frac{\partial^2 \mathcal{L}}{\partial y \partial y} & \frac{\partial^2 \mathcal{L}}{\partial z \partial y} & \frac{\partial^2 \mathcal{L}}{\partial w \partial y} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial z} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial z} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_3 \partial z} & \frac{\partial^2 \mathcal{L}}{\partial x \partial z} & \frac{\partial^2 \mathcal{L}}{\partial y \partial z} & \frac{\partial^2 \mathcal{L}}{\partial z \partial z} & \frac{\partial^2 \mathcal{L}}{\partial w \partial z} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial w} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial w} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_3 \partial w} & \frac{\partial^2 \mathcal{L}}{\partial x \partial w} & \frac{\partial^2 \mathcal{L}}{\partial y \partial w} & \frac{\partial^2 \mathcal{L}}{\partial z \partial w} & \frac{\partial^2 \mathcal{L}}{\partial w \partial w} \end{pmatrix}$$

Using the module below we obtain the Hessian matrix:

$$H\mathcal{L}(\lambda_1, \lambda_2, \lambda_3, x, y, z, w) = \begin{pmatrix} 0 & 0 & 0 & -2x & -2y & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ -2x & -1 & -1 & 2-2\lambda 1 & 0 & 0 & 0 \\ -2y & -1 & 1 & 0 & 2-2\lambda 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

```
(* Find the Hessian matrix of the Lagrangian function *)

Module[{n,k},

Clear[λ1,λ2,λ3,x,y,z,w];

n=4;

k=3;

X[1]=λ1;

X[2]=λ2;
```

```
X[3]=λ3;

X[4]=x;

X[5]=y;

X[6]=z;

X[7]=w;

Table[D[lagrangian[X[1],X[2],X[3],X[4],X[5],X[6],X[7]],X[i],X[j]],
{i,1,n+k},{j,1,n+k}]//MatrixForm

]
```

To compute the Hessian matrix at a critical point, we write another function:

```
(* Find the Hessian matrix of the Lagrangian function at a critical point  *)

Clear[hessianLagrangian]

hessianLagrangian[lagrangian_,
{m1_,m2_,m3_,a1_,a2_,a3_,a4_}]:=Module[{n,k},

Clear[λ1,λ2,λ3,x,y,z,w];

n=4;

k=3;

X[1]=λ1;

X[2]=λ2;

X[3]=λ3;

X[4]=x;
```

X[5]=y;

X[6]=z;

X[7]=w;

Table[D[lagrangian[X[1],X[2],X[3],X[4],X[5],X[6],X[7]],X[i],X[j]],
{i,1,n+k},{j,1,n+k}]/.{λ1->m1,λ2->m2,λ3->m3,x->a1,y->a2,z->a3,w->a4}

]

Next, we investigate the Hessian matrix for the two critical points obtained above.

Check critical point 1:

$$\lambda_1 = 2 + \frac{1}{\sqrt{2}},\ \lambda_2 = 1 + \sqrt{2},\ \lambda_3 = 0,$$
$$x = -\frac{1}{\sqrt{2}},\ y = -\frac{1}{\sqrt{2}},\ z = \frac{1}{2}\left(1 + \sqrt{2}\right),\ w = \frac{1}{2}\left(1 + \sqrt{2}\right)$$

.

(* Compute the Hessian matrix of the Lagrangian function at critical point 1 *)

hessianLagrangian[lagrangian,{2+1/Sqrt[2],1+Sqrt[2],0,-(1/Sqrt[2]),-(1/Sqrt[2]),1/2 (1+Sqrt[2]),1/2 (1+Sqrt[2])}]//MatrixForm

Since n=4 and k=3, there is 1 (n-k=1) sequence of number by equation (5.9):

$$(-1)^3 d_7 = (-1)\begin{vmatrix} 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ \sqrt{2} & -1 & -1 & 2-2\left(2+\frac{1}{\sqrt{2}}\right) & 0 & 0 & 0 \\ \sqrt{2} & -1 & 1 & 0 & 2-2\left(2+\frac{1}{\sqrt{2}}\right) & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -16\sqrt{2}$$

```
(-1)^3*Det[{{0,0,0,Sqrt[2],Sqrt[2],0,0},{0,0,0,-1,-1,-1,-1},{0,0,0,-1,1,-1,1},
{Sqrt[2],-1,-1,2-2 (2+1/Sqrt[2]),0,0,0},{Sqrt[2],-1,1,0,2-2
(2+1/Sqrt[2]),0,0},{0,-1,-1,0,0,2,0},{0,-1,1,0,0,0,2}}]
```

There is only one sequence of number here. By Theorem 5.7.(iii) it is a constrained saddle point. It can be verified using FindMinimum and Find Maximum below:

```
(* Critical point 1: x, y, z, w *)

{-(1/Sqrt[2]),-(1/Sqrt[2]),1/2 (1+Sqrt[2]),1/2 (1+Sqrt[2])}//N
```

```
FindMinimum[{f[x,y,z,w],g1[x,y,z,w]==1,g2[x,y,z,w]==1,g3[x,y,z,w]==0},
{{x,-0.7},{y,-0.7},{z,1.1},{w,1.1}}]
```

```
FindMaximum[{f[x,y,z,w],g1[x,y,z,w]==1,g2[x,y,z,w]==1,g3[x,y,z,w]==0},
{{x,-0.7},{y,-0.7},{z,1.1},{w,1.1}}]
```

Since in one direction it is a local minimum and another direction a local maximum, this suggests that the critical point at (x,y,z,w)=(-(1/Sqrt[2]),-(1/Sqrt[2]),1/2 (1+Sqrt[2]),1/2 (1+Sqrt[2])) is a saddle point. The extreme value at this point is 3.91421.

f[-(1/Sqrt[2]),-(1/Sqrt[2]),1/2 (1+Sqrt[2]),1/2 (1+Sqrt[2])]//N

Check critical point 2:

$$\lambda_1 = 2 - \frac{1}{\sqrt{2}},\ \lambda_2 = 1 - \sqrt{2},\ \lambda_3 = 0,$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}},\ z = \frac{1}{2}\left(1 - \sqrt{2}\right),\ w = \frac{1}{2}\left(1 - \sqrt{2}\right)$$

(* Compute the Hessian matrix of the Lagrangian function at critical point 2 *)

hessianLagrangian[lagrangian,{2-1/Sqrt[2],1-Sqrt[2],0,1/Sqrt[2],1/Sqrt[2],1/2 (1-Sqrt[2]),1/2 (1-Sqrt[2])}]//MatrixForm

$$(-1)^3 d_7 = (-1)\begin{vmatrix} 0 & 0 & 0 & -\sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ -\sqrt{2} & -1 & -1 & 2-2\left(2-\frac{1}{\sqrt{2}}\right) & 0 & 0 & 0 \\ -\sqrt{2} & -1 & 1 & 0 & 2-2\left(2-\frac{1}{\sqrt{2}}\right) & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= 16\sqrt{2}$$

(-1)^3*Det[{{0,0,0,-Sqrt[2],-Sqrt[2],0,0},{0,0,0,-1,-1,-1,-1},
{0,0,0,-1,1,-1,1},{-Sqrt[2],-1,-1,2-2 (2-1/Sqrt[2]),0,0,0},{-Sqrt[2],-1,1,0,2-2
(2-1/Sqrt[2]),0,0},{0,-1,-1,0,0,2,0},{0,-1,1,0,0,0,2}}]

The sequence consists of only one positive number. By Theorem 5.7.(iii) it is a constrained saddle point. We verify this with FindMinimum and FindMaximum using start point near critical point 2:

(* Critical point 2: x, y, z, w *)

{1/Sqrt[2],1/Sqrt[2],1/2 (1-Sqrt[2]),1/2 (1-Sqrt[2])}//N

FindMinimum[{f[x,y,z,w],g1[x,y,z,w]==1,g2[x,y,z,w]==1,g3[x,y,z,w]==0},
{{x,0.7},{y,0.7},{z,-0.15},{w,-0.15}}]

FindMaximum[{f[x,y,z,w],g1[x,y,z,w]==1,g2[x,y,z,w]==1,g3[x,y,z,w]==0},
{{x,0.7},{y,0.7},{z,-0.15},{w,-0.15}}]

Using the same argument as the first critical point, the point at (x,y,z,w)= (1/Sqrt[2],1/Sqrt[2],1/2 (1-Sqrt[2]),1/2 (1-Sqrt[2])) is also a saddle point of f. The extreme value at this point is 1.08579.

f[1/Sqrt[2],1/Sqrt[2],1/2 (1-Sqrt[2]),1/2 (1-Sqrt[2])]//N

Thus, the two critical points are saddle points of the function f(x,y,z,w)=x^2+y^2+z^2+w^2 subject to the constraints:

x^2+y^2=1, x+y+z+w=1, x-y+z-w=0.

Example 5.15. A line is given by the equation

y=-x+4.

Find the minimum distance between a point on the ellipse x^2+2y^2=1 and a point on this line.

Solution:

Step 1. Write out the Lagrange equations.

We may consider the distance between a point on the ellipse and a point on the line as a function of four variables, two on the line and the other two on the ellipse such that it is optimized to be minimum. For algebraic simplicity we consider the function as the square of the distance rather than the actual distance. Thus, the function is given by

$$f(x_1, y_1, x_2, y_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

.

In this problem we have two constraints in order to minimize $f(x_1, y_1, x_2, y_2)$:

$$\begin{cases} g_1(x_1, y_1, x_2, y_2) \equiv x_1 + y_1 = 4, \\ g_2(x_2, y_2, x_2, y_2) \equiv x_2^2 + 2y_2^2 = 1. \end{cases}$$

The Lagrange condition for two constraints:

$$\nabla f(x_1, y_1, x_2, y_2) = \lambda_1 \nabla g_1(x_1, y_1, x_2, y_2) + \lambda_2 \nabla g_2(x_2, y_2, x_2, y_2)$$

The gradient of f is

$$\nabla f(x_1, y_1, x_2, y_2) = (-2(x_2 - x_1), -2(y_2 - y_1), \\ 2(x_2 - x_1), 2(y_2 - y_1))$$

$$\begin{cases} \nabla g_1(x_1, y_1, x_2, y_2) = (1, 1, 0, 0), \\ \nabla g_2(x_1, y_1, x_2, y_2) = (0, 0, 2x_2, 4y_2). \end{cases}$$

The Lagrange condition becomes:

$$(-2(x_2 - x_1), -2(y_2 - y_1), 2(x_2 - x_1), 2(y_2 - y_1)) \\ = \lambda_1 (1, 1, 0, 0) + \lambda_2 (0, 0, 2x_2, 4y_2)$$

So, the Lagrange equations to solve are

$$\begin{cases} -2(x_2 - x_1) = \lambda_1, \\ -2(y_2 - y_1) = \lambda_1, \\ 2(x_2 - x_1) = 2\lambda_2 x_2, \\ 2(y_2 - y_1) = 4\lambda_2 y_2, \end{cases}$$

together with two constraints:

$$\begin{cases} x_1 + y_1 = 4, \\ x_2^2 + 2y_2^2 = 1. \end{cases}$$

Step 2. Solve for λ_1 , λ_2 , x_1, y_1, x_2, y_2 .

Solve[{-2(x2-x1)==λ1,-2(y2-y1)==λ1,2(x2-x1)==2λ2 x2,2(y2-y1)==4λ2 y2,x1+y1==4,x2^2+2y2^2==1}]

Solve command returns all solutions including complex ones. Since, we only need real solutions we can add Reals option in Solve:

Solve[{-2(x2-x1)==λ1,-2(y2-y1)==λ1,2(x2-x1)==2λ2 x2,2(y2-y1)==4λ2

y2,x1+y1==4,x2^2+2y2^2==1},Reals]//Simplify

So, there are two constrained critical points including the Lagrange multipliers at

$$\lambda_1 = 4 + \sqrt{\frac{3}{2}}\,,\ \lambda_2 = \frac{3}{4} + \sqrt{6}\,,$$

$$x_1 = 2 - \frac{1}{2\sqrt{6}}\,,\ y_1 = 2 + \frac{1}{2\sqrt{6}}\,,\ x_2 = -\sqrt{\frac{2}{3}}\,,\ y_2 = -\frac{1}{\sqrt{6}}\,;$$

and

$$\lambda_1 = 4 - \sqrt{\frac{3}{2}}\,,\ \lambda_2 = \frac{3}{4} - \sqrt{6}\,,$$

$$x_1 = 2 + \frac{1}{2\sqrt{6}}\,,\ y_1 = 2 - \frac{1}{2\sqrt{6}}\,,\ x_2 = \sqrt{\frac{2}{3}}\,,\ y_2 = \frac{1}{\sqrt{6}}\,.$$

Step 3. Second Derivative Test for constrained local extrema

We construct the Lagrangian function using equation (5.7):

$$\begin{aligned}\mathcal{L}(\lambda_1, \lambda_2, x_1, y_1, x_2, y_2) &= f(x_1, y_1, x_2, y_2)\\ &\quad -\lambda_1\,[g_1(x_1, y_1, x_2, y_2) - c_1]\\ &\quad -\lambda_2\,[g_2(x_1, y_1, x_2, y_2) - c_2],\\ \mathcal{L}(\lambda_1, \lambda_2, x_1, y_1, x_2, y_2) &= (x_2 - x_1)^2 + (y_2 - y_1)^2\\ &\quad -\lambda_1\,(x_1 + y_1) + 4\,\lambda_1 - \lambda_2\left(x_2^{\,2} + 2\,y_2^{\,2}\right) + \lambda_2.\end{aligned}$$

By equation (5.8), the Hessian matrix of $\mathcal{L}$ is

$$H\mathcal{L}(\lambda_1, \lambda_2, x_1, y_1, x_2, y_2) =$$

$$\begin{pmatrix}
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial y_1 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial y_2 \partial \lambda_1} \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial y_1 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda_2} & \frac{\partial^2 \mathcal{L}}{\partial y_2 \partial \lambda_2} \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial y_1 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial y_2 \partial x_1} \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial y_1} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial y_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial y_1} & \frac{\partial^2 \mathcal{L}}{\partial y_1 \partial y_1} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial y_1} & \frac{\partial^2 \mathcal{L}}{\partial y_2 \partial y_1} \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial y_1 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} & \frac{\partial^2 \mathcal{L}}{\partial y_2 \partial x_2} \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial y_2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial y_2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial y_2} & \frac{\partial^2 \mathcal{L}}{\partial y_1 \partial y_2} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial y_2} & \frac{\partial^2 \mathcal{L}}{\partial y_2 \partial y_2}
\end{pmatrix}$$

To construct the Hessian matrix of $\mathcal{L}$ we need the second partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = -(x_1 + y_1) + 4, \quad \frac{\partial \mathcal{L}}{\partial \lambda_2} = -\left(x_2^2 + 2\, y_2^2\right) + 1,$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2\,(x_2 - x_1) - \lambda_1, \quad \frac{\partial \mathcal{L}}{\partial x_2} = 2\,(x_2 - x_1) + 2\,\lambda_2\, x_2.$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_1} = 0, \quad \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_1} = 0,$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda_1} = -1, \quad \frac{\partial^2 \mathcal{L}}{\partial y_1 \partial \lambda_1} = -1, \quad \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda_1} = 0, \quad \frac{\partial^2 \mathcal{L}}{\partial y_2 \partial \lambda_1} = 0,$$

etc...

The computation is rather tedious, so we write a Mathematica module to complete the task. The result is given here:

$$H\mathcal{L}(\lambda_1, \lambda_2, x_1, y_1, x_2, y_2) =$$

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_2 & -4y_2 \\ -1 & 0 & 2 & 0 & -2 & 0 \\ -1 & 0 & 0 & 2 & 0 & -2 \\ 0 & -2x_2 & -2 & 0 & 2-2\lambda_2 & 0 \\ 0 & -4y_2 & 0 & -2 & 0 & 2-4\lambda_2 \end{pmatrix}$$

```
Clear[f,g1,g2,c1,c2,lagrangian]

f[x1_,y1_,x2_,y2_]:=(x2-x1)^2+(y2-y1)^2

g1[x1_,y1_,x2_,y2_]:=x1+y1

g2[x1_,y1_,x2_,y2_]:=x2^2+2y2^2

c1=4;

c2=1;

lagrangian[λ1_,λ2_,x1_,y1_,x2_,y2_]:=f[x1,y1,x2,y2]-λ1*(g1[x1,y1,x2,y2]-
c1)-λ2*(g2[x1,y1,x2,y2]-c2)


(* Find the Hessian matrix of the Lagrangian function *)

Module[{n,k},

Clear[λ1,λ2,x1,y1,x2,y2];

n=4;

k=2;

X[1]=λ1;

X[2]=λ2;
```

```
X[3]=x1;

X[4]=y1;

X[5]=x2;

X[6]=y2;

Table[D[lagrangian[X[1],X[2],X[3],X[4],X[5],X[6]],X[i],X[j]],{i,1,n+k},
{j,1,n+k}]//MatrixForm

]
```

We need to test all critical points. Here is a module to compute the Hessian matrix of the Lagrangian function at a critical point:

```
(* Find the Hessian matrix of the Lagrangian function at a critical point *)

Clear[hessianLagrangian]

hessianLagrangian[lagrangian_,
{m1_,m2_,a1_,a2_,a3_,a4_}]:=Module[{n,k},

Clear[λ1,λ2,x1,y1,x2,y2];

n=4;

k=2;

X[1]=λ1;

X[2]=λ2;

X[3]=x1;

X[4]=y1;
```

X[5]=x2;

X[6]=y2;

Table[D[lagrangian[X[1],X[2],X[3],X[4],X[5],X[6]],X[i],X[j]],{i,1,n+k},
{j,1,n+k}]/.{λ1->m1,λ2->m2,x1->a1,y1->a2,x2->a3,y2->a4}

]

Check critical point 1:

$$\lambda_1 = 4 + \sqrt{\frac{3}{2}},\ \lambda_2 = \frac{3}{4} + \sqrt{6},$$

$$x_1 = 2 - \frac{1}{2\sqrt{6}},\ y_1 = 2 + \frac{1}{2\sqrt{6}},\ x_2 = -\sqrt{\frac{2}{3}},\ y_2 = -\frac{1}{\sqrt{6}};$$

We compute the Hessian matrix and the sequence of numbers, equation (5.9) for this critical point:

hessianLagrangian[lagrangian,{4+Sqrt[3/2],3/4+Sqrt[6],2-
1/(2Sqrt[6]),2+1/(2Sqrt[6]),-Sqrt[(2/3)],-(1/Sqrt[6])}]//MatrixForm

Since n=4 and k=2, there are 2 (n-k=2) sequence of numbers by equation (5.9):

$$(-1)^2 d_5 = \begin{vmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2/3} \\ -1 & 0 & 2 & 0 & -2 \\ -1 & 0 & 0 & 2 & 0 \\ 0 & 2\sqrt{2/3} & -2 & 0 & 2 - 2\left(\frac{3}{4} + \sqrt{6}\right) \end{vmatrix} = \frac{32}{3},$$

$$(-1)^2 d_6 = \begin{vmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sqrt{2/3} & 2\sqrt{2/3} \\ -1 & 0 & 2 & 0 & -2 & 0 \\ -1 & 0 & 0 & 2 & 0 & -2 \\ 0 & 2\sqrt{2/3} & -2 & 0 & 2-2\left(\frac{3}{4}+\sqrt{6}\right) & 0 \\ 0 & 2\sqrt{2/3} & 0 & -2 & 0 & 2-4\left(\frac{3}{4}+\sqrt{6}\right) \end{vmatrix}$$

$$= -204.767$$

By Theorem 5.7.(iii) the critical point:

$$(x_1, y_1, x_2, y_2) = \left(2 - \frac{1}{2\sqrt{6}}, 2 + \frac{1}{2\sqrt{6}}, -\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right)$$

is a constrained saddle point. So, it is not the minimum point we are looking for.

(* Compute d5 for critical point 1 *)

```
Det[{{0,0,-1,-1,0},{0,0,0,0,2 Sqrt[2/3]},{-1,0,2,0,-2},{-1,0,0,2,0},{0,2
Sqrt[2/3],-2,0,2-2 (3/4+Sqrt[6])}}]
```

(* Compute d6 for critical point 1 *)

```
Det[{{0,0,-1,-1,0,0},{0,0,0,0,2 Sqrt[2/3],2 Sqrt[2/3]},{-1,0,2,0,-2,0},
{-1,0,0,2,0,-2},{0,2 Sqrt[2/3],-2,0,2-2 (3/4+Sqrt[6]),0},{0,2
Sqrt[2/3],0,-2,0,2-4 (3/4+Sqrt[6])}}]//N
```

Check critical point 2:

$$\lambda_1 = 4 - \sqrt{\frac{3}{2}}\,,\ \lambda_2 = \frac{3}{4} - \sqrt{6}\,,$$

$$x_1 = 2 + \frac{1}{2\sqrt{6}}\,,\ y_1 = 2 - \frac{1}{2\sqrt{6}}\,,\ x_2 = \sqrt{\frac{2}{3}}\,,\ y_2 = \frac{1}{\sqrt{6}}\,.$$

We compute the Hessian matrix and the sequence of numbers for this critical point:

```
hessianLagrangian[lagrangian,{4-Sqrt[3/2],3/4-Sqrt[6],2+1/(2Sqrt[6]),2-
1/(2Sqrt[6]),Sqrt[2/3],1/Sqrt[6]}]//MatrixForm
```

```
(* Compute d5 for critical point 2 *)

Det[{{0,0,-1,-1,0},{0,0,0,0,-2 Sqrt[2/3]},{-1,0,2,0,-2},{-1,0,0,2,0},{0,-2
Sqrt[2/3],-2,0,2-2 (3/4-Sqrt[6])}}]
```

```
(* Compute d6 for critical point 2 *)

Det[{{0,0,-1,-1,0,0},{0,0,0,0,-2 Sqrt[2/3],-2 Sqrt[2/3]},{-1,0,2,0,-2,0},
{-1,0,0,2,0,-2},{0,-2 Sqrt[2/3],-2,0,2-2 (3/4-Sqrt[6]),0},{0,-2
Sqrt[2/3],0,-2,0,2-4 (3/4-Sqrt[6])}}]//N
```

and obtain

$$(-1)^2\, d_5 = \frac{32}{3}, \quad (-1)^2\, d_6 = 108.767$$

.

By Theorem 5.7. (i) the critical point:

$$(x_1, y_1, x_2, y_2) = \left(2 + \frac{1}{2\sqrt{6}},\ 2 - \frac{1}{2\sqrt{6}},\ \sqrt{\frac{2}{3}},\ \frac{1}{\sqrt{6}}\right)$$

is a local minimum. Thus, it is verified that at these coordinates the distance between the line y=-x+4 and the ellipse x^2+2y^2=1 is minimum.

The minimum distance is 1.9624.

```
Sqrt[f[2+1/(2Sqrt[6]),2-1/(2Sqrt[6]),Sqrt[2/3],1/Sqrt[6]]]//N
```

```
Module[{plot,x1,y1,x2,y2},

{x1,y1}={2+1/(2Sqrt[6]),2-1/(2Sqrt[6])};

{x2,y2}={Sqrt[2/3],1/Sqrt[6]};

plot[1]=ContourPlot[{x+y==4,x^2+2y^2==1},{x,-1,4},
{y,-1,4},ContourStyle->AbsoluteThickness[2]];

plot[2]=Graphics[{PointSize[0.03],Red,Point[{{x1,y1},{x2,y2}}]}];

plot[3]=Graphics[{AbsoluteThickness[2],Dashed,Line[{{x1,y1},
{x2,y2}}]}];

Show[Array[plot,3]]

]
```

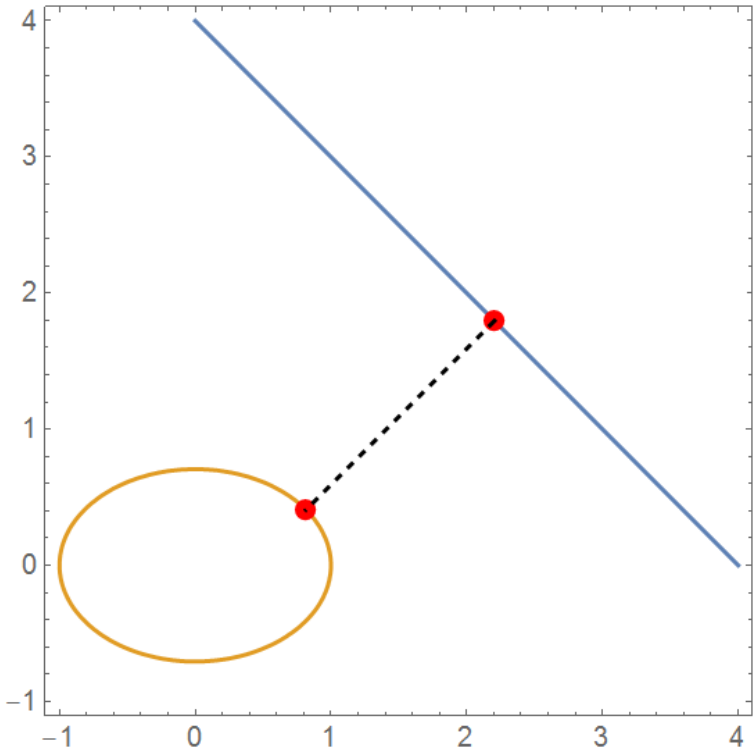


Figure 5.17. The minimum distance between the line y=-x+4 and the ellipse x^2+2y^2=1.

Here again we use FindMinimum and FindMaximum to verify our results:

```
FindMinimum[{f[x1,y1,x2,y2],g1[x1,y1,x2,y2]==4,g2[x1,y1,x2,y2]==1},
{x1,y1,x2,y2}]
```

```
Sqrt[3.85012]
```

The minimum of actual distance is $\sqrt{3.85102} \simeq 1.962$ which is equal to our result to three decimal places.

Example 5.16. Find the extrema of

f(x,y)=(1 + x^2 + x y)/(1 + x^2 + y^4),

subject to the constraint x^2+y^2/4=1.

Solution:

Step 1. Write out the Lagrange equations.

The Lagrange condition is

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$,

where

$$\nabla f(x, y) = \left(\frac{y - x^2 y + 2 x y^4 + y^5}{\left(1 + x^2 + y^4\right)^2}, \frac{-4 y^3 \left(1 + x^2 + x y\right) + x \left(1 + x^2 + y^4\right)}{\left(1 + x^2 + y^4\right)^2} \right)$$

,

and $$\nabla g(x, y) = \left(2 x, \frac{y}{2}\right)$$.

So, we solve the Lagrange equations:

$$\begin{cases} \frac{y - x^2 y + 2 x y^4 + y^5}{\left(1 + x^2 + y^4\right)^2} = 2 \lambda x, \\ \frac{-4 y^3 \left(1 + x^2 + x y\right) + x \left(1 + x^2 + y^4\right)}{\left(1 + x^2 + y^4\right)^2} = \lambda \frac{y}{2}. \end{cases}$$

Clear[f,g]

f[x_,y_]:=(1+x^2+x y)/(1+x^2+y^4)

```
g[x_,y_]:=x^2+y^2/4
```

```
Grad[f[x,y],{x,y},"Cartesian"]//Simplify
```

```
Grad[g[x,y],{x,y},"Cartesian"]//Simplify
```

Step 2. Solve for λ , x, y.

There are no close form solutions for this system of equations so we resort to numerical solutions using NSolve, with additional Reals option to obtain real values.

```
NSolve[{Grad[f[x,y],{x,y},"Cartesian"]==λ Grad[g[x,y],
{x,y},"Cartesian"],g[x,y]==1}//Thread,Reals]
```

or equivalently

```
NSolve[{(y-x^2 y+2 x y^4+y^5)/(1+x^2+y^4)^2==2λ x,(-4 y^3 (1+x^2+x
y)+x (1+x^2+y^4))/(1+x^2+y^4)^2==λ y/2,g[x,y]==1},Reals]
```

There are four constrained critical points:

$$\lambda_1 = -0.108124, \quad x_1 = 0.395591, \quad y_1 = -1.83685,$$
$$\lambda_2 = -0.108124, \quad x_2 = -0.395591, \quad y_2 = 1.83685,$$
$$\lambda_3 = 0.0410612, \quad x_3 = 0.957548, \quad y_3 = 0.576545,$$

$$\lambda_4 = 0.0410612, \quad x_4 = -0.957548, \quad y_4 = -0.576545.$$

Step 3. Second Derivative Test for constrained local extrema

We construct the Lagrangian function using equation (5.7):

$$\mathcal{L}(\lambda, x, y) = f(x, y) - \lambda\,(g(x, y) - c)$$

,

$$\mathcal{L}(\lambda, x, y) = \frac{1 + x^2 + x\,y}{1 + x^2 + y^4} - \lambda\left(x^2 + \frac{y^2}{4} - 1\right)$$

.

By equation (5.8), the Hessian matrix of $\mathcal{L}$ is

$$H\,\mathcal{L}(\lambda, x, y) = \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial \lambda\,\partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial x\,\partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial y\,\partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda\,\partial x} & \frac{\partial^2 \mathcal{L}}{\partial x\,\partial x} & \frac{\partial^2 \mathcal{L}}{\partial y\,\partial x} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda\,\partial y} & \frac{\partial^2 \mathcal{L}}{\partial x\,\partial y} & \frac{\partial^2 \mathcal{L}}{\partial y\,\partial y} \end{pmatrix}$$

.

It is found to be

$$H\,\mathcal{L}(\lambda, x, y) =$$

$$\begin{pmatrix} 0 & -2x & -\frac{y}{2} \\ -2x & \frac{8x^2\left(1+x^2+x\,y\right)}{\left(1+x^2+y^4\right)^3} - \frac{4x\,(2x+y)}{\left(1+x^2+y^4\right)^2} - \frac{2\left(1+x^2+x\,y\right)}{\left(1+x^2+y^4\right)^2} + \frac{2}{1+x^2+y^4} - 2\lambda & -\frac{-1+x^4-8x^3y^3+2y^4-12x^2y^4+3y^8+8x\,y^3\left(-1+y^4\right)}{\left(1+x^2+y^4\right)^3} \\ -\frac{y}{2} & -\frac{-1+x^4-8x^3y^3+2y^4-12x^2y^4+3y^8+8x\,y^3\left(-1+y^4\right)}{\left(1+x^2+y^4\right)^3} & \frac{32y^6\left(1+x^2+x\,y\right)}{\left(1+x^2+y^4\right)^3} - \frac{8x\,y^3}{\left(1+x^2+y^4\right)^2} - \frac{12y^2\left(1+x^2+x\,y\right)}{\left(1+x^2+y^4\right)^2} - \frac{\lambda}{2} \end{pmatrix}$$

Clear[lagrangian]

```
c=1;

lagrangian[λ_,x_,y_]:=f[x,y]-λ*(g[x,y]-c)


(* Find the Hessian matrix of the Lagrangian function *)

Module[{n,k},

Clear[λ,x,y];

n=2;

k=1;

X[1]=λ;

X[2]=x;

X[3]=y;

Table[D[lagrangian[X[1],X[2],X[3]],X[i],X[j]],{i,1,n+k},
{j,1,n+k}]//Simplify//MatrixForm

]
```

Here is a module to compute the Hessian matrix of the Lagrangian function at a critical point:

```
(* Find the Hessian matrix of the Lagrangian function at a critical point *)

Clear[hessianLagrangian]

hessianLagrangian[lagrangian_,{m_,a_,b_}]:=Module[{n,k},

Clear[λ,x,y];
```

n=2;

k=1;

X[1]=λ;

X[2]=x;

X[3]=y;

Table[D[lagrangian[X[1],X[2],X[3]],X[i],X[j]],{i,1,n+k},{j,1,n+k}]/.{λ->m,x->a,y->b}

]

Find the Hessian matrix at critical points:

Check critical point 1:

$\lambda_1 = -0.108124,\ x_1 = 0.395591,\ y_1 = -1.83685$

,

We compute the Hessian matrix and the sequence of numbers for this critical point:

hessianLagrangian[lagrangian, {-0.108124,0.395591,-1.83685}]//MatrixForm

Since there are two variables and one constraint we have n=2 and k=1, respectively. So, there is 1 (n-k=1) sequence of number by equation (5.9):

$$(-1)^1 d_3 = -\begin{vmatrix} 0 & -0.791182 & 0.918425 \\ -0.791182 & 0.381059 & -0.0956321 \\ 0.918425 & -0.0956321 & 0.336009 \end{vmatrix}$$

$$= 0.392776$$

(* Compute d3 for critical point 1 *)

(-1)^1*Det[{{0,-0.791182,0.918425},{-0.791182,0.381059,-0.0956321},
{0.918425,-0.0956321,0.336009}}]

By Theorem 5.7 the critical point:

$(x_1, y_1) = (0.395591, -1.83685)$

is a constrained local minimum. Let us verify this by using FindMinimum and FindMaximum:

FindMinimum[{f[x,y],g[x,y]==1},{{x,0.4},{y,-1.8}}]

FindMaximum[{f[x,y],g[x,y]==1},{{x,0.4},{y,-1.8}}]

The results from FindMinimum and FindMaximum suggest that the critical point:

$(x_1, y_1) = (0.395591, -1.83685)$

is a constrained local minimum.

Check critical point 2:

$\lambda_2 = -0.108124,\ x_2 = -0.395591,\ y_2 = 1.83685$

.

Here again, we compute the Hessian matrix and the sequence of numbers for this critical point:

```
hessianLagrangian[lagrangian,
{-0.108124,-0.395591,1.83685}]//MatrixForm
```

```
(* Compute d3 for critical point 2 *)

(-1)^1*Det[{{0,0.791182,-0.918425},{0.791182,0.381059,-0.0956321},
{-0.918425,-0.0956321,0.336009}}]
```

$$(-1)^1 d_3 = -\begin{vmatrix} 0 & 0.791182 & -0.918425 \\ 0.791182 & 0.381059 & -0.0956321 \\ -0.918425 & -0.0956321 & 0.336009 \end{vmatrix}$$

$$= 0.392776$$

By Theorem 5.7 the critical point:

$(x_2, y_2) = (-0.395591,\ 1.83685)$

is also a constrained local minimum. Let us verify this by using FindMinimum and FindMaximum:

```
FindMinimum[{f[x,y],g[x,y]==1},{{x,-0.4},{y,1.8}}]
```

```
FindMaximum[{f[x,y],g[x,y]==1},{{x,-0.4},{y,1.8}}]
```

Check critical point 3:

$\lambda_3 = 0.0410612,\ x_3 = 0.957548,\ y_3 = 0.576545$

hessianLagrangian[lagrangian,
{0.0410612,0.957548,0.576545}]//MatrixForm

(* Compute d3 for critical point 3 *)

(-1)^1*Det[{{0,-1.9151,-0.288273},{-1.9151,-0.445547,0.45233},
{-0.288273,0.45233,-2.42549}}]

$$(-1)^1 d_3 = -\begin{vmatrix} 0 & -1.9151 & -0.288273 \\ -1.9151 & -0.445547 & 0.45233 \\ -0.288273 & 0.45233 & -2.42549 \end{vmatrix}$$

$$= -9.43221$$

There is only one sequence of number. By using FindMinimum and FindMaximum it is verified that the critical point:

$(x_3,\ y_3) = (0.957548,\ 0.576545)$

is a constrained local maximum.

FindMaximum[{f[x,y],g[x,y]==1},{{x,0.9},{y,0.5}}]

FindMinimum[{f[x,y],g[x,y]==1},{{x,0.9},{y,0.5}}]

Check critical point 4:
$\lambda_4 = 0.0410612,\ x_4 = -0.957548,\ y_4 = -0.576545$

hessianLagrangian[lagrangian,
{0.0410612,-0.957548,-0.576545}]//MatrixForm

(* Compute d3 for critical point 4 *)

(-1)^1*Det[{{0,1.9151,0.288273},{1.9151,-0.445547,0.45233},
{0.288273,0.45233,-2.42549}}]

$$(-1)^1 d_3 = -\begin{vmatrix} 0 & 1.9151 & 0.288273 \\ 1.9151 & -0.445547 & 0.45233 \\ 0.288273 & 0.45233 & -2.42549 \end{vmatrix}$$

$$= -9.43221$$

Using FindMinimum and FindMaximum, the critical point:

$(x_4, y_4) = (-0.957548, -0.576545)$

is found to be a constrained local maximum.

FindMaximum[{f[x,y],g[x,y]==1},{{x,-0.9},{y,-0.5}}]

FindMinimum[{f[x,y],g[x,y]==1},{{x,-0.9},{y,-0.5}}]

Let us summarize our results:

Local minimum:

$\lambda_1 = -0.108124,\ x_1 = 0.395591,\ y_1 = -1.83685,$
$f(x_1, y_1) = 0.034277.$

Local minimum:

$\lambda_2 = -0.108124,\ x_2 = -0.395591,\ y_2 = 1.83685,$
$f(x_2, y_2) = 0.034277.$

Local maximum:

$\lambda_3 = 0.0410612,\ x_3 = 0.957548,\ y_3 = 0.576545,$
$f(x_3, y_3) = 1.21781.$

Local maximum:

$\lambda_4 = 0.0410612,\ x_4 = -0.957548,\ y_4 = -0.576545,$
$f(x_4, y_4) = 1.21781.$

f[0.395591,-1.83685]

f[-0.395591,1.83685]

f[0.957548,0.576545]

f[-0.957548,-0.576545]

Without using Lagrange Multipliers method we can also use FindMinimum command to search for local minimum:

FindMinimum[{f[x,y],g[x,y]==1},{x,y}]

Indeed, Mathematica gives the same result for the second minimum.

However, it only returns one local minimum. To verify our results, we can give additional information to Mathematica to search for other minimum near to the one obtained from our analysis:

```
FindMinimum[{f[x,y],g[x,y]==1},{{x,0.35},{y,-1.5}}]
```

Similarly, for local maxima:

```
FindMaximum[{f[x,y],g[x,y]==1},{x,y}]
```

```
FindMaximum[{f[x,y],g[x,y]==1},{{x,-0.9},{y,-0.5}}]
```

```
Module[{plot,x1,y1,x2,y2,x3,y3,x4,y4},

{x1,y1}={0.395591,-1.83685};

{x2,y2}={-0.395591,1.83685};

{x3,y3}={0.957548,0.576545};

{x4,y4}={-0.957548,-0.576545};

plot[1]=Plot3D[f[x,y]//Evaluate,{x,-3,3},{y,-3,3}];

plot[2]=Graphics3D[{{Red,PointSize[0.025],Point[{{x1,y1,f[x1,y1]},
{x2,y2,f[x2,y2]}}]},{Black,PointSize[0.025],Point[{{x3,y3,f[x3,y3]},
{x4,y4,f[x4,y4]}}]}}];

Show[Array[plot,2],AxesLabel->{"x","y","z"},ViewPoint->{-3,3,8}]

]
```

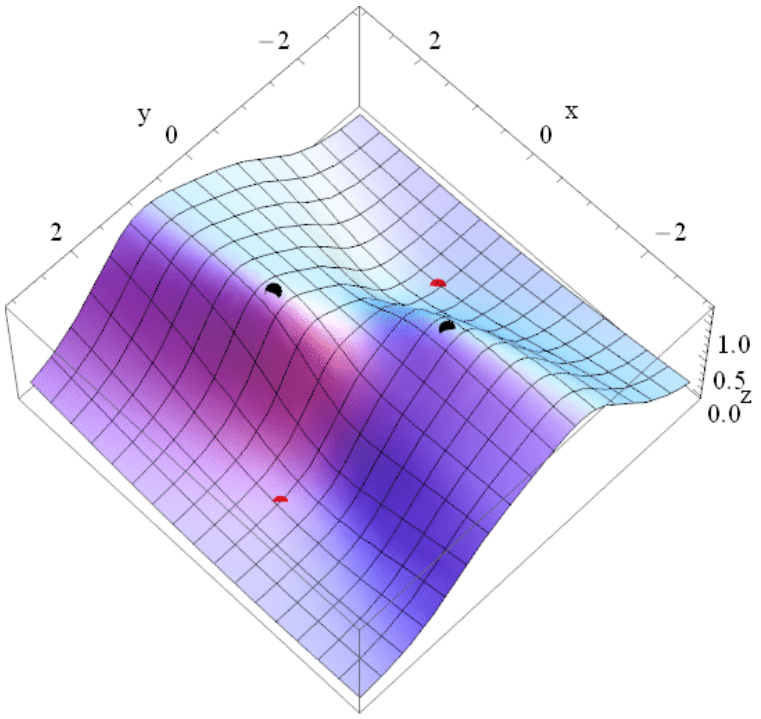


Figure 5.18. The graph f(x,y)=(1 + x^2 + x y)/(1 + x^2 + y^4) subject to the constraint x^2+y^2/4=1 with local maxima (black) and minima (red) .

```
Module[{plot,x1,y1,x2,y2,x3,y3,x4,y4},

{x1,y1}={0.395591,-1.83685};

{x2,y2}={-0.395591,1.83685};

{x3,y3}={0.957548,0.576545};

{x4,y4}={-0.957548,-0.576545};

plot[1]=ContourPlot[f[x,y],{x,-3,3},{y,-3,3},Contours->
{0.034277,1.21781},ContourShading->None,ContourStyle->
{{AbsoluteThickness[2],Green},{AbsoluteThickness[2],Orange}}];

plot[2]=ContourPlot[g[x,y]==1,{x,-3,3},{y,-3,3},ContourStyle->
{AbsoluteThickness[2]}];
```

```
plot[3]=Graphics[{{Red,PointSize[0.02],Point[{{x1,y1},{x2,y2}}]},
{Black,PointSize[0.02],Point[{{x3,y3},{x4,y4}}]}}];

Show[Array[plot,3]]

]
```

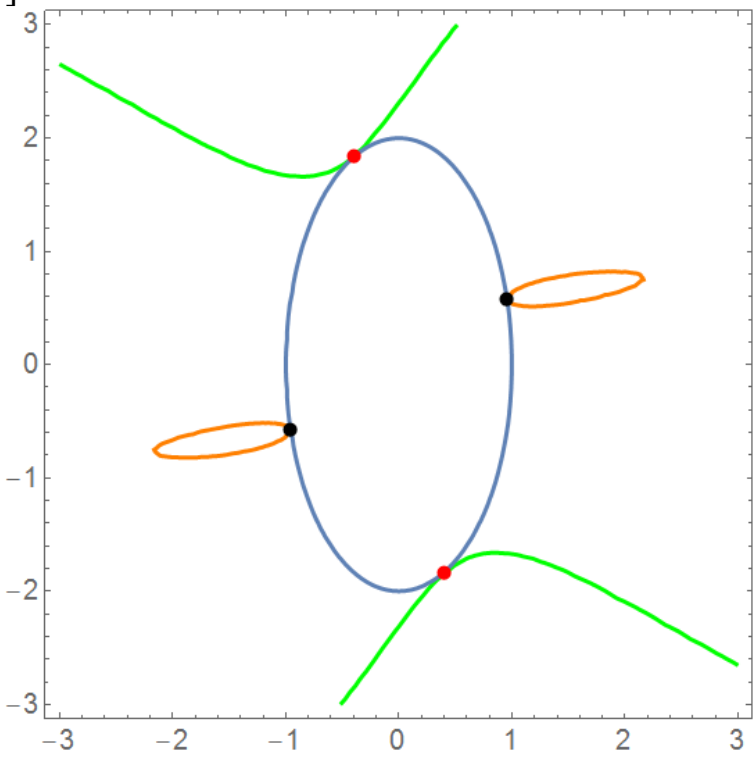


Figure 5.19. The level curves of the function f(x,y)=(1 + x^2 + x y)/(1 + x^2 + y^4) optimized at local maxima (orange curves) and local minima (green curves). They touch the level curve of the constraint g(x,y)=x^2+y^2/4=1 at their respective maxima (black) and minima (red).

Example 5.17. Find the extreme values of

f(x,y,z)=x+y+z

subject to the constraints z^3+z=E^x and x^2+y^2=1 by using FindMinimum and FindMaximum.

Hint: this problem is rather difficult to solve using Lagrange Multipliers method due to the transcendental nature of the constraints.

Solution:

```
Clear[f,g1,g2]

f[x_,y_,z_]:=x+y+z

g1[x_,y_,z_]:=-E^x+z^3+z

g2[x_,y_,z_]:=x^2+y^2


(* NSolve cannot find the solutions using Lagrange Multipliers method *)

NSolve[{Grad[f[x,y,z],{x,y,z},"Cartesian"]==λ1 Grad[g1[x,y,z],
{x,y,z},"Cartesian"]+λ2 Grad[g2[x,y,z],
{x,y,z},"Cartesian"],g1[x,y,z]==0,g2[x,y,z]==1}//Thread,Reals]


FindMinimum[{f[x,y,z],g1[x,y,z]==0,g2[x,y,z]==1},{x,y,z}]


FindMaximum[{f[x,y,z],g1[x,y,z]==0,g2[x,y,z]==1},{x,y,z}]
```

The minimum value -1.01006 occurs at (x,y,z)=
(-0.794731,-0.606962,0.391635).

The maximum value 2.45745 occurs at (x,y,z)=(0.835099,0.5501,1.07225).

```
Module[{plot,point,points},

point[1]={-0.794731,-0.606962,0.391635};

point[2]={0.835099,0.5501,1.07225};

plot[1]=ContourPlot3D[f[x,y,z]//Evaluate,{x,-1,1},{y,-1,1},
{z,-1,2},Contours->{-1.01006,2.45745}];

plot[2]=ContourPlot3D[g1[x,y,z]==0//Evaluate,{x,-1,1},{y,-1,1},
{z,-1,2},Mesh->None,ContourStyle->{Green,Opacity[0.2]}];

plot[3]=ContourPlot3D[g2[x,y,z]==1//Evaluate,{x,-1,1},{y,-1,1},
{z,-1,2},Mesh->None,ContourStyle->{Yellow,Opacity[0.3]}];

points=Graphics3D[{{Red,PointSize[0.02],Point[point[1]]},
{Blue,PointSize[0.03],Point[point[2]]}}];

Show[Array[plot,3],points,AxesLabel->{"x","y","z"}]

]
```

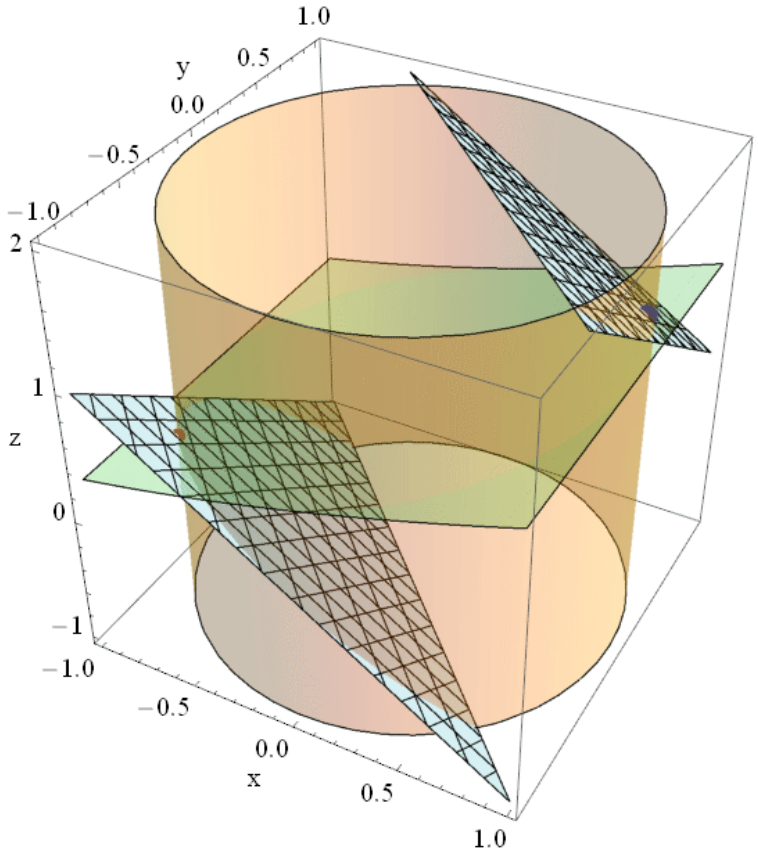


Figure 5.20. The maximum and minimum of the function f(x,y,z)=x+y+z plotted as level surfaces of f(x,y,z)=-1.01006 and f(x,y,z)=2.45745, respectively; subject to the constraints z^3+z=E^x and x^2+y^2=1.

Example 5.18. Optimization Inside a Region

Find the extrema and value of the global extremum of the function f(x,y)=x^4+y^4-x^2-y^2+4 subject to the inequality constraint g(x,y)=x^2+y^2-2x-3<=0.

Solution:

Geometrically, the constraint represents a region $\mathcal{D}$ in the xy-plane, with boundary defined by the curve g(x,y)=x^2+y^2-2x-3=0. To solve this problem, we can break the constraint into two cases:

(a) For points on the boundary curve: g(x,y)=x^2+y^2-2x-3=0.

The curve x^2+y^2-2x-3=0 is actually a circle of radius 2 centered at (1,0), for we can re-write it as

(x-1)^2+y^2=4.

We use Lagrange Multipliers method to find the critical points on the boundary and the extremum value of f(x,y).

(b) For points inside the region: g(x,y)=x^2+y^2-2x-3<0.

We can use the techniques discussed in Section 5.2 Global Extrema to find the critical points and the extremum value of f(x,y).

The true global extremum is found by comparing the two answers from part (a) and part (b).

```
Clear[f,g]

f[x_,y_]:=x^4+y^4-x^2-y^2+4

g[x_,y_]:=x^2+y^2-2x-3

ContourPlot[g[x,y],{x,-2,4},{y,-3,3},ContourStyle->
{AbsoluteThickness[3],Blue},Contours->{0},ImageSize-
>300,ContourShading->{Hue[0.5],GrayLevel[0.8]},FrameLabel->
{"x","y"}]
```

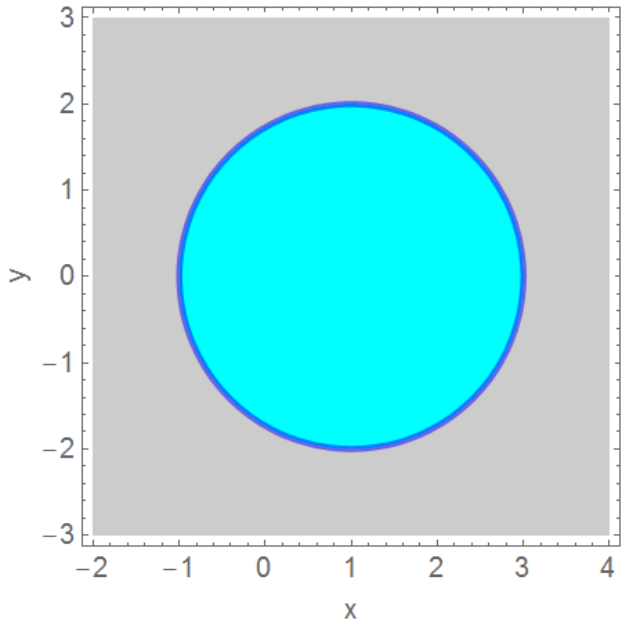


Figure 5.21. The region $\mathcal{D}$ is a disk of radius 2 centered at (1,0).

(a) For points on the boundary curve: g(x,y)=x^2+y^2-2x-3=0.

We use Lagrange Multipliers method to find the extreme values of

f(x,y)=x^4+y^4-x^2-y^2+4,

subject to the given constraint:

g(x,y)=x^2+y^2-2x-3=0.

Step 1. Write out the Lagrange equations.

$$\nabla f = \left(4x^3 - 2x,\ 4y^3 - 2y\right)$$

,

$$\nabla g = (2x - 2,\ 2y)$$.

Using Lagrange conditions:

$\nabla f(x, y) = \lambda \nabla g(x, y)$,

the Lagrange equations are given by

$$\begin{cases} 4x^3 - 2x = \lambda(2x - 2), \\ 4y^3 - 2y = 2\lambda y. \end{cases}$$

Step 2. Solve for λ , x and y with constraint using Solve command.

Solve[{4x^3-2x==λ(2x-2),4y^3-2y==2λ y,x^2+y^2-2x-3==0},Reals]//N

We have four critical points on the boundary of the region:

$\lambda_1 = 0.5; \qquad (x_1, y_1) = (-1, 0)$

,

$\lambda_2 = 25.5; \qquad (x_2, y_2) = (3, 0)$

,

$\lambda_3 = 0.195288; \quad (x_3, y_3) = (-0.844548, -0.773074)$

,

$\lambda_4 = 0.195288; \quad (x_4, y_4) = (-0.844548, 0.773074)$

.

The extreme values corresponding to the critical points above are

f(-1,0)=4,

f(3,0)=76,

f(-0.844548,-0.773074)=3.55501,

f(-0.844548,0.773074)=3.55501.

Whether it is a local maximum or minimum is not important at this moment as we seek global extrema.

```
Clear[x1,y1,x2,y2,x3,y3,x4,y4]

{x1,y1}={-1,0};

{x2,y2}={3,0};

{x3,y3}={-0.844548,-0.773074};

{x4,y4}={-0.844548,0.773074};

f[x1,y1]

f[x2,y2]

f[x3,y3]

f[x4,y4]


Module[{plot},
```

```
plot[1]=ContourPlot[g[x,y],{x,-2,4},{y,-3,3},ContourStyle->
{AbsoluteThickness[3],Blue},Contours->{0},ImageSize-
>300,ContourShading->{Hue[0.5],GrayLevel[0.8]},FrameLabel->
{"x","y"}];

plot[2]=Graphics[{PointSize[0.03],Red,Point[{{x1,y1},{x2,y2},{x3,y3},
{x4,y4}}]}];

Show[Array[plot,2]]

]
```

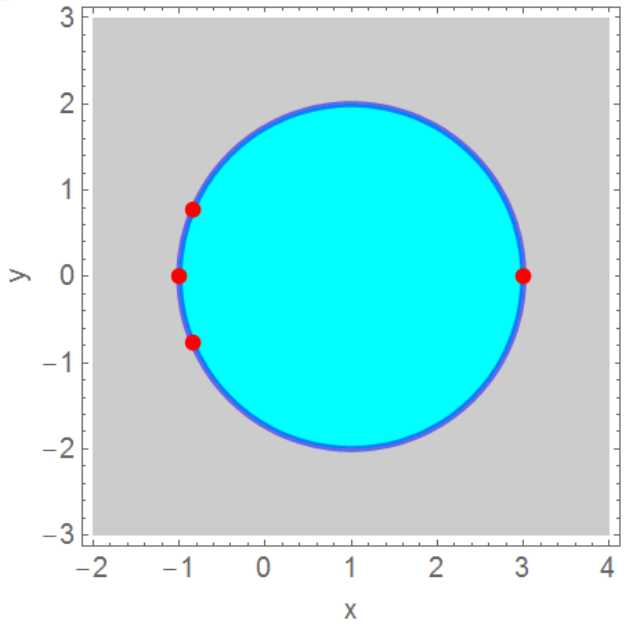


Figure 5.22. The region $\mathcal{D}$ with boundary critical points.

(b) For points inside the region: g(x,y)=x^2+y^2-2x-3<0.

We solve for the inequalities involving x and y using Reduce command:

The solution is given by

-1

Note that the term inside the square root has to be positive, hence it yields the inequality for x.

Reduce[g[x,y]<0,y]

These are the points that satisfy the interior constraint g(x,y)<0. The points are located inside the shaded area of the graph below

Plot[{-Sqrt[3+2 x-x^2],Sqrt[3+2 x-x^2]}//Evaluate,{x,-1,3},PlotStyle->{{AbsoluteThickness[3],Dashed,Blue}},Filling->{1->{2}},FillingStyle->Hue[0.5],AxesLabel->{"x","y"},AspectRatio->Automatic,ImageSize->200]

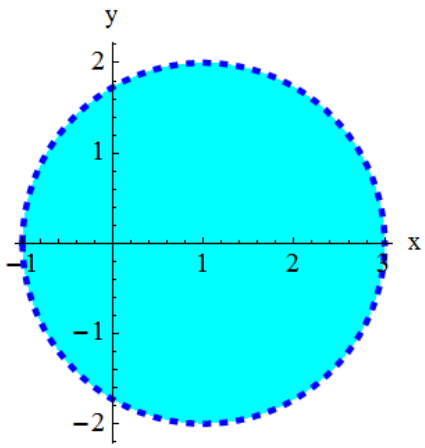


Figure 5.23. The open region of $\mathcal{D}$: g(x,y)=x^2+y^2-2x-3<0.

Our next task is to find the points inside the region of the disk bounded by the curve -Sqrt[3+2x-x^2] and Sqrt[3+2x-x^2], together with -1

The critical points of f can be found by solving:

$$\nabla f(x, y) = \left(4x^3 - 2x, 4y^3 - 2y\right) = \mathbf{0}$$

.

Solve[Grad[f[x,y],{x,y},"Cartesian"]=={0,0}//Thread,{x,y}]

There are nine critical points as possible extrema:

$$(x_5, y_5) = \left(-\frac{1}{\sqrt{2}}, 0\right), (x_6, y_6) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), (x_7, y_7) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

,

$$(x_8, y_8) = \left(\frac{1}{\sqrt{2}}, 0\right), \quad (x_9, y_9) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad (x_{10}, y_{10}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

,

$$(x_{11}, y_{11}) = (0, 0), \quad (x_{12}, y_{12}) = \left(0, -\frac{1}{\sqrt{2}}\right), \quad (x_{13}, y_{13}) = \left(0, \frac{1}{\sqrt{2}}\right)$$

.

The reason for numbering the coordinates starting from 5 is because we have already obtained four critical points from part (a) above. Note that all these points are interior points of $\mathcal{D}$.

The extreme values corresponding to the critical points above are

$$f(x_5, y_5) = \left(-\frac{1}{\sqrt{2}}, 0\right) = 3.75,$$
$$f(x_6, y_6) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 3.5,$$
$$f(x_7, y_7) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 3.5,$$
$$f(x_8, y_8) = \left(\frac{1}{\sqrt{2}}, 0\right) = 3.75,$$
$$f(x_9, y_9) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 3.5,$$
$$f(x_{10}, y_{10}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 3.5,$$
$$f(x_{11}, y_{11}) = (0, 0) = 4,$$
$$f(x_{12}, y_{12}) = \left(0, -\frac{1}{\sqrt{2}}\right) = 3.75,$$
$$f(x_{13}, y_{13}) = \left(0, \frac{1}{\sqrt{2}}\right) = 3.75.$$

```
Clear[x5,y5,x6,y6,x7,y7,x8,y8,x9,y9,x10,y10,x11,y11,x12,y12,x13,y13]

{x5,y5}={-(1/Sqrt[2]),0};

{x6,y6}={-(1/Sqrt[2]),-(1/Sqrt[2])};

{x7,y7}={-(1/Sqrt[2]),1/Sqrt[2]};

{x8,y8}={1/Sqrt[2],0};

{x9,y9}={1/Sqrt[2],-(1/Sqrt[2])};

{x10,y10}={1/Sqrt[2],1/Sqrt[2]};

{x11,y11}={0,0};

{x12,y12}={0,-(1/Sqrt[2])};

{x13,y13}={0,1/Sqrt[2]};
```

```
{f[x5,y5],f[x6,y6],f[x7,y7],f[x8,y8],f[x9,y9],f[x10,y10],f[x11,y11],f[x12,y12
],f[x13,y13]}//N


Module[{plot},

plot[1]=Plot[{-Sqrt[3+2 x-x^2],Sqrt[3+2 x-x^2]}//Evaluate,
{x,-1,3},PlotStyle->{{AbsoluteThickness[3],Dashed,Blue}},Filling->{1->
{2}},FillingStyle->Hue[0.5],AxesLabel->{"x","y"},AspectRatio-
>Automatic,ImageSize->200];

plot[2]=Graphics[{PointSize[0.03],Point[{{x5,y5},{x6,y6},{x7,y7},
{x8,y8},{x9,y9},{x10,y10},{x11,y11},{x12,y12},{x13,y13}}]}];

Show[Array[plot,2]]

]
```

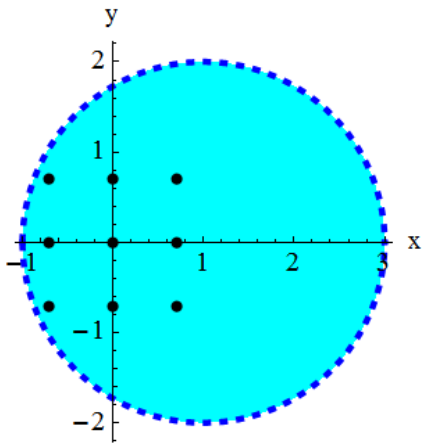


Figure 5.24. The region $\mathcal{D}$ with interior critical points.

Comparing the results above, we have the global maximum occurs at $(x_2, y_2) = (3, 0)$ (this is a boundary point) with value of f(3,0)=76.

The global minimum value of f is 3.5 at four interior critical points

$$(x_6, y_6) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), (x_7, y_7) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

,

$$(x_9, y_9) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad (x_{10}, y_{10}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

.

```
Module[{plot},

plot[1]=Plot3D[f[x,y]//Evaluate,{x,-1,3},{y,-3,3},PlotRange-
>All,AxesLabel->{"x","y","z"},PlotStyle->Opacity[0.3]];

plot[2]=Graphics3D[{{Red,PointSize[0.02],Point[{x2,y2,f[x2,y2]}]},
{Blue,PointSize[0.02],Point[{{x6,y6,f[x6,y6]},{x7,y7,f[x7,y7]},
{x9,y9,f[x9,y9]},{x10,y10,f[x10,y10]}}]},
{Black,PointSize[0.02],Point[{{x1,y1,f[x1,y1]},{x3,y3,f[x3,y3]},
{x4,y4,f[x4,y4]},{x5,y5,f[x5,y5]},{x8,y8,f[x8,y8]},{x11,y11,f[x11,y11]},
{x12,y12,f[x12,y12]},{x13,y13,f[x13,y13]}}]}}];

Show[Array[plot,2]]

]
```

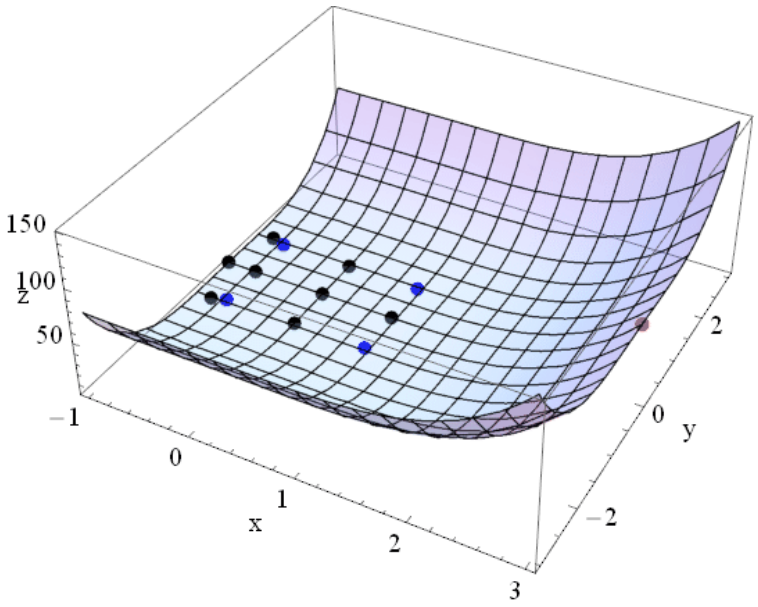


Figure 5.25. Critical points of the function f(x,y)=x^4+y^4-x^2-y^2+4 subject to the inequality constraint g(x,y)=x^2+y^2-2x-3<=0. Global maximum (red) occurs at $(x_2, y_2) = (3, 0)$. There are four global minima (blue) at $(x_6, y_6) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $(x_7, y_7) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(x_9, y_9) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $(x_{10}, y_{10}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Chapter 6. Multiple Integration

In this chapter we discuss integration involving functions of several variables. Such integrals are called multiple integrals which are a generalization of the notion of one variable integrals studied in first year calculus courses. In one variable, the domain of integration is simply an interval [a,b]. However, it is more complicated in the case of multiple integrals. It is the domain of integration that plays a prominent role in the multivariable case as we shall see in the rest of this chapter.

Initialization

```
Clear["Global`*"]
```

```
SetOptions[Plot3D,PlotTheme->"Classic"];
```

```
SetOptions[ContourPlot3D,PlotTheme->"Classic"];
```

```
SetOptions[Plot,PlotTheme->"Classic",PlotStyle->AbsoluteThickness[2]];
```

```
SetOptions[ParametricPlot3D,PlotTheme->"Classic",PlotStyle-
>AbsoluteThickness[2]];
```

6.1 Integration in Two Variables

We begin our discussion with simple domain of integration which forms a rectangular shape.

6.1.1 The Integral over a Rectangle

Suppose that (x,y)->f(x,y) is a non-negative-valued continuous function whose domain contains the closed rectangle

D={ (x,y)|a<=x<=b,c<=y<=d },

or written as D=[a,b]x[c,d].

We shall use the term domain and region interchangeably since the domain of integration is the same region in $\mathbb{R}^2$ over which the integration of f(x,y) takes place.

The integral of a function of two variables f(x,y) are called double integral. It is denoted by

$$\iint_D f(x, y)\, dA$$

.

When $\iint_D f(x, y)\, dA$ exists, we say that f(x,y) is integrable on D.

We state without proof the result of the Fubini's Theorem that the double integrals under certain assumptions can be calculated by using iterated integrals and, moreover, that the order of integration for the iterated integral does not matter.

Theorem 6.1. Fubini's Theorem for Double Integrals

Let f(x,y) be a continuous function defined over a rectangle domain D= [a,b]x[c,d]. The double integral of f over D is equal to the iterated integral given by

(6.1)

$$\iint_D f(x, y)\, dA = \int_{x=a}^{b}\int_{y=c}^{d} f(x, y)\, dy\, dx$$
$$= \int_{y=c}^{d}\int_{x=a}^{b} f(x, y)\, dx\, dy$$

.

To evaluate the iterated integral, for instance

$$\int_{x=a}^{b}\left(\int_{y=c}^{d} f(x, y)\, dy\right) dx$$.

First, hold x fixed and evaluate the inner integral with respect to y. This gives the result as a function in x alone. Then integrate the resulting function with respect to x.

Geometrically, the double integral $\iint_D f(x, y)\, dA$ defines the volume V of the solid region between the graph of a positive function z=f(x,y) and the rectangle D in the xy-plane given by

$$V \equiv \iint_D f(x, y)\, dA$$.

If f(x,y) takes on both positive and negative values, the double integral defines a signed volume. A special case when f(x,y)=1, we have

$$A \equiv \iint_D 1\, dA$$,

where A is the area of the region D.

Example 6.1. Evaluate $\iint_D x^2 \sin y\, dA$, where D=[-1,1]x[0,π] and shows that changing the integration order yields the same result, and thus verifies Theorem 6.1.

Solution:

Write

$$\begin{aligned}
\iint_D x^2 \sin y\, dA &= \int_{x=-1}^{1} \int_{y=0}^{\pi} x^2 \sin y\, dy\, dx \\
&= \int_{x=-1}^{1} x^2 \left(\int_{y=0}^{\pi} \sin y\, dy \right) dx \\
&= \int_{x=-1}^{1} x^2 (-\cos y) \Big]_0^{\pi} dx \\
&= 2 \int_{x=-1}^{1} x^2\, dx \\
&= \frac{2}{3} x^3 \Big]_{-1}^{1} = \frac{4}{3}
\end{aligned}$$

Integrate[x^2 Sin[y],{x,-1,1},{y,0,π}]

Note the position of the limits of integration in the Integrate command. It follows the same positions as the iterated integrals.

Next, we show that the change of order of integration for the rectangular domain does not alter the result.

$$\iint_D x^2 \sin y \, dA = \int_{y=0}^{\pi} \int_{x=-1}^{1} x^2 \sin y \, dx \, dy$$
$$= \frac{1}{3} \int_{y=0}^{\pi} x^3 \Big]_{-1}^{1} \sin y \, dy$$
$$= \frac{2}{3} \int_0^{\pi} \sin y \, dy$$
$$= \frac{2}{3} (-\cos y) \Big]_0^{\pi} = \frac{4}{3}$$

Integrate[x^2 Sin[y],{y,0,π},{x,-1,1}]

The graph of the function and the domain is given below:

```
Module[{f,range,vertex,plot},
f[x_,y_]:=x^2 Sin[y];
range[1]={-1,1};
range[2]={0,π};
vertex[1]={range[1][[1]],range[2][[1]],0};
vertex[2]={range[1][[1]],range[2][[2]],0};
vertex[3]={range[1][[2]],range[2][[1]],0};
vertex[4]={range[1][[2]],range[2][[2]],0};
vertex[5]={range[1][[1]],range[2][[1]],f[range[1][[1]],range[2][[1]]]};
vertex[6]={range[1][[1]],range[2][[2]],f[range[1][[1]],range[2][[2]]]};
```

```
vertex[7]={range[1][[2]],range[2][[1]],f[range[1][[2]],range[2][[1]]]};

vertex[8]={range[1][[2]],range[2][[2]],f[range[1][[2]],range[2][[2]]]};

plot[1]=Plot3D[f[x,y],{x,-1,1},{y,0,π},PlotRange->{{-1.5,1.5},{-1,4},
{0,1.5}},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{Red,PointSize[0.015],Point[Array[vertex,8]]}];

plot[3]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[2]
},{vertex[2],vertex[4]},{vertex[4],vertex[3]},{vertex[3],vertex[1]}}]}];

plot[4]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[5]
},{vertex[2],vertex[6]},{vertex[3],vertex[7]},{vertex[4],vertex[8]}}]}];

Show[Array[plot,4]]

]
```

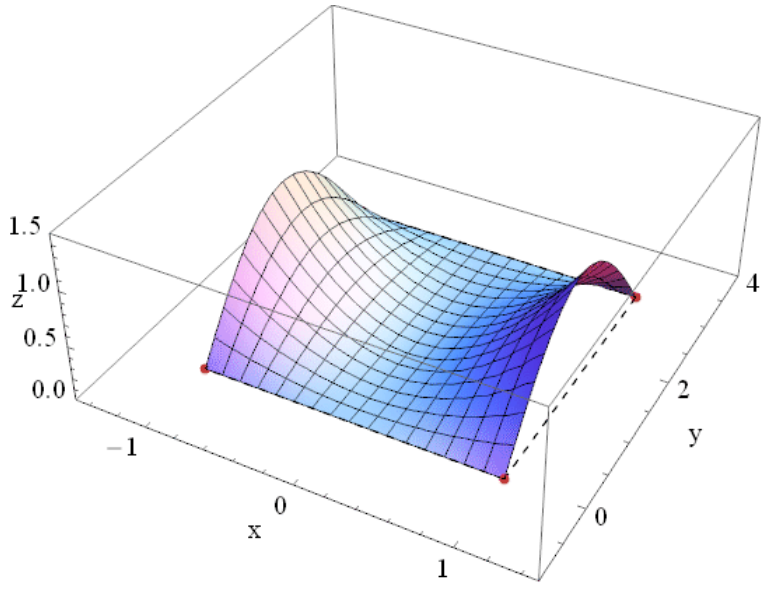


Figure 6.1. The solid region between the function f(x,y)=x^2 sin y and the rectangle D=[-1,1]x[0,π] in Example 6.1.

Example 6.2. Evaluate $\int\int_D \frac{1}{x+y}\,dA$, where D={ (x,y)|0<=x<=2,1<=y<=E }.

Solution:

$$\begin{aligned}\int\int_D \frac{1}{x+y}\,dA &= \int_{x=0}^{2}\int_{y=1}^{e} \frac{1}{x+y}\,dy\,dx \\ &= \int_{x=0}^{2} \ln(x+y)\Big]_1^e\,dx \\ &= \int_{x=0}^{2} [\ln(x+e) - \ln(x+1)]\,dx \\ &= [-x + e\ln(x+e) + x\ln(x+e) \\ &\qquad + x - \ln(x+1) - x\ln(x+1)]_0^2 \\ &= [(x+e)\ln(x+e) - (x+1)\ln(x+1)]_0^2 \\ &= (2+e)\ln(2+e) - 3\ln 3 - e\end{aligned}$$

Integrate[1/(x+y),{x,0,2},{y,1,E}]

We plot the the graph using the following module:

```
Module[{f,range,vertex,plot},
f[x_,y_]:=1/(x+y);
range[1]={0,2};
range[2]={1,E};
vertex[1]={range[1][[1]],range[2][[1]],0};
vertex[2]={range[1][[1]],range[2][[2]],0};
```

```
vertex[3]={range[1][[2]],range[2][[1]],0};

vertex[4]={range[1][[2]],range[2][[2]],0};

vertex[5]={range[1][[1]],range[2][[1]],f[range[1][[1]],range[2][[1]]]};

vertex[6]={range[1][[1]],range[2][[2]],f[range[1][[1]],range[2][[2]]]};

vertex[7]={range[1][[2]],range[2][[1]],f[range[1][[2]],range[2][[1]]]};

vertex[8]={range[1][[2]],range[2][[2]],f[range[1][[2]],range[2][[2]]]};

plot[1]=Plot3D[f[x,y],{x,0,2},{y,1,E},PlotRange->{{-0.5,2.5},{0,3},
{0,1}},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{Red,PointSize[0.015],Point[Array[vertex,8]]}];

plot[3]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[2]
},{vertex[2],vertex[4]},{vertex[4],vertex[3]},{vertex[3],vertex[1]}}]}];

plot[4]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[5]
},{vertex[2],vertex[6]},{vertex[3],vertex[7]},{vertex[4],vertex[8]}}]}];

Show[Array[plot,4]]

]
```

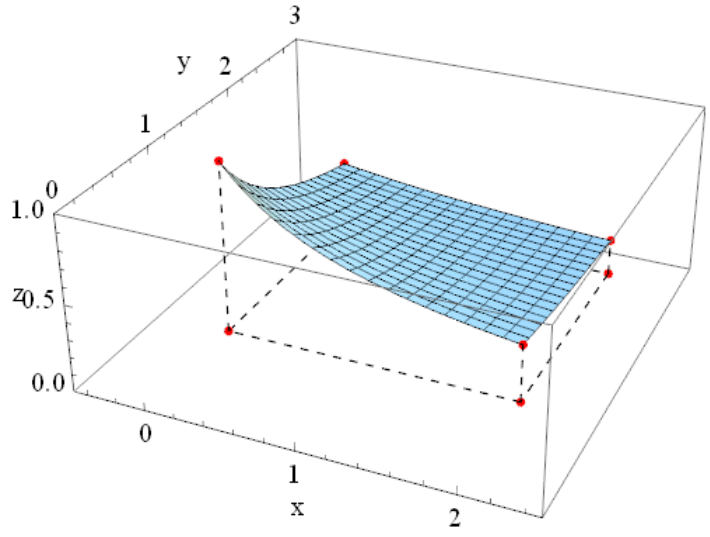


Figure 6.2. The solid region between the function f(x,y)=1/(x + y) and the rectangle D=[0,2]x[1,E] in Example 6.2

Example 6.3. Find the volume of the solid region bounded on top by the plane

z=x+3y+1,

on the bottom by the xy-plane, and on the sides by the planes x=0, x=3, y=1, y=2.

Solution:

First we determine the domain of integration which is a rectangle given by

D={ (x,y)|0<=x<=3,1<=y<=2 }.

```
Module[{f,range,vertex,plot},
f[x_,y_]:=x+3y+1;
range[1]={0,3};
range[2]={1,2};
vertex[1]={range[1][[1]],range[2][[1]],0};
vertex[2]={range[1][[1]],range[2][[2]],0};
vertex[3]={range[1][[2]],range[2][[1]],0};
vertex[4]={range[1][[2]],range[2][[2]],0};
```

```
vertex[5]={range[1][[1]],range[2][[1]],f[range[1][[1]],range[2][[1]]]};

vertex[6]={range[1][[1]],range[2][[2]],f[range[1][[1]],range[2][[2]]]};

vertex[7]={range[1][[2]],range[2][[1]],f[range[1][[2]],range[2][[1]]]};

vertex[8]={range[1][[2]],range[2][[2]],f[range[1][[2]],range[2][[2]]]};

plot[1]=Plot3D[f[x,y],{x,0,3},{y,1,2},PlotRange->{{-0.5,3.5},{0,3},
{0,11}},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{Red,PointSize[0.015],Point[Array[vertex,8]]}];

plot[3]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[2]
},{vertex[2],vertex[4]},{vertex[4],vertex[3]},{vertex[3],vertex[1]}}]}];

plot[4]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[5]
},{vertex[2],vertex[6]},{vertex[3],vertex[7]},{vertex[4],vertex[8]}}]}];

Show[Array[plot,4]]

]
```

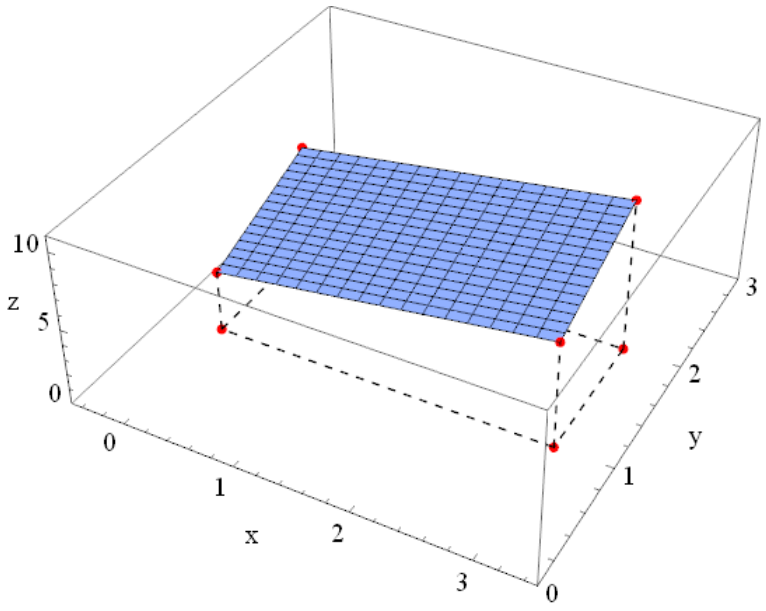


Figure 6.3. The solid region bounded on top by the plane z=x+3y+1, on the bottom by the xy-plane, and on the sides by the planes x=0, x=3, y=1, y=2.

Thus, the volume of the solid region is

$$V = \iint_D f(x, y)\, dA = \int_{x=0}^{3}\int_{y=1}^{2}(x + 3y + 1)\, dy\, dx$$
$$= 21$$

Integrate[x+3y+1,{x,0,3},{y,1,2}]

Example 6.4. Find the volume of the solid region bounded by the surface

f(x,y)=2x^2+y^4 sin π x,

the xy-plane, and the planes x=0, x=1, y=-1, y=2.

Solution:

The domain of integration is a rectangle given by

D={ (x,y)|0<=x<=1,-1<=y<=2 }.

Module[{f,range,vertex,plot},

f[x_,y_]:=2x^2+y^4 Sin[π x];

range[1]={0,1};

range[2]={-1,2};

vertex[1]={range[1][[1]],range[2][[1]],0};

```
vertex[2]={range[1][[1]],range[2][[2]],0};

vertex[3]={range[1][[2]],range[2][[1]],0};

vertex[4]={range[1][[2]],range[2][[2]],0};

vertex[5]={range[1][[1]],range[2][[1]],f[range[1][[1]],range[2][[1]]]};

vertex[6]={range[1][[1]],range[2][[2]],f[range[1][[1]],range[2][[2]]]};

vertex[7]={range[1][[2]],range[2][[1]],f[range[1][[2]],range[2][[1]]]};

vertex[8]={range[1][[2]],range[2][[2]],f[range[1][[2]],range[2][[2]]]};

plot[1]=Plot3D[f[x,y],{x,0,1},{y,-1,2},PlotRange->{{-0.5,1.5},{-1.5,2.5},
{0,20}},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{Red,PointSize[0.015],Point[Array[vertex,8]]}];

plot[3]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[2]
},{vertex[2],vertex[4]},{vertex[4],vertex[3]},{vertex[3],vertex[1]}}]}];

plot[4]=Graphics3D[{Dashed,Thickness[0.003],Line[{{vertex[1],vertex[5]
},{vertex[2],vertex[6]},{vertex[3],vertex[7]},{vertex[4],vertex[8]}}]}];

Show[Array[plot,4]]

]
```

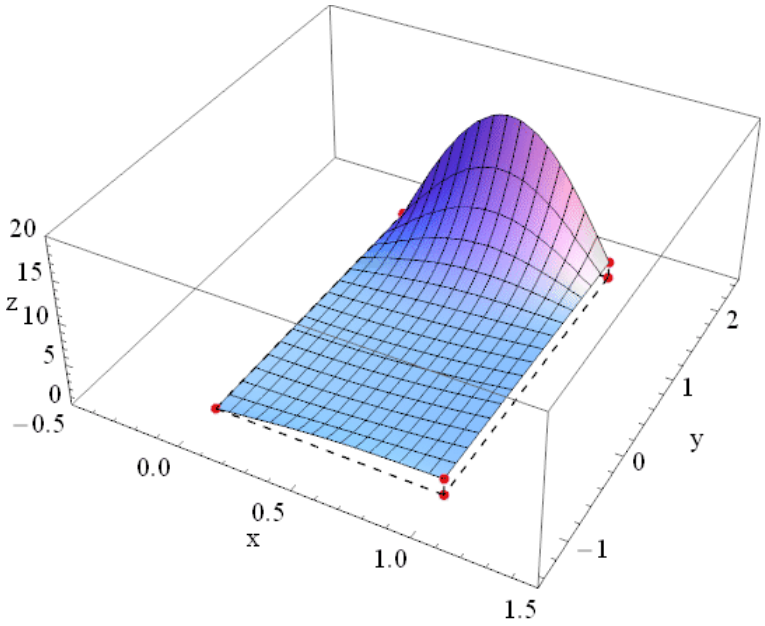


Figure 6.4. The solid region bounded on top by the surface f(x,y)=2x^2+y^4 sin π x, the xy-plane, and on the sides by the planes x=0, x=1, y=-1, y=2.

The volume of the solid region:

$$V = \iint_D f(x, y)\, dA = \int_{x=0}^{1}\int_{y=-1}^{2} \left(2x^2 + y^4 \sin \pi x\right) dy\, dx$$

$$= 2 + \frac{66}{5\pi}$$

Integrate[2x^2+y^4 Sin[π x],{x,0,1},{y,-1,2}]

6.1.2 Double Integrals over General Regions

In previous section, we have considered only domains of integration with rectangular shape. Our next step is to define the integral of a function of two variables over an arbitrary region D in the plane whose

boundaries are simple closed curves. A curve is simple if it does not intersect itself. Of such, there are infinitely many regions one can possibly imagine. Therefore we shall restrict our discussion in this book only certain type of regions, and in addition we will assume that the integrand f is continuous over the domain of integration. The types of regions we shall consider in this book are called elementary regions.

Let us consider some simple regions as are defined below:

Definition 6.2

The region D in the plane is called elementary region if it can be described as a subset of $\mathbb{R}^2$ of one of the following three types:

Type 1 or vertically simple region (see Figure 6.6):

(6.2) $D=\{ (x,y)|a<=x<=b,\gamma(x)<=y<=\delta(x) \},$

where a, b constants; both γ and δ are continuous functions on [a,b].

Type 2 or horizontally simple region (see Figure 6.8):

(6.3) $D=\{ (x,y)|\alpha(y)<=x<=\beta(y),c<=y<=d \},$

where c, d constants; both α and β are continuous functions on [c,d].

Type 3 D is both type 1 and type 2 (see Figure 6.9 or 6.10).

Using Definition 6.2 we can establish an important result below:

Theorem 6.3

Let D be an elementary region in $\mathbb{R}^2$ as defined in Definition 6.2 and f(x,y) a continuous function over D.

If D is of type 1, then

(6.4)

$$\iint_D f \, dA = \int_{x=a}^{b} \int_{y=\gamma(x)}^{\delta(x)} f(x, y) \, dy \, dx$$

.

If D is of type 2, then

(6.5)

$$\iint_D f \, dA = \int_{y=c}^{d} \int_{x=\alpha(y)}^{\beta(y)} f(x, y) \, dx \, dy$$

.

For type 1 elementary region, $\gamma(x)$ and $\delta(x)$ are the lower boundary and upper boundary curves, respectively.

Where as in type 2 elementary region, $\alpha(y)$ and $\beta(y)$ are the left and right boundary curves, respectively.

Since the domain of integration plays a crucial role in determining the integral, we provide Mathematica modules for visualizing elementary regions, viewRegion1 and viewRegion2 for type 1 and type 2 regions, respectively. The parameters for these modules are given such that they are the same with those needed to enter into the Integrate command in order to compute the integral.

```
(* Graph the shaded area of type one elementary region *)

viewRegion1[{x_,a_,b_},
{y_,f_,g_},opt___]:=Module[{boundaryCurves},boundaryCurves=Plot[{f,g
},{x,a,b},PlotStyle->{{Blue,Thickness[0.006]},
{Red,Thickness[0.006]}},PlotRange->All,AxesLabel->{"x","y"}];

Show[boundaryCurves,Table[Graphics[{GrayLevel[0.8],Line[{{x,f},
{x,g}}]}],{x,a,b,(b-a)/50}],opt]

]


(* Graph the shaded area of type two elementary region *)

viewRegion2[{y_,c_,d_},
{x_,f_,g_},opts___]:=Module[{boundaryCurves},boundaryCurves=Parame
tricPlot[Evaluate[{{f,y},{g,y}}],{y,c,d},PlotStyle->
{{Blue,Thickness[0.006]},{Red,Thickness[0.006]}},AspectRatio-
>Automatic,AxesLabel->{"x","y"}];

Show[boundaryCurves,Table[Graphics[{GrayLevel[0.8],Line[{{f,y},
{g,y}}]}],{y,c,d,(d-c)/50}],opts]

]
```

For instance, if the region D is of type 1, then by Theorem 6.3 the double integral takes the form of equation (6.4):

$$\iint_D f \, dA = \int_{x=a}^{b} \int_{y=\gamma(x)}^{\delta(x)} f(x, y) \, dy \, dx$$

.

The outer limits of this integral is

x=a and x=b.

and the inner limits of the integral is performed first which are

y=γ(x) and y=δ(x).

Such double integral in Mathematica is entered as

Integrate[f[x,y],{x,a,b},{y,γ(x),δ(x)}]

To view the domain of this double integral we shall write

viewRegion1[{x,a,b},{y,γ(x),δ(x)}]

Let us see some examples to illustrate the usage.

Example 6.5. (a) Shade the area of the region D bounded by the curve y=1/x and y=Sqrt[x], and the line x=1 and x=3. Write the description of D.

(b) Find the area of D.

(c) Evaluate $\iint_D x^2 y \, dA$, where D is the region in part (a).

Solution:

This is a type 1 elementary region or vertically simple region according to Definition 6.2.

Using ContourPlot we plot the curves to determine which one is the lower boundary and which is upper boundary curve.

```
ContourPlot[{y==1/x,y==Sqrt[x],x==1,x==3}//Evaluate,{x,0,4},
{y,0,2},Frame->False,Axes->True,AxesLabel->{"x","y"},ContourStyle->
{Blue,Red,Green,Black},PlotLegends->"Expressions"]
```

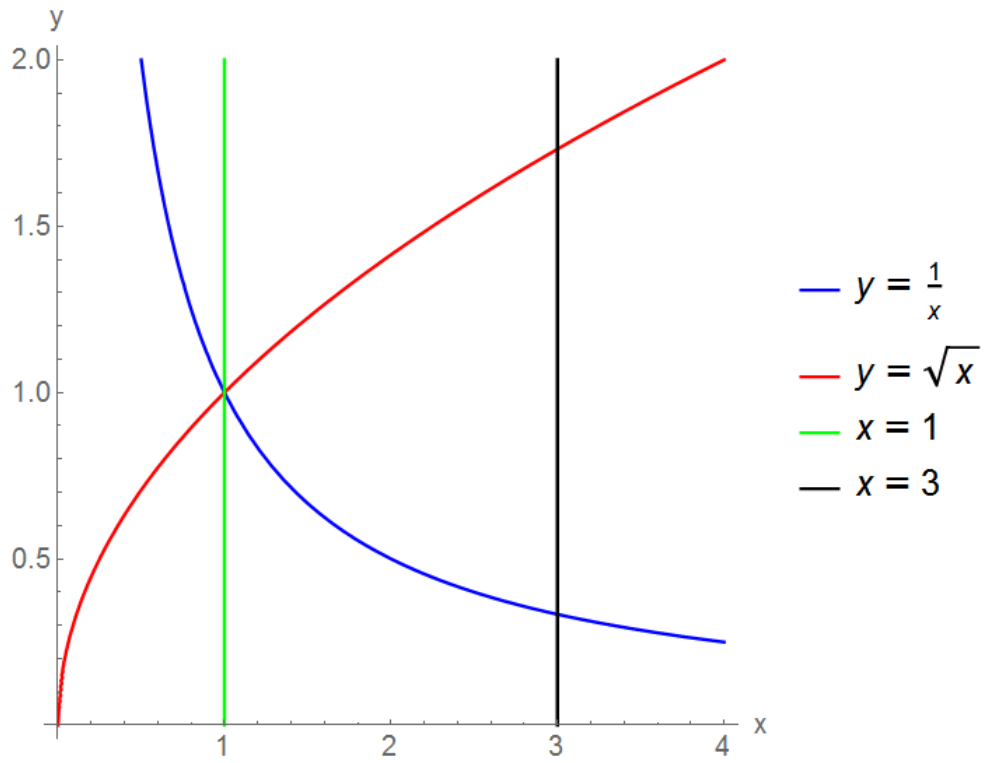


Figure 6.5. The boundary curves of the elementary region.

So, the shaded area can be determined by using viewRegion1.

```
viewRegion1[{x,1,3},{y,1/x,Sqrt[x]},AspectRatio->Automatic]
```

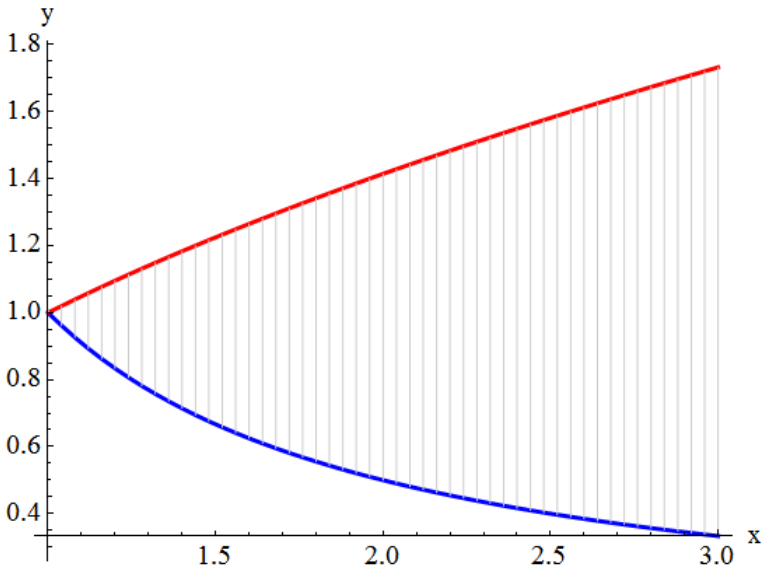


Figure 6.6. Type 1 elementary region (vertically simple region) described by D={ (x,y)|1<=x<=3,1/x<=y<=Sqrt[x] }.

The domain of integration is given by

D={ (x,y)|1<=x<=3,1/x<=y<=Sqrt[x] }.

(b)

Note that the parameters needed to supply to the viewRegion1 command are the same with those we need to supply to the Integrate command.

So, the area of D is

$$\iint_D dA = \int_{x=1}^{3} \int_{y=1/x}^{\sqrt{x}} dy\, dx = 1.69882$$

.

```
Integrate[1,{x,1,3},{y,1/x,Sqrt[x]}]//N
```

(c)

$$\iint_D x^2 y \, dA = \int_{x=1}^{3} \int_{y=1/x}^{\sqrt{x}} x^2 y \, dy \, dx = 9$$

.

```
Integrate[x^2 y,{x,1,3},{y,1/x,Sqrt[x]}]
```

Example 6.6. Integrate the function f(x,y)=x+y over the region bounded by x+y=2 and y^2-2y-x=0.

Solution:

Step 1. Describe the domain of integration D.

First, we plot the boundary curves to determine the region using ContourPlot.

```
ContourPlot[{x+y==2,y^2-2y-x==0}//Evaluate,{x,-2,4},{y,-2,4},Frame->False,Axes->True,AxesLabel->{"x","y"},PlotLegends->"Expressions"]
```

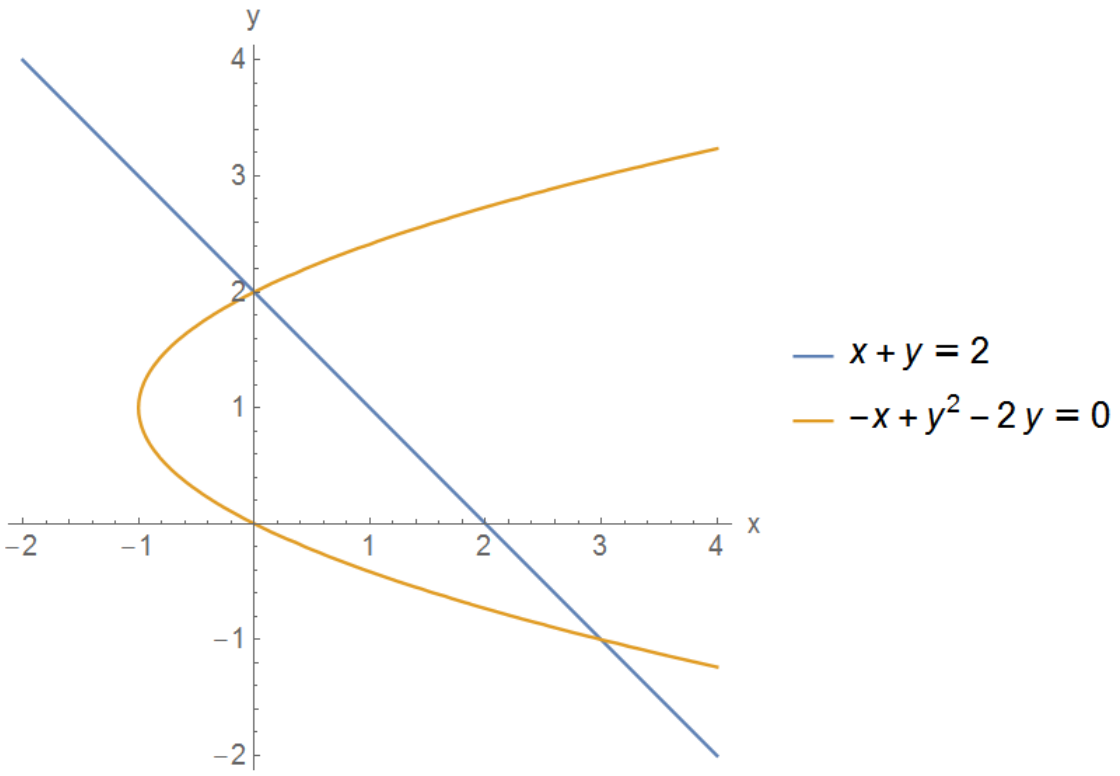


Figure 6.7. The region D is bounded by x+y=2 and y^2-2y-x=0.

The curves intersect where

2-y=y^2-2y,

which gives

y=-1 and y=2.

Solve[2-y==y^2-2y,y]

By Definition 6.2, this is a type 2 elementary region or horizontally simple region.

The limits of integration with respect to x is

x=y^2-2y and x=2-y,

and the limits of integration with respect to y is

y=-1 and y=2.

Thus, the description of D is given by

D={ (x,y)|y^2-2y<=x<=2-y, -1<=y<=2 }.

viewRegion2[{y,-1,2},{x,y^2-2y,2-y}]

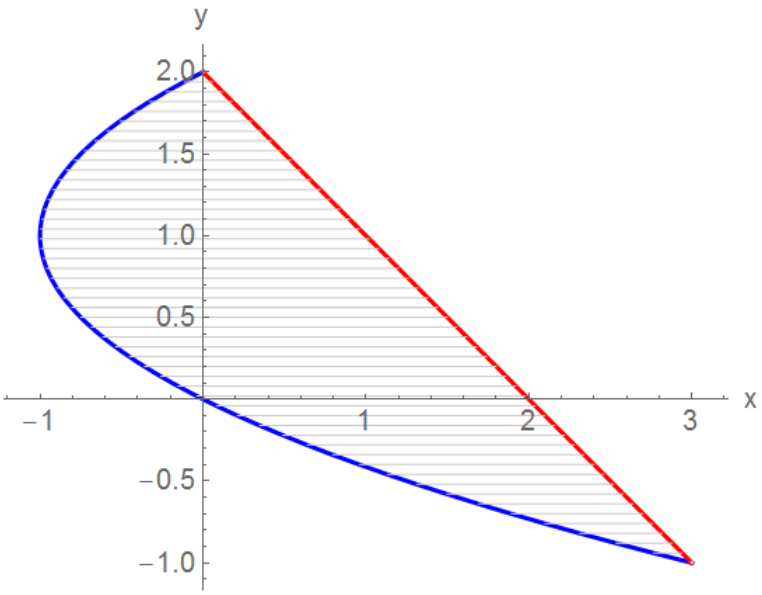


Figure 6.8. Type 2 elementary region (horizontally simple region) D={ (x,y)|y^2-2y<=x<=2-y, -1<=y<=2 }.

Step 2. Set up the iterated integral.

$$\iint_D (x+y)\,dA = \int_{y=-1}^{2}\int_{x=y^2-2y}^{2-y} x^2 y\,dy\,dx = \frac{99}{20}$$

.

Step 3. Compute the iterated integral.

Integrate[x+y,{y,-1,2},{x,y^2-2y,2-y}]

Example 6.7. Compute $\int_0^1\int_0^{\sqrt{1-x^2}} 3\,dy\,dx$.

Solution:

Note that the limits of integration with respect to x is

x=0 and x=1.

The limits of integration with respect to y is

y=0 and y=Sqrt[1-x^2].

So, the domain of integration D_1 is given by

D_1 ={ (x,y)|0<=x<=1,0<=y<=Sqrt[1-x^2] }.

With this description, clearly the region is of type 1 elementary region. So, we can sketch it using viewRegion1 module.

viewRegion1[{x,0,1},{y,0,Sqrt[1-x^2]},AspectRatio->1,ImageSize->300]

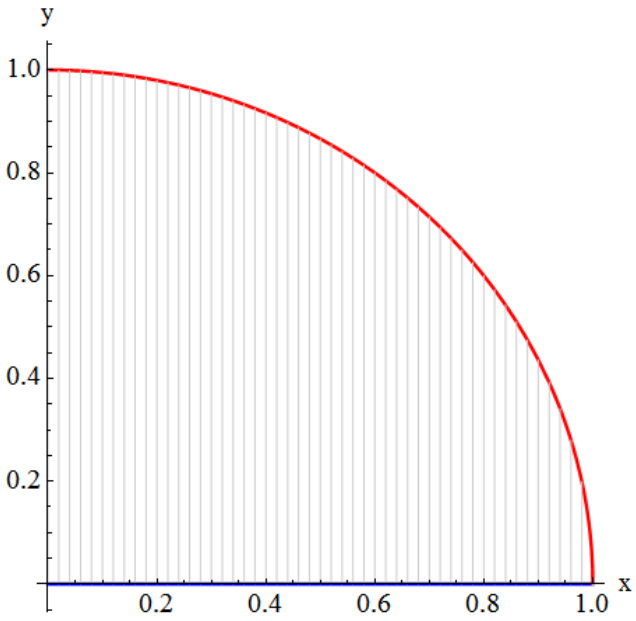


Figure 6.9. Type 3 elementary region. The region is both type 1 and type 2.

Thus,

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} 3\,dy\,dx = \frac{3\pi}{4}$$

.

Integrate[3,{x,0,1},{y,0,Sqrt[1-x^2]}]

However, we can also describe the region as type 2 elementary region using the limits of integration as in the following

D_2 ={ (x,y)|0<=x<=Sqrt[1-y^2],0<=y<=1 }.

D_2 can be graphed using viewRegion2:

viewRegion2[{y,0,1},{x,0,Sqrt[1-y^2]},ImageSize->300]

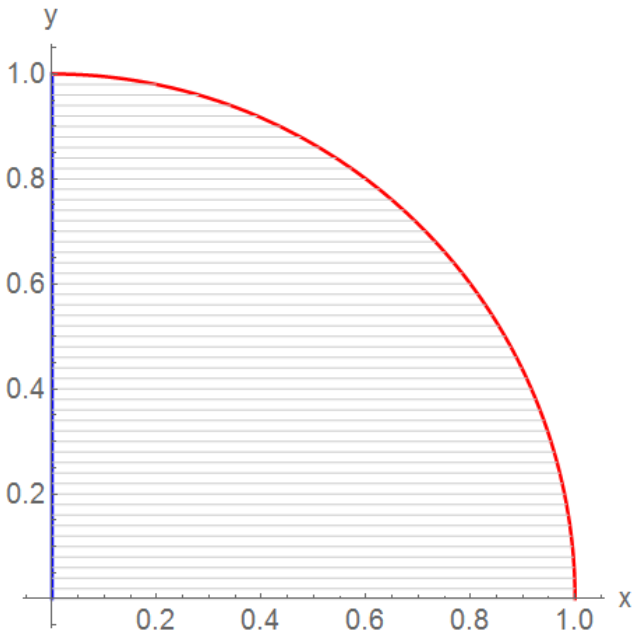


Figure 6.10. Type 3 elementary region. The region is both type 1 and type 2.

So, with new description D_2 , the integral becomes

$$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} 3\,dx\,dy = \frac{3\pi}{4}$$

.

It yields the same result as we have expected.

```
Integrate[3,{y,0,1},{x,0,Sqrt[1-y^2]}]
```

The last example provides an initial view about change of order of integration when the domain is not a rectangle. Recall in Section 6.1.1 when the domain of integration is rectangle, the order in which we integrate has no significance. However, when the domain is not rectangle, then we must use the limits of integration in the original iterated integral to identify the region D over which the integration takes place, then change the order of integration by re-assigning the appropriate variables as we have done in Example 6.7. We shall look into more examples about changing the order of integration in the next section. Before doing that, here are some examples where the domains of integrations are not elementary. In most cases though, we can subdivide the regions into finitely many of elementary types.

Example 6.8. Calculate the double integral of f(x,y)=x^2+y^2 over the region D, where D is the region in the first quadrant bounded by y=x, y=3x, and x y=3.

Solution:

First we need to describe the domain of integration D. To do so, we plot the boundary curves to determine the region using ContourPlot.

```
plot=ContourPlot[{y==x,y==3x,x y==3},{x,-4,4},{y,-4,4},Frame->False,Axes->True,ContourStyle->{Blue,Red,Green},PlotLegends->"Expressions"]
```

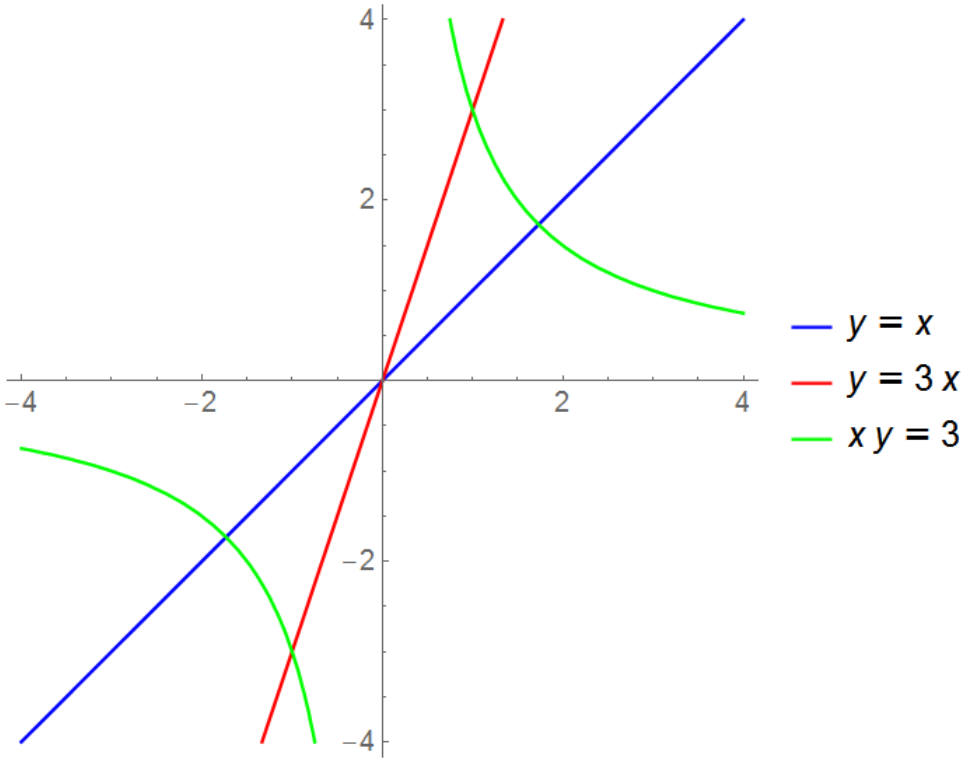


Figure 6.11.a The region D bounded by y=x, y=3x, and x y=3

Note that the region is not elementary.

Using the fact that double integrals is additive with respect to the domain, we can decompose D into two type 2 elementary regions, D_1 and D_2 , respectively. The two subregions are divided by the horizontal line passing through the intersection point between the line y=x and the curve x y=3 in the first quadrant.

So first, we find the intersection point between the line y=x and the curve x y=3:

Substituting y=x into x y=3, we have

x^2=3,

which gives

x=±Sqrt[3] and y=±Sqrt[3].

Since we consider only the region in the first quadrant, so the horizontal line that passes through this intersection point is y=Sqrt[3].

Hence, the double integral is decomposed into

$$\iint_D (x^2 + y^2)\, dA = \iint_{D_1} (x^2 + y^2)\, dA_1 + \iint_{D_2} (x^2 + y^2)\, dA_2$$

.

```
line=ContourPlot[y==Sqrt[3],{x,-4,4},{y,-4,4},Frame->False,Axes->True,ContourStyle->{Dashed,Red},PlotLegends->"Expressions"];

Show[plot,line]
```

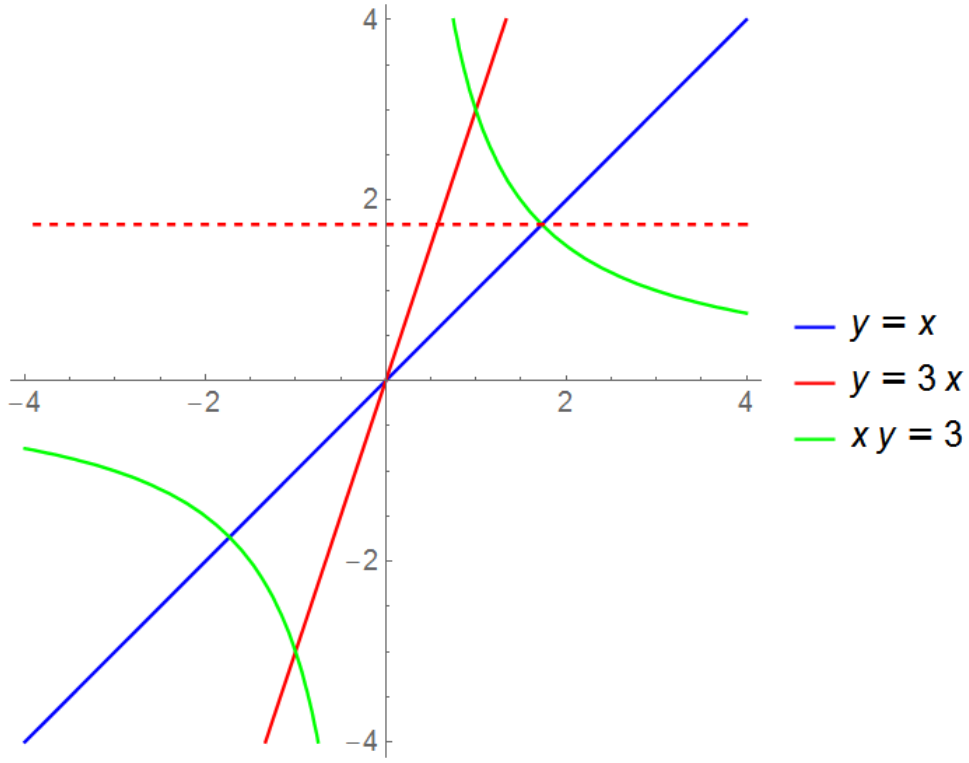


Figure 6.11.b The region D is decomposed into two subregions separated by the horizontal line y=Sqrt[3].

Consider the subregion D_1 .

Let D_1 be the subregion bounded by the line y=x, y=3x, and the horizontal line y=Sqrt[3]. The description D_1 is given by

D_1 ={ (x,y)|y/3<=x<=y,0<=y<=Sqrt[3] }.

Module[{text},

D1=viewRegion2[{y,0,Sqrt[3]},{x,y/3,y},AspectRatio->1,ImageSize->300];

text=Graphics[Text["D1",{0.7,1.2}]];

Show[D1,text]

]

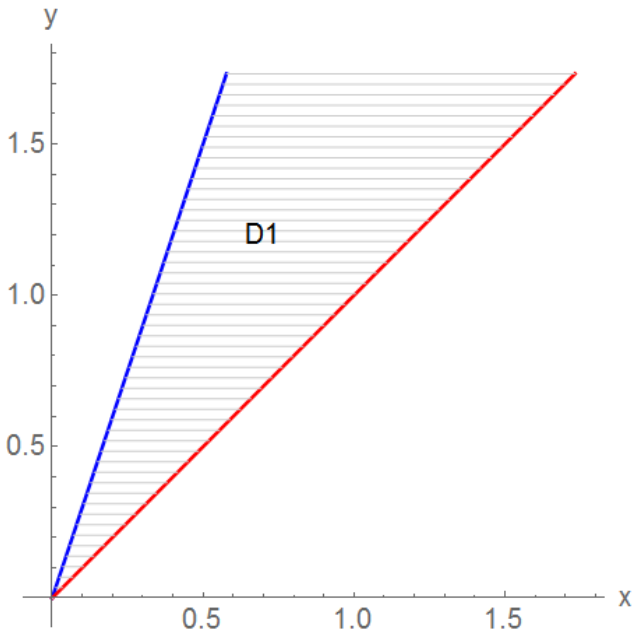


Figure 6.12. The region D_1 bounded by y=x, y=3x, and y=Sqrt[3]

For the subregion D_1 :

$$\iint_{D_1}(x^2+y^2)\,dA_1 = \int_{y=0}^{\sqrt{3}}\int_{x=y/3}^{y}(x^2+y^2)\,dx\,dy = \frac{20}{9}$$

.

Integrate[x^2+y^2,{y,0,Sqrt[3]},{x,y/3,y}]

Next, consider the subregion D_2 .

Let D_2 be the subregion bounded by the line y=3x, the curve x y=3, and the bottom line y=Sqrt[3].

The limits of integration are the bottom line y=Sqrt[3] and the intersection point between the line y=3x and the curve x y=3.

To find this intersection point, we solve for y from these two equations:

y=3x and x y=3

y=3*3/y=9/y

y^2=9, y=±3.

For first quadrant, we choose y=3.

So, the description of D_2 is given by

D_2 ={ (x,y)|y/3<=x<=3/y,Sqrt[3]<=y<=3 }.

Module[{text},

D2=viewRegion2[{y,Sqrt[3],3},{x,y/3,3/y},AspectRatio->1];

text=Graphics[Text["D2",{1,2.2}]];

Show[D2,text,AxesOrigin->{0,0},PlotRange->{{0,2},{0,3}}]

]

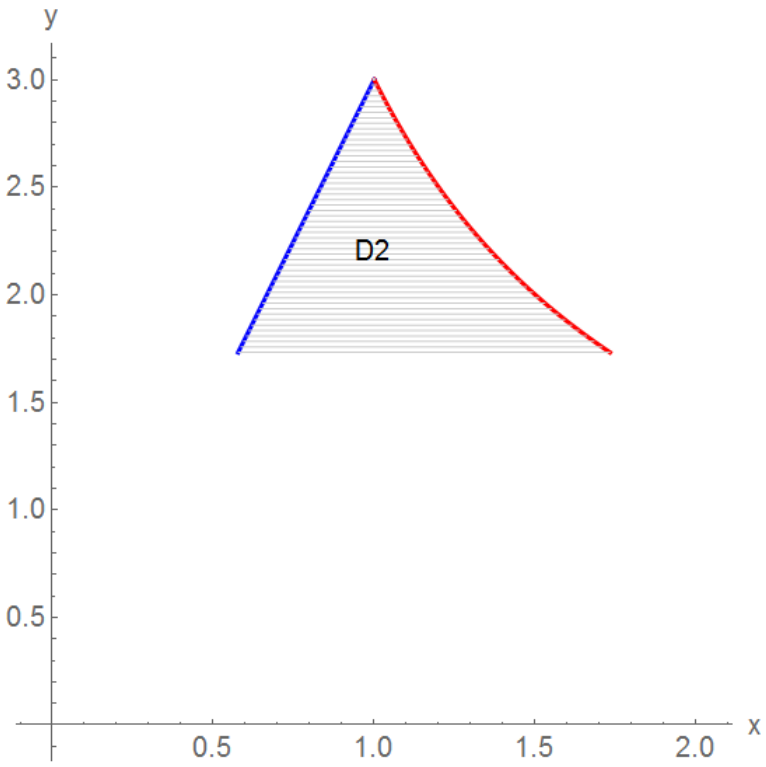


Figure 6.13. The region D_2 bounded on the left and right by y=3x and the curve x y=3, respectively; horizontally by y=Sqrt[3] and y=3.

For the subregion D_2 :

$$\iint_{D_2}(x^2+y^2)\,dA_1 = \int_{y=\sqrt{3}}^{3}\int_{x=y/3}^{3/y}(x^2+y^2)\,dx\,dy = \frac{34}{9}$$

.

Integrate[x^2+y^2,{y,Sqrt[3],3},{x,y/3,3/y}]

The double integral:

$$\iint_D (x^2+y^2)\,dA = \iint_{D_1} (x^2+y^2)\,dA_1 + \iint_{D_2} (x^2+y^2)\,dA_2$$
$$= \frac{20}{9} + \frac{34}{9} = \frac{54}{9} = 6$$

Show[plot,D1,D2]

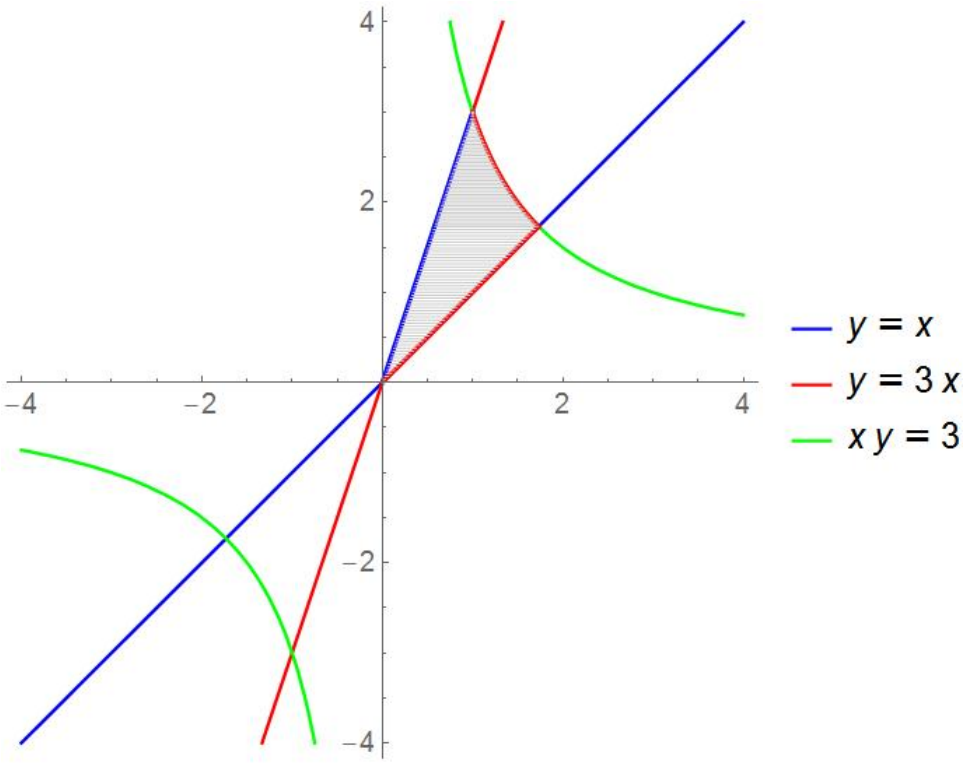


Figure 6.14. The region D in the first quadrant is bounded by y=x, y=3x, and x y=3.

Example 6.9. Calculate $\iint_D f(x, y)\,dA$ for the function f given by

f(x,y)=x+y,

and the region D is in the first quadrant that is below y=1+x^2 and to the left of y=5(x-1).

Solution:

To describe the domain of integration D, we plot the boundary lines to determine the region using ContourPlot.

```
plot=ContourPlot[{y==1+x^2,y==5(x-1)},{x,0,2},{y,0,6},Axes->True,ContourStyle->{Red,Blue},PlotLegends->"Expressions"]
```

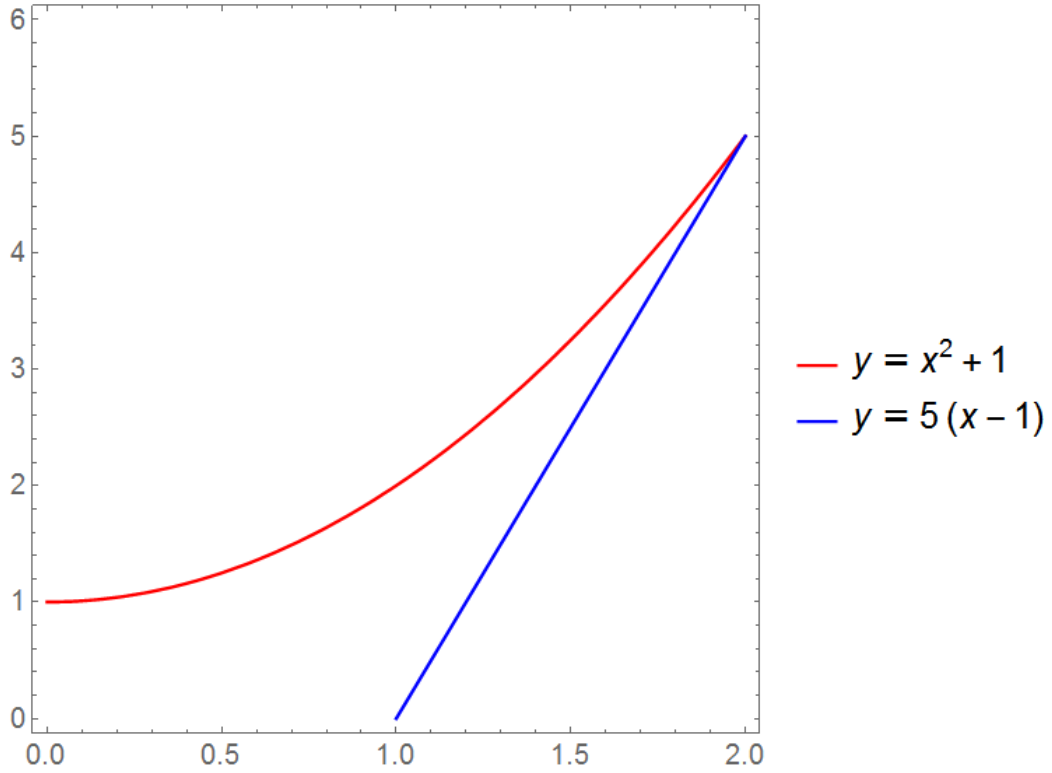


Figure 6.15. The region D is in the first quadrant that is below y=1+x^2 and to the left of y=5(x-1).

D is not an elementary region. However, we can decompose it into two type 1 elementary regions, D_1 and D_2 , respectively. The two subregions are

divided by the vertical line passing through x=1.

So, we have

D_1 ={ (x,y)|0<=x<=1,0<=y<=x^2+1 }

and D_2 ={ (x,y)|1<=x<=2,5(x-1)<=y<=x^2+1 }.

The two subregions are represented by the following graphs:

Module[{text},

D1=viewRegion1[{x,0,1},{y,0,x^2+1}];

text=Graphics[Text["D1",{0.5,0.5}]];

Show[D1,text,PlotRange->{{0,2},{0,6}}]

]

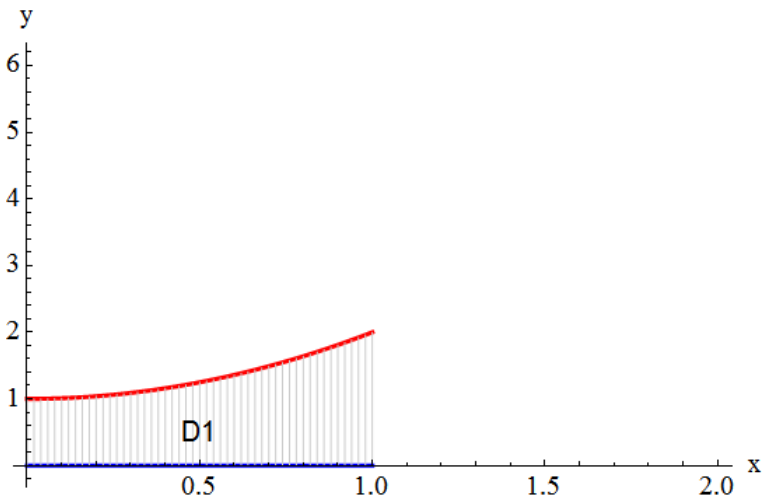


Figure 6.16. D_1 is of type 1 elementary region bounded by y=0 and y=x^2+1; vertically by x=0 and x=1. D_1 ={ (x,y)|0<=x<=1,0<=y<=x^2+1

}.

```
Module[{text},
D2=viewRegion1[{x,1,2},{y,5(x-1),x^2+1}];
text=Graphics[Text["D2",{1.3,2}]];
Show[D2,text,AxesOrigin->{0,0},PlotRange->{{0,2},{0,6}}]
]
```

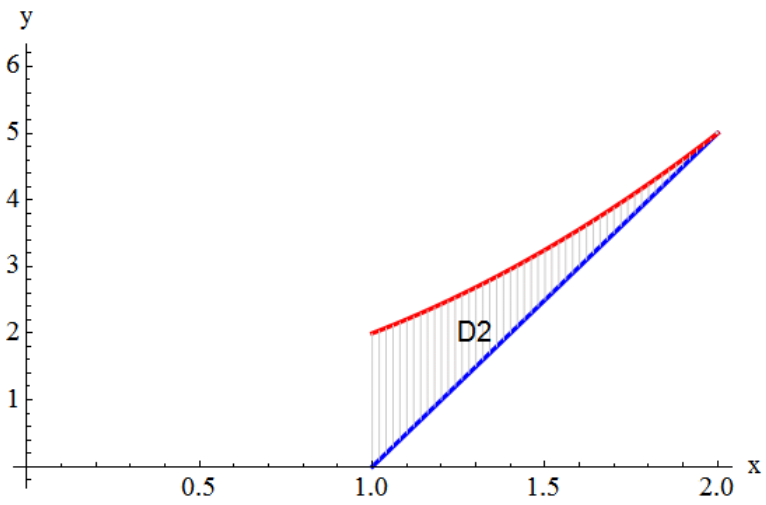


Figure 6.17. D_2 is of type 1 elementary region bounded by y=5(x-1) and y=x^2+1; vertically by x=1 and x=2. D_2 ={ (x,y)|1<=x<=2,5(x-1) <=y<=x^2+1 }

So, the double integral is given by

$$\iint_D (x+y)\,dA = \iint_{D_1} (x+y)\,dA_1 + \iint_{D_2} (x+y)\,dA_2$$

$$= \frac{68}{15}$$

Integrate[x+y,{x,0,1},{y,0,x^2+1}]+Integrate[x+y,{x,1,2},{y,5(x-1),x^2+1}]

Show[plot,D1,D2]

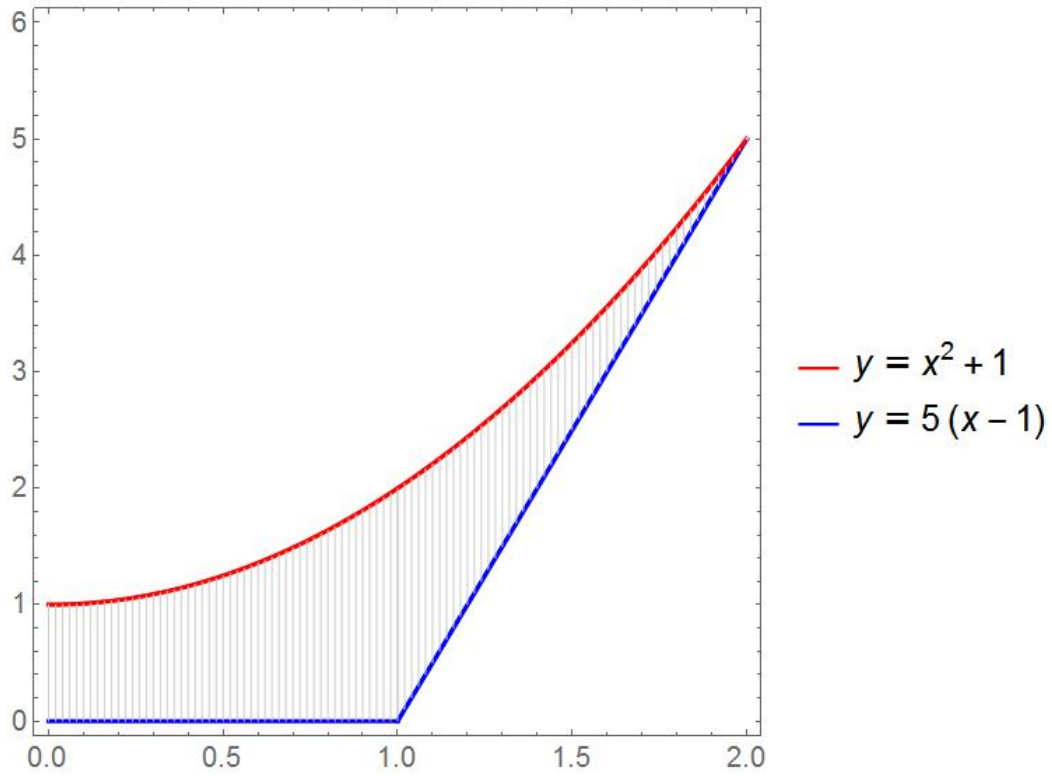


Figure 6.18. Region D is in the first quadrant that is below y=1+x^2 and to the left of y=5(x-1).

Example 6.10. (a) Write an iterated integral to compute the volume under the hyperbolic paraboloid z=x^2-y^2+5 and over the region D given by

D={ (x,y)|x^2+y^2<=4 }.

(b) Evalute the integral.

Solution:

(a). Geometrically $\iint_D f(x, y)\, dA$ represents the volume under the graph of f over D.

To determine the description of D, we plot the disk given by x^2+y^2<=4.

```
ContourPlot[x^2+y^2,{x,-3,3},{y,-3,3},ContourStyle->Dashed,Contours->
{4},ContourShading->{Hue[0.5],GrayLevel[1]},FrameLabel->
{"x","y"},ImageSize->300]
```

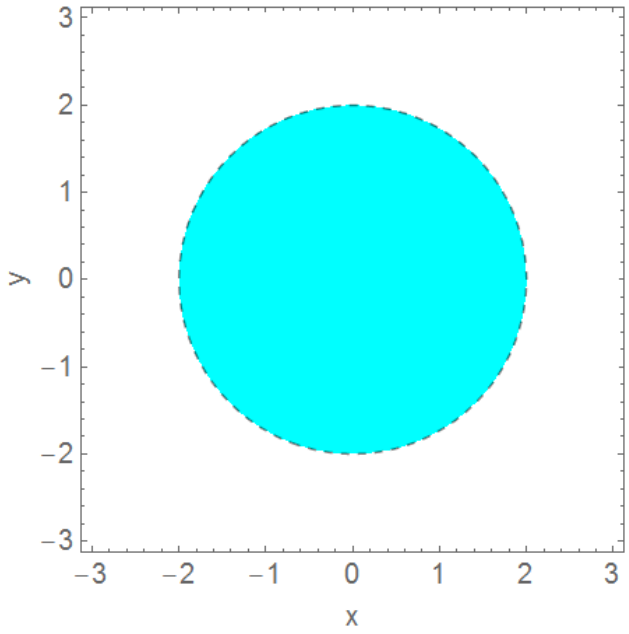


Figure 6.19. Region D is a disk x^2+y^2<=4.

The region D is both type 1 and type 2 elementary regions. So, we choose it to be type 1 elementary region whose description is given by

D={ (x,y)|-2<=x<=2,-Sqrt[2-x^2]<=y<=Sqrt[2-x^2] }.

The hyperbolic paraboloid is a surface that can be rewritten as

z=f(x,y)=x^2-y^2+5.

So, the iterated integral is

$$\iint_D f(x, y)\, dA = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 - y^2 + 5)\, dy\, dx$$

$$= 20\pi$$

.

(b)

Integrate[x^2-y^2+5,{x,-2,2},{y,-Sqrt[4-x^2],Sqrt[4-x^2]}]

6.1.3 Changing the Order of Integration

In Section 6.1.1 we see that when the region of integration is a rectangle, the order in which we integrate does not matter and still yields the same result. In other words, we get the same result by a simple exchange of the limits of integration. However, in Example 6.7 we encountered a region which is both type 1 and type 2 elementary regions but not a rectangle. We could change the order of integration to compute the integral after giving an appropriate description of the region. This flexibility can be of advantage when changing the order of integration makes a difficult or even impossible calculation easier. Let us consider some examples to illustrate this point.

Example 6.11. (a) Use Integrate command to compute the integral

$$\int_0^3 \int_{x^2}^9 x \sin(y^2)\, dy\, dx$$

as it is written. Use Timing command to check how long it takes for Mathematica to deliver the answer.

(b) Reverse the order of integration and evaluate the new iterated integral and note how long it takes for Mathematica to deliver the answer.

Solution:

Below is a short explanation about the Timing command:

?Timing

```
Timing[Integrate[x Sin[y^2],{x,0,3},{y,x^2,9}]]
```

It takes some time in seconds to execute this integration depending on the resources of the computer. The result is 1/2 sin^2(81/2).

(b)

When changing the order of integration, it is helpful to draw the region as an aid to determine a description of the region. So, we plot the limits of integrations in the following:

ContourPlot[{x==0,x==3,y==x^2,y==9},{x,0,4},{y,0,10},Frame->False,Axes->True,AxesLabel->{"x","y"},ContourStyle->{Black,Green,Blue,Red},PlotLegends->"Expressions",AspectRatio->Automatic]

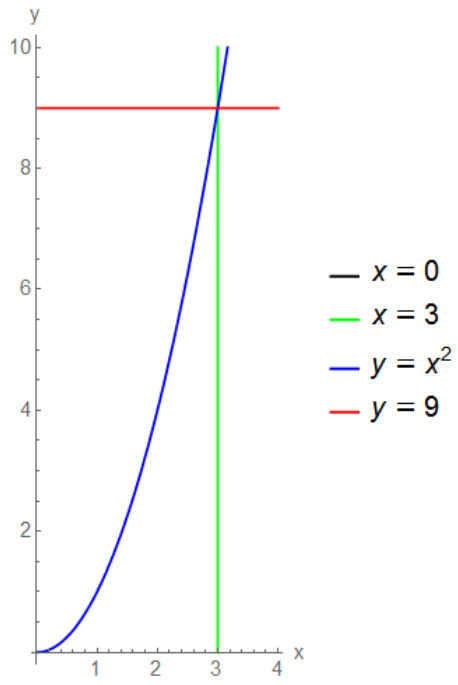


Figure 6.20. The domain of integration D is bounded by the curve y=x^2 and the horizontal line y=9; vertically by x=0 and x=3

The original region is of type 1 elementary region given by

D'={ (x,y)|0<=x<=3,x^2<=y<=9 }

viewRegion1[{x,0,3},{y,x^2,9},AspectRatio->Automatic]

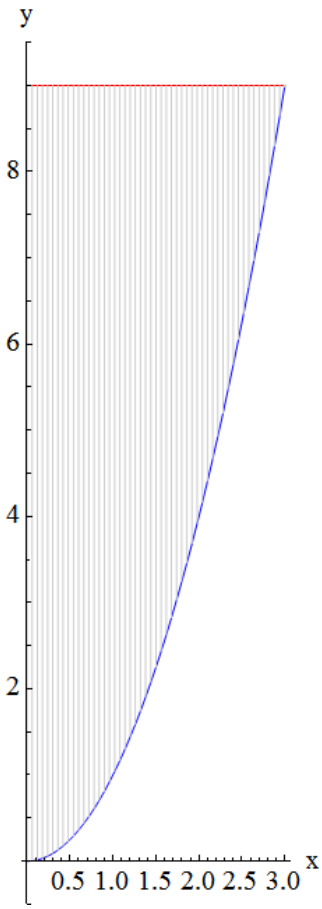


Figure 6.21. The region D'={ (x,y)|0<=x<=3,x^2<=y<=9 }.

If we change the order of integration to first integrate with respect to y and then with respect to x, the description of the new region D becomes

D={ (x,y)|0<=x<=Sqrt[y],0<=y<=9 }.

Note that it is now a type 2 elementary region. We shall verify this using viewRegion2.

viewRegion2[{y,0,9},{x,0,Sqrt[y]},AspectRatio->Automatic]

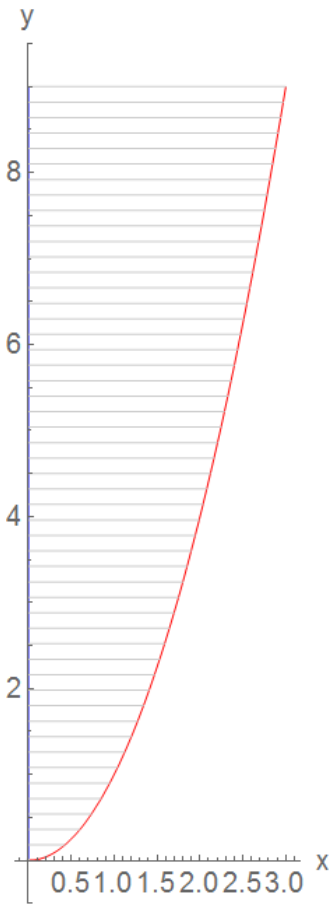


Figure 6.22. The new region description is D={ (x,y)|0<=x<=Sqrt[y],0<=y<=9 }.

Now, let us evaluate the integral using the new region description D:

Timing[Integrate[x Sin[y^2],{y,0,9},{x,0,Sqrt[y]}]]

This time it takes Mathematica to deliver the answer few times faster than it did in part (a).

Example 6.12. (a) Use Mathematica to evaluate the following integral

$$\int_0^1 \int_{\sin^{-1} y}^{\pi/2} e^{\cos x}\, dx\, dy$$

as it is written.

(b) Reverse the order of integration and use Timing command to check how long it takes for Mathematica to deliver the answer.

Solution:

The domain of integration is a type 2 elementary region which description is given by

D={ (x,y)|sin^-1 y<=x<=π/2, 0<=y<=1}.

viewRegion2[{y,0,1},{x,ArcSin[y],π/2},Ticks->{{0,π/4,π/2},Automatic}]

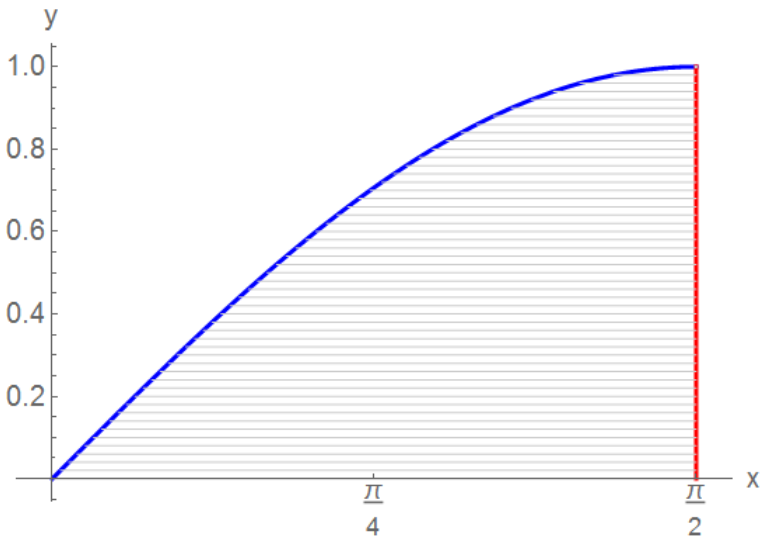


Figure 6.23. The region D={ (x,y)|sin^-1 y<=x<=π/2, 0<=y<=1}.

We evaluate the integral directly using Mathematica:

$$\int_0^1 \int_{\sin^{-1} y}^{\pi/2} e^{\cos x} \, dx \, dy$$

```
Integrate[E^Cos[x],{y,0,1},{x,ArcSin[y],π/2}]
```

However, Mathematica has not been able to evaluate the integral, so it returns the integral as it is written.

(b)

First, we plot the limits of integrations to determine a description of the region.

```
ContourPlot[{y==0,y==1,x==ArcSin[y],x==π/2},{x,0,2},{y,0,1.5},Frame->False,Axes->True,AxesLabel->{"x","y"},ContourStyle->{Black,Green,Blue,Red},PlotLegends->"Expressions",AspectRatio->Automatic,Ticks->{{0,π/4,π/2},Automatic}]
```

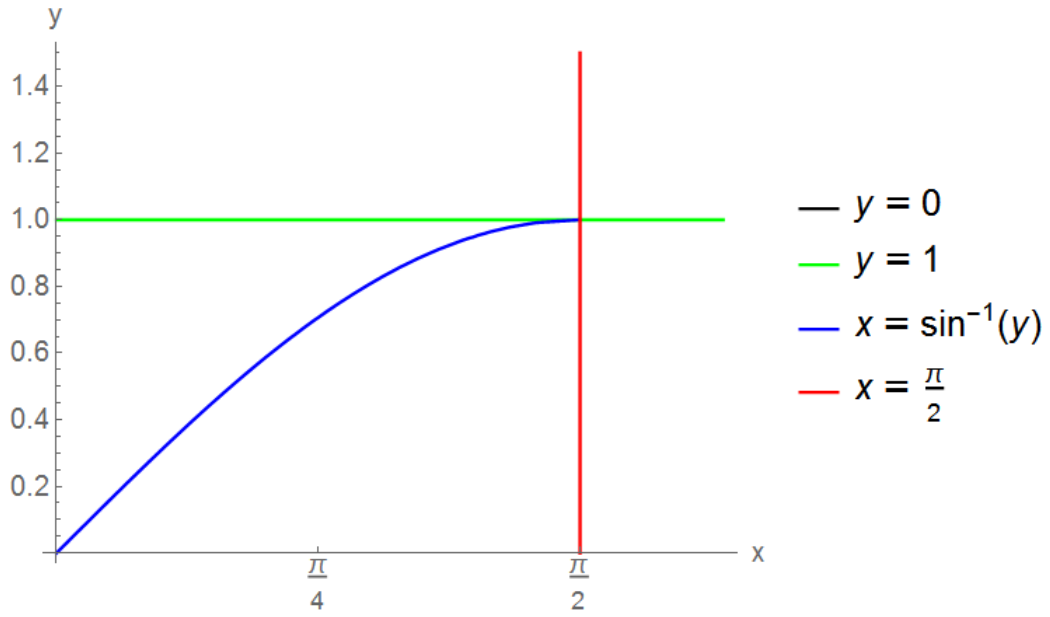


Figure 6.24. The limits of integration of the original integral: sin^-1 y<=x<=π/2 and 0<=y<=1.

Using Figure 6.24 the new description for our change of order of integration becomes

D={ (x,y)|0<=x<=π/2,0<=y<=sin x }.

Now, it becomes a type 1 elementary region. Let us verify this using viewRegion1.

viewRegion1[{x,0,π/2},{y,0,Sin[x]},Ticks->{{0,π/4,π/2},Automatic}]

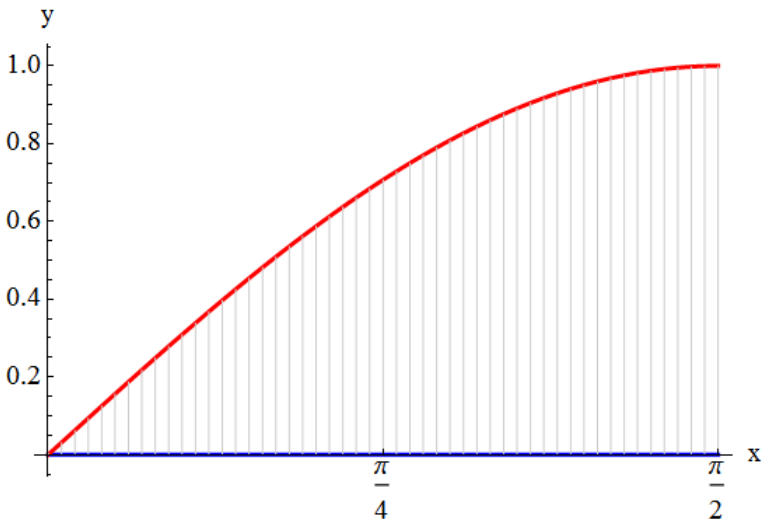


Figure 6.25. The new region description: D={ (x,y)|0<=x<=π/2,0<=y<=sin x }.

The integral after changing order of integration becomes

$$\int_{x=0}^{\pi/2} \int_{y=0}^{\sin x} e^{\cos x}\, dy\, dx = e - 1$$

.

Timing[Integrate[E^Cos[x],{x,0,π/2},{y,0,Sin[x]}]]

Mathematica can now evaluate this integral easily within fraction of seconds!

6.2 Triple Integrals

Let f(x,y,z) be a function of three variables. A triple integral is a straightforward generalization of double integrals. Instead of integration over a region in the plane $\mathbb{R}^2$, the domain of a triple integral is a solid region in space $\mathbb{R}^3$ and the integrand is a function of three variables. First, we start with integral over box-shaped regions and then proceed to define integral over more general solid regions.

6.2.1 The Integral over a Box

Consider a box:

W=[a,b]x[c,d]x[p,q]

consisting of all points in $\mathbb{R}^3$ such that

a<=x<=b, c<=y<=d, p<=z<=q.

To evaluate the triple integral over W we use the result from Fubini's Theorem for Triple Integrals as is stated in the following:

Theorem 6.4. Fubini's Theorem for Triple Integrals

Let f(x,y,z) be a continuous function defined over a box W= [a,b]x[c,d]x[p,q]. The triple integral of f over W is equal to the iterated integral given by

(6.6)

$$\iiint_W f(x, y, z)\, dV = \int_{x=a}^{b}\int_{y=c}^{d}\int_{z=p}^{q} f(x, y)\, dz\, dy\, dx$$

.

Furthermore, the iterated integral may be evaluated in any order.

Recall from Section 6.1 a double integral represents a signed volume of the three-dimensional solid region between a graph z=f(x,y) and the region D in the xy-plane. However, the graph of a function f(x,y,z) of three variables resides in four-dimensional space, so a triple integral represents a four-dimensional volume. Only in special case where f(x,y,z)=1, we can interpret it as a volume of the solid region of the domain of integration:

$$V = \iiint_W 1\, dV$$.

In many applications, triple integrals represent some physical quantities. For instance, if f(x,y,z) represents a mass density function, then the triple integral of f over W gives the total mass of W.

Example 6.13. Calculate the triple integral:

$$\iiint_W x\, e^{y-2z}\, dV$$,

where W=[0,2]x[0,1]x[0,1].

Solution:

The domain of integration is a box W whose description is given by

W={ (x,y,z)|0<=x<=2,0<=y<=1,0<=z<=1 }.

Using Theorem 6.4 the triple integral can be written as an iterated integral:

$$\iiint_W x\, e^{y-2z}\, dV = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1} x\, e^{y-2z}\, dz\, dy\, dx$$

.

Step 1. Evaluate the inner integral with respect to z, holding x and y constant.

$$\begin{aligned}\int_{z=0}^{1} x\, e^{y-2z}\, dz &= -\frac{1}{2}\, x\, e^{y-2z}\Big]_0^1 \\ &= -\frac{1}{2}\, x\, e^{y-2} + \frac{1}{2}\, x\, e^{y} \\ &= \frac{1}{2}\left(1-e^{-2}\right) x\, e^{y}\end{aligned}$$

Step 2. Evaluate the middle integral with respect to y, holding x constant.

$$\begin{aligned}\int_{y=0}^{1} \frac{1}{2}\left(1-e^{-2}\right) x\, e^{y}\, dy &= \frac{1}{2}\left(1-e^{-2}\right) x \int_{y=0}^{1} e^{y}\, dy \\ &= \frac{1}{2}\left(1-e^{-2}\right)(e-1)\, x\end{aligned}$$

Step 3. Evaluate the outer integral with respect to x.

$$\begin{aligned}\iiint_W x\, e^{y-2z}\, dV &= \frac{1}{2}\left(1-e^{-2}\right)(e-1)\int_{x=0}^{2} x\, dx \\ &= \frac{1}{4}\left(1-e^{-2}\right)(e-1)\, x^2\Big]_0^2 \\ &= \left(1-e^{-2}\right)(e-1)\end{aligned}$$

$$\iiint_W x\, e^{y-2z}\, dV = e - \frac{1}{e} - 1 + \frac{1}{e^2}.$$

Integrate[x E^(y-2z),{x,0,2},{y,0,1},{z,0,1}]//Expand

6.2.2 Triple Integrals over Arbitrary Solid Regions

In previous section, we see how to calculate triple integrals over a box-shaped region. Much like the case in double integrals, we seek to calculate integrals over arbitrary solid regions. However, we shall consider only certain types of elementary regions and treat integrals over more general regions by subdividing them into solid regions of elementary type. Thus, we need some definitions of elementary region in space which is given below:

Definition 6.5

The solid region W in space is called an elementary region if it can be described as a subset of $\mathbb{R}^3$ of one of the following four types:

Type 1:

(a)

$$W = \{(x, y, z) \mid a \le x \le b,\ y_1(x) \le y \le y_2(x),\ z_1(x, y) \le z \le z_2(x, y)\}$$

,

where a, b constants; both y_1 and y_2 are continuous functions on [a,b]; both z_1 and z_2 are continuous functions on $D \equiv [a, b] \times [y_1, y_2]$ in the xy-plane (see Figure 6.26).

Or

(b)

$$W = \{(x, y, z) \mid x_1(y) \le x \le x_2(y),\ c \le y \le d,\ z_1(x, y) \le z \le z_2(x, y)\}$$

,

where c, d constants; both x_1 and x_2 are continuous functions on [c,d]; both z_1 and z_2 are continuous functions on $D \equiv [x_1, x_2] \times [c, d]$ in the xy-plane (see Figure 6.27)

Type 2:

(a)

$$W = \{(x, y, z) \mid x_1(y, z) \le x \le x_2(y, z),\ y_1(z) \le y \le y_2(z),\ p \le z \le q\}$$

,

where p, q constants; both y_1 and y_2 are continuous functions on [p,q]; both x_1 and x_2 are continuous functions on $D \equiv [y_1, y_2] \times [p, q]$ in the yz-plane (see Figure 6.29).

Or

(b)

$$W = \{(x, y, z) \mid x_1(y, z) \le x \le x_2(y, z),\ c \le y \le d,\ z_1(y) \le z \le z_2(y)\}$$

,

where c, d constants; both z_1 and z_2 are continuous functions on [c,d]; both x_1 and x_2 are continuous functions on $D \equiv [c, d] \times [z_1, z_2]$ in the yz-plane (see Figure 6.31).

Type 3:

(a)

$$W = \{(x, y, z) \mid x_1(z) \le x \le x_2(z),\ y_1(x, z) \le y \le y_2(x, z),\ p \le z \le q\}$$

,

where p, q constants; both x_1 and x_2 are continuous functions on [p,q]; both y_1 and y_2 are continuous functions on $D \equiv [x_1, x_2] \times [p, q]$ in the xz-plane (see Figure 6.33).

Or

(b)

$$W = \{(x, y, z) \mid a \le x \le b,\ y_1(x, z) \le y \le y_2(x, z),\ z_1(x) \le z \le z_2(x)\}$$

,

where a, b constants; both z_1 and z_2 are continuous functions on [a,b]; both y_1 and y_2 are continuous functions on $D \equiv [a, b] \times [z_1, z_2]$ in the xz-plane (see Figure 6.35).

Type 4: W is all three types described above.

Corresponding to the above types of region, the triple integrals can be calculated as iterated integrals as stated in the following theorem:

Theorem 6.6 The triple integral of a continuous function f(x,y,z) over the solid region of

Type 1:

(a)

$$W = \{(x, y, z) \mid a \le x \le b,\ y_1(x) \le y \le y_2(x),\ z_1(x, y) \le z \le z_2(x, y)\}$$

,

where a, b constants; both y_1 and y_2 are continuous functions on [a,b]; both z_1 and z_2 are continuous functions on $D \equiv [a, b] \times [y_1, y_2]$ in the xy-plane, is equal to the iterated integral

(6.7)

$$\iiint_W f(x, y, z)\, dV = \iint_D \left(\int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right) dA$$
$$= \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

(see Figure 6.26)

(b)

$$W = \{(x, y, z) \mid x_1(y) \le x \le x_2(y),\ c \le y \le d,\ z_1(x, y) \le z \le z_2(x, y)\}$$

,

where c, d constants; both x_1 and x_2 are continuous functions on [c,d]; both z_1 and z_2 are continuous functions on $D \equiv [x_1, x_2] \times [c, d]$ in the xy-plane, is equal to the iterated integral

(6.8)

$$\iiint_W f(x, y, z)\, dV = \iint_D \left(\int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right) dA$$
$$= \int_{y=c}^{d} \int_{x=x_1(y)}^{x_2(y)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dx\, dy$$

(see Figure 6.27)

Type 2:

(a)

$$W = \{(x, y, z) \mid x_1(y, z) \le x \le x_2(y, z),\ y_1(z) \le y \le y_2(z),\ p \le z \le q\}$$

,

where p, q constants; both y_1 and y_2 are continuous functions on [p,q]; both x_1 and x_2 are continuous functions on $D \equiv [y_1, y_2] \times [p, q]$ in the yz-plane, is equal to the iterated integral

(6.9)

$$\iiint_W f(x, y, z)\, dV = \iint_D \left(\int_{x=x_1(y,z)}^{x_2(y,z)} f(x, y, z)\, dx \right) dA$$
$$= \int_{z=p}^{q} \int_{y=y_1(z)}^{y_2(z)} \int_{x=x_1(y,z)}^{x_2(y,z)} f(x, y, z)\, dx\, dy\, dz$$

(see Figure 6.29)

(b)

$$W = \{(x, y, z) \mid x_1(y, z) \le x \le x_2(y, z),\ c \le y \le d,\ z_1(y) \le z \le z_2(y)\}$$

,

where c, d constants; both z_1 and z_2 are continuous functions on [c,d]; both x_1 and x_2 are continuous functions on $D \equiv [c, d] \times [z_1, z_2]$ in the yz-plane, is equal to the iterated integral

(6.10)

$$\iiint_W f(x, y, z)\, dV = \iint_D \left(\int_{x=x_1(y,z)}^{x_2(y,z)} f(x, y, z)\, dx \right) dA$$
$$= \int_{y=c}^{d} \int_{z=z_1(y)}^{z_2(y)} \int_{x=x_1(y,z)}^{x_2(y,z)} f(x, y, z)\, dx\, dz\, dy$$

(see Figure 6.31)

Type 3:

(a)

$$W = \{(x, y, z) \mid x_1(z) \le x \le x_2(z),\ y_1(x, z) \le y \le y_2(x, z),\ p \le z \le q\}$$

,

where p, q constants; both x_1 and x_2 are continuous functions on [p,q]; both y_1 and y_2 are continuous functions on $D \equiv [x_1, x_2] \times [p, q]$ in the xz-plane, is equal to the iterated integral

(6.11)

$$\iiint_W f(x, y, z)\, dV = \iint_D \left(\int_{y=y_1(x,z)}^{y_2(x,z)} f(x, y, z)\, dy \right) dA$$
$$= \int_{z=p}^{q} \int_{x=x_1(z)}^{x_2(z)} \int_{y=y_1(x,z)}^{y_2(x,z)} f(x, y, z)\, dy\, dx\, dz$$

(see Figure 6.33)

(b)

$$W = \{(x, y, z) \mid a \le x \le b,\ y_1(x, z) \le y \le y_2(x, z),\ z_1(x) \le z \le z_2(x)\}$$

,

where a, b constants; both z_1 and z_2 are continuous functions on [a,b]; both y_1 and y_2 are continuous functions on $D \equiv [a, b] \times [z_1, z_2]$ in the xz-plane.

(6.12)

$$\iiint_W f(x, y, z)\, dV = \iint_D \left(\int_{y=y_1(x,z)}^{y_2(x,z)} f(x, y, z)\, dy \right) dA$$
$$= \int_{x=a}^{b} \int_{z=z_1(x)}^{z_2(x)} \int_{y=y_1(x,z)}^{y_2(x,z)} f(x, y, z)\, dy\, dz\, dx$$

(see Figure 6.35)

Example 6.14. (a) Draw the solid region whose volume is given by the iterated integral

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dz\, dy\, dx$$

.

(b) Evaluate the integral in part (a).

(c) Rewrite the integral using the integration order $dz\, dx\, dy$ and draw the solid.

(d) Evaluate the integral in (c) and verify the result with the one obtained in (b).

Solution:

(a)

The solid region W is given by

W={(x,y,z)|0<=x<=3,0<=y<=Sqrt[9-x^2],0<=z<=6-x-y}.

Clearly, W is type 1.(a) elementary solid region by Definition 6.5, where D is elementary region of type 1 in the xy-plane by Definition 6.2:

D={(x,y)|0<=x<=3,0<=y<=Sqrt[9-x^2]}.

Note that D is a circle of radius 1 centered at the origin in the first quadrant.

To draw the solid region we write a module viewSolid1a whose parameters are the same as those entered into the Integrate command:

```
(* Draw the solid of type 1.(a) elementary region *)

viewSolid1a[{x_,a_,b_},{y_,f_,g_},{z_,F_,G_},opts___]:=

Module[{u,v,xx,yy,surfs,pplot},

xx=a+u*(b-a);

yy=f+v*(g-f)/.x->xx;

surfs={{x,y,F},{x,y,G}}/.{x->xx,y->yy};

pplot=ParametricPlot3D[Evaluate[surfs],{u,0,1},{v,0,1},PlotStyle->
{Blue,Red},AxesLabel->{"x","y","z"}];

Show[pplot,Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1,0.05},
{v,0,1}],Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1},
{v,0,1,0.05}],opts]

]


Module[{plot,text},

plot[1]=viewRegion1[{x,0,3},{y,0,Sqrt[9-x^2]},AspectRatio->Automatic];

text=Graphics[Text["D",{1.3,1.5}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid1a[{x,0,3},{y,0,Sqrt[9-x^2]},{z,0,6-x-y}];
```

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]

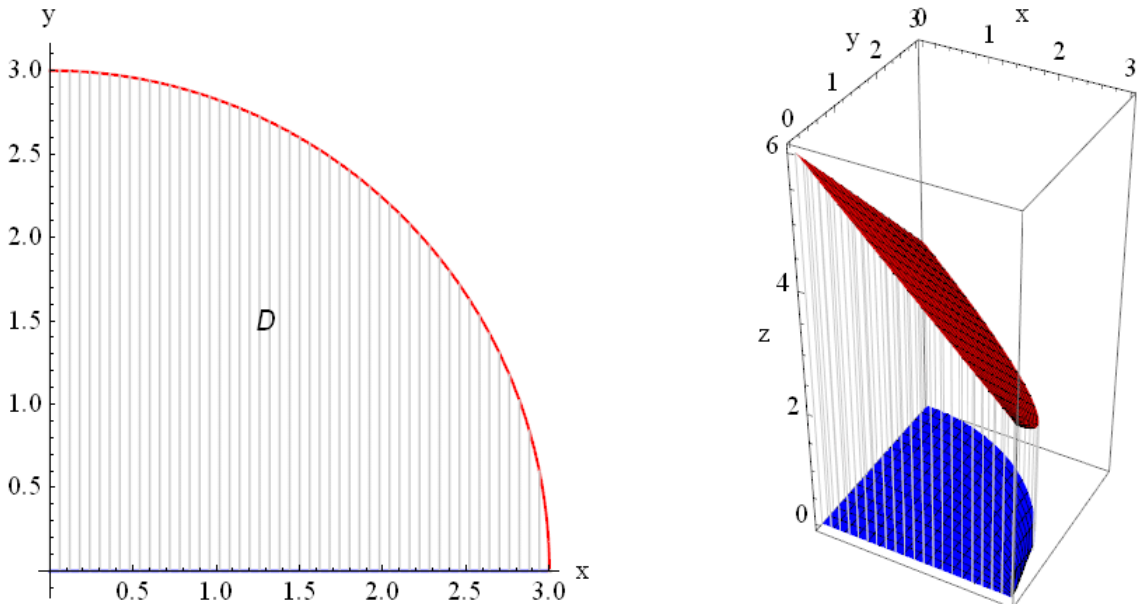


Figure 6.26. The region D={(x,y)|0<=x<=3,0<=y<=Sqrt[9-x^2]} (left) and the solid region W={(x,y,z)|0<=x<=3,0<=y<=Sqrt[9-x^2],0<=z<=6-x-y} (right) of type 1.(a).

(b)

$$\int_{x=0}^{3}\int_{y=0}^{\sqrt{9-x^2}}\int_{z=0}^{6-x-y} dz\,dy\,dx = \frac{9}{2}(3\pi-4)$$

.

Integrate[1,{x,0,3},{y,0,Sqrt[9-x^2]},{z,0,6-x-y}]

(c)

Using Figure 6.26 we can describe the region D as

D={(x,y)|0<=x<=Sqrt[9-y^2],0<=y<=3}.

So, by Definition 6.5 W is of type 1.(b) as described by

W={(x,y)|0<=x<=Sqrt[9-y^2],0<=y<=3,0<=z<=6-x-y}.

The integral becomes

$$\int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}}\int_{0}^{6-x-y} dz\,dx\,dy$$

.

Similarly we have a module viewSolid1b to draw the solid region and again the module takes as parameters that are the same as those supplied to the Integrate command:

(* Draw the solid of type 1.(b) elementary region *)

```
viewSolid1b[{y_,a_,b_},{x_,f_,g_},
{z_,F_,G_},opts___]:=Module[{u,v,xx,yy,surfs,pplot},

yy=a+u*(b-a);

xx=f+v*(g-f)/.y->yy;

surfs={{x,y,F},{x,y,G}}/.{x->xx,y->yy};

pplot=ParametricPlot3D[Evaluate[surfs],{u,0,1},{v,0,1},PlotStyle->
{Blue,Red}];

Show[pplot,Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1,0.05},
{v,0,1}],Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1},
{v,0,1,0.05}],AxesLabel->{"x","y","z"},opts]
```

```
]

Module[{plot,text},

plot[1]=ContourPlot[x==Sqrt[9-y^2],{x,0,3},{y,0,3},Frame->False,Axes->True,AxesLabel->{"x","y"}];

plot[1]=viewRegion2[{y,0,3},{x,0,Sqrt[9-y^2]},AspectRatio->Automatic];

text=Graphics[Text["D",{1.3,1.5}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid1b[{y,0,3},{x,0,Sqrt[9-y^2]},{z,0,6-x-y}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]
```

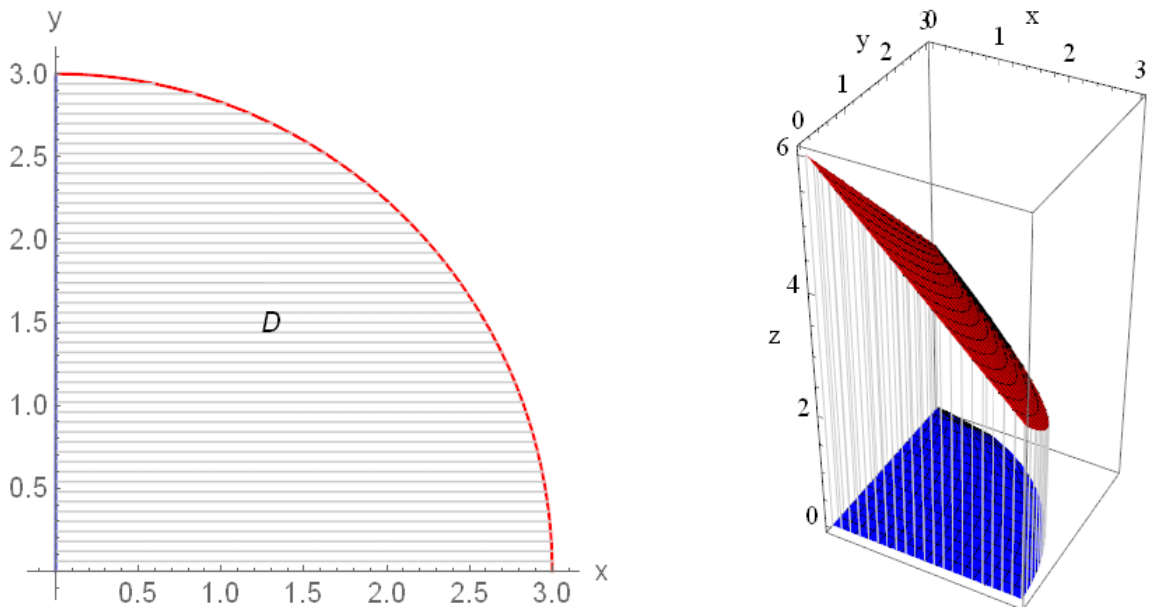


Figure 6.27. The region D={(x,y)|0<=x<=Sqrt[9-y^2],0<=y<=3} (left) and the solid region W={(x,y)|0<=x<=Sqrt[9-y^2],0<=y<=3,0<=z<=6-x-y} of type 1.(b).

Note that although the geometric figures in both Figure 6.26 and Figure 6.27 are the same as they should be, their domain descriptions are different.

(d)

$$\int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}}\int_{0}^{6-x-y} dz\,dx\,dy = \frac{9}{2}(3\pi - 4)$$

.

Integrate[1,{y,0,3},{x,0,Sqrt[9-y^2]},{z,0,6-x-y}]

Both integrals yield the same result as they should.

Example 6.15. (a) Draw the solid whose volume is given by the iterated integral:

$$\int_{0}^{2}\int_{2x}^{4}\int_{0}^{\sqrt{y^2-4x^2}} dz\,dy\,dx$$

.

(b) Evaluate the integral in part (a).

(c) Rewrite the integral using the integration order $dx\,dy\,dz$ and draw the solid.

(d) Evaluate the integral in (c) and verify the result with the one obtained in (b).

Solution:

(a)

The solid region W is given by

W={(x,y,z)|0<=x<=2,2x<=y<=4,0<=z<=Sqrt[y^2-4x^2]}.

By Definition 6.5 W is type 1.(a) elementary solid region, where D is elementary region of type 1 in the xy-plane by Definition 6.2:

D={(x,y)|0<=x<=2,2x<=y<=4}.

To draw the solid region we use the module viewSolid1a:

```
Module[{plot,text},

plot[1]=viewRegion1[{x,0,2},{y,2x,4},AspectRatio->Automatic];

text=Graphics[Text["D",{0.7,3}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid1a[{x,0,2},{y,2x,4},{z,0,Sqrt[y^2-4x^2]},ViewPoint->
{-1,-2,1}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]
```

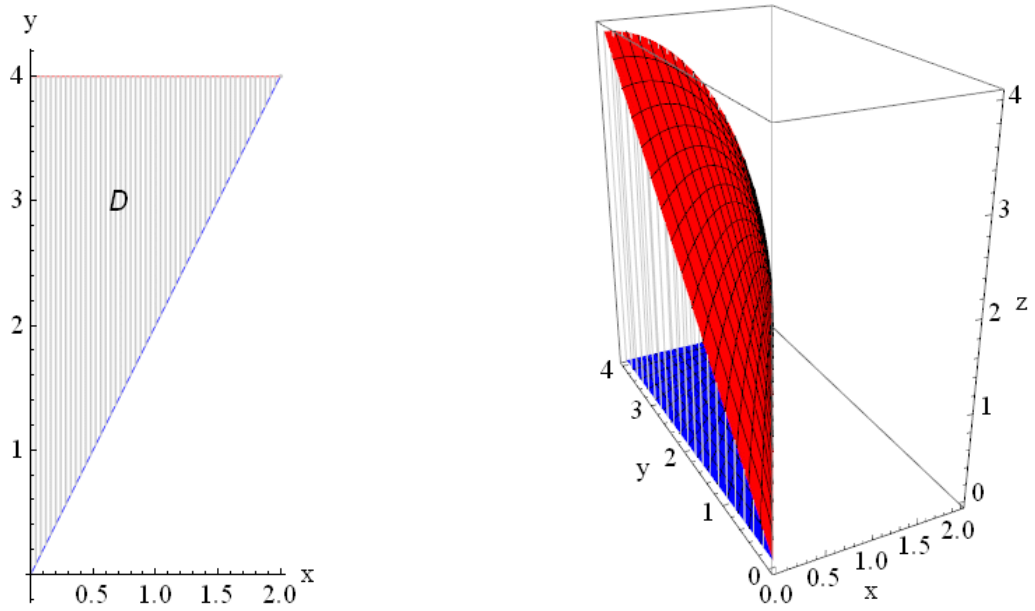


Figure 6.28. The region D={(x,y)|0<=x<=2,2x<=y<=4} (left) and the solid region W={(x,y,z)|0<=x<=2,2x<=y<=4,0<=z<=Sqrt[y^2-4x^2]} (right) of type 1.(a).

(b)

$$\int_{x=0}^{2}\int_{y=2x}^{4}\int_{z=0}^{\sqrt{y^2-4x^2}} dz\,dy\,dx = \frac{8\pi}{3}$$

.

Integrate[1,{x,0,2},{y,2x,4},{z,0,Sqrt[y^2-4x^2]}]

(c)

We want to rewrite the integral using the integration order $dx\,dy\,dz$. So, the solid becomes elementary region of type 2.(a) by Definition 6.5. In this case, D is the region in the yz-plane, where the equation relating y and z can be obtained by setting x=0 in z=Sqrt[y^2-4x^2].

Thus, y=z.

So, D is described by (Figure 6.29-left)

D={(y,z)|z<=y<=4,0<=z<=4}.

The variable x in terms of y and z can be obtained from the surface equation z=Sqrt[y^2-4x^2].

x=1/2 Sqrt[y^2-z^2].

Thus, the description of W is given by (Figure 6.29-right)

W={(x,y,z)|0<=x<=1/2 Sqrt[y^2-z^2],z<=y<=4,0<=z<=4}.

The integral becomes

$$\int_{z=0}^{4}\int_{y=z}^{4}\int_{x=0}^{\frac{1}{2}\sqrt{y^2-z^2}} dx\,dy\,dz$$

.

For the purpose of viewing the solid region of type 2.(a) we provide a module called viewSolid2a defined below:

(* Draw the solid of type 2.(a) elementary region *)

```
viewSolid2a[{z_,a_,b_},{y_,f_,g_},
{x_,F_,G_},opts___]:=Module[{u,v,zz,yy,surfs,pplot},

zz=a+u*(b-a);
```

```
yy=f+v*(g-f)/.z->zz;

surfs={{F,y,z},{G,y,z}}/.{z->Abs[zz],y->Abs[yy]};

pplot=ParametricPlot3D[Evaluate[surfs],{u,0,1},{v,0,1},PlotStyle->
{Blue,Red}];

Show[pplot,Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1,0.05},
{v,0,1}],Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1},
{v,0,1,0.05}],AxesLabel->{"x","y","z"},opts]

]



Module[{plot,text},

plot[1]=viewRegion1[{z,0,4},{y,z,4},AxesLabel->{"z","y"},AspectRatio-
>Automatic];

text=Graphics[Text["D",{1,2.5}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid2a[{z,0,4},{y,z,4},{x,0,1/2 Sqrt[y^2-z^2]},ViewPoint->
{-1,-2,1}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]
```

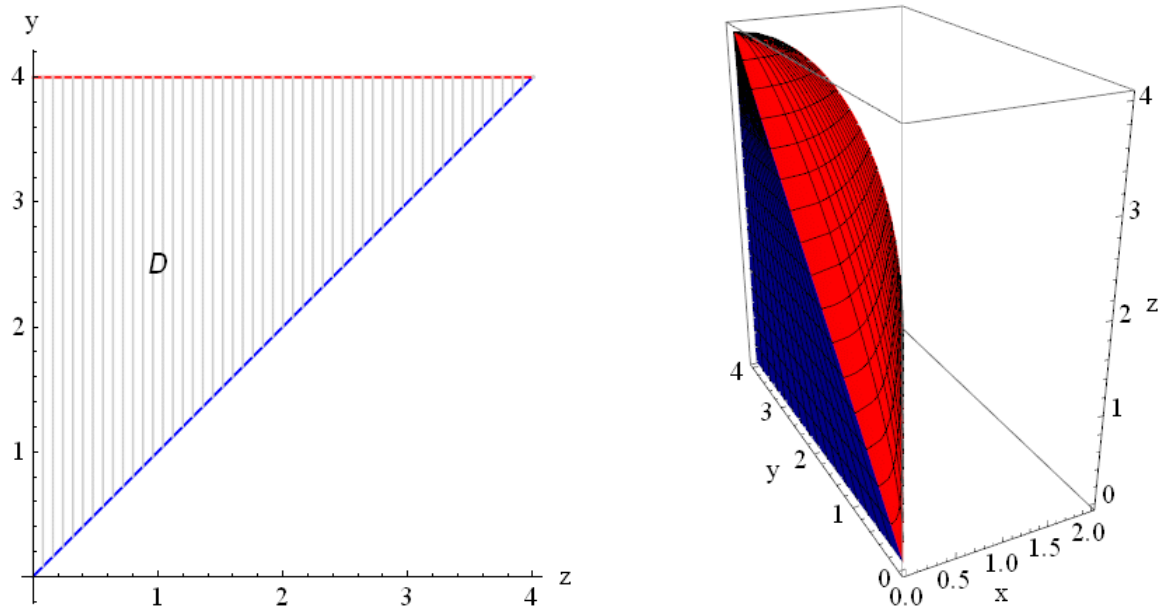


Figure 6.29. The new region description D={(y,z)|z<=y<=4,0<=z<=4} (left) and the solid region of type 2.(a) W={(x,y,z)|0<=x<=1/2 Sqrt[y^2-z^2],z<=y<=4,0<=z<=4} (right).

Note that we have used viewRegion1 to draw the region D by replacing the horizontal axis by z instead of x. And also the option AxesLabel-> {"z","y"} to overwrite the default AxesLabel->{"x","y"}.

(d)

$$\int_{z=0}^{4}\int_{y=z}^{4}\int_{x=0}^{\frac{1}{2}\sqrt{y^2-z^2}} dx\,dy\,dz = \frac{8\pi}{3}$$

.

Integrate[1,{z,0,4},{y,z,4},{x,0,1/2 Sqrt[y^2-z^2]}]

The integral using the new region description gives the same result as it should.

Example 6.16. (a) Draw the solid whose volume is given by the iterated integral

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} dz\,dx\,dy$$

(b) Evaluate the integral in part (a).

(c) Rewrite the integral using the integration order $dx\,dz\,dy$ and draw the solid.

(d) Evaluate the integral in (c) and verify the result with the one obtained in (b).

Solution:

(a)

The solid region W is described by

W={(x,y,z)|y^2<=x<=1,-1<=y<=1,0<=z<=1-x}.

By Definition 6.5 this is type 1.(b) elementary solid region, where D is elementary region of type 2 in the xy-plane by Definition 6.2:

D={(x,y)|y^2<=x<=1,-1<=y<=1}.

Let us draw the solid region using module viewSolid1b:

```
Module[{plot,text},
plot[1]=viewRegion2[{y,-1,1},{x,y^2,1},AspectRatio->Automatic];
text=Graphics[Text["D",{0.7,3}]];
plot[1]=Show[plot[1],text];
plot[2]=viewSolid1b[{y,-1,1},{x,y^2,1},{z,0,1-x},ViewPoint->{-0.5,-2,1}];
GraphicsGrid[{Array[plot,2]},ImageSize->600]
]
```

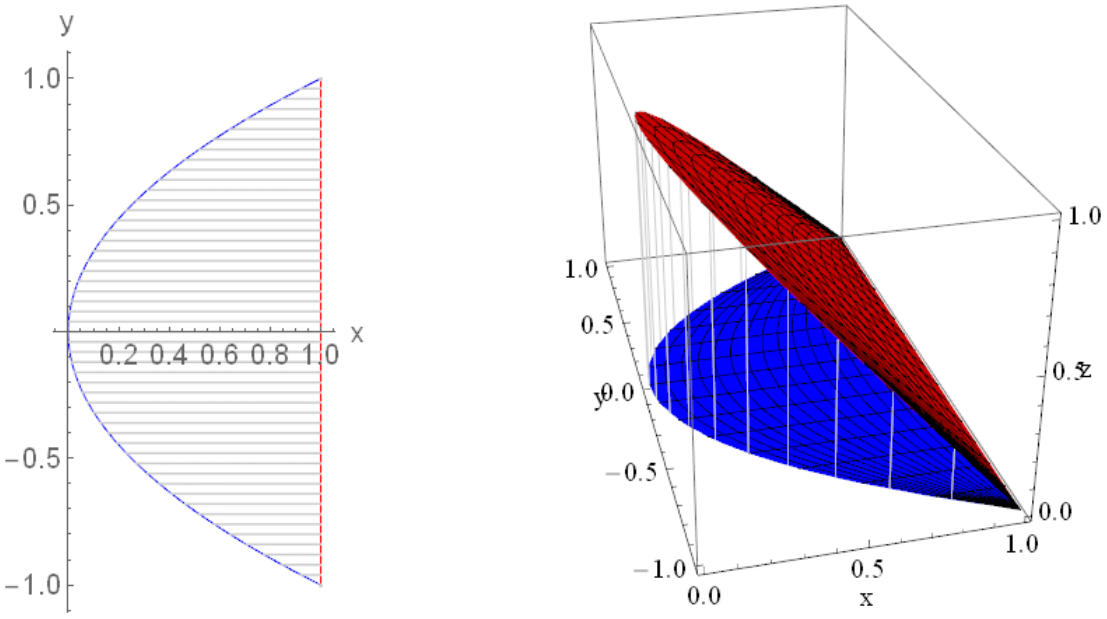


Figure 6.30. The region D={(x,y)|y^2<=x<=1,-1<=y<=1} (left) and the solid region W={(x,y,z)|y^2<=x<=1,-1<=y<=1,0<=z<=1-x} (right) of type 1.(b).

(b)

$$\int_{y=-1}^{1}\int_{x=y^2}^{1}\int_{z=0}^{1-x} dz\,dx\,dy = \frac{8}{15}$$

.

Integrate[1,{y,-1,1},{x,y^2,1},{z,0,1-x}]

(c)

We want to rewrite the integral using the integration order $dx\,dz\,dy$. So, the solid becomes elementary region of type 2.(b) by Definition 6.5. In this case, D is the region in the yz-plane, where the equation relating y and z can be obtained by substituting x=y^2 into

z=1-x.

Thus, z=1-y^2,

We can consider the graph z=1-y^2 as the "shadow" of the solid lying on the yz-plane (try moving Figure 6.31-right with your mouse to view the "vertical" axis z vs the "horizontal" axis y).

So, the new D is described by (Figure 6.31-left):

D={(y,z)|-1<=y<=1,0<=z<=1-y^2}.

The variable x in terms of y and z are the lower and upper boundary graphs (respectively) given by

x=y^2,

x=1-z.

Thus, the new description of W is

W={(x,y,z)|y^2<=x<=1-z,-1<=y<=1,0<=z<=1-y^2}.

The integral becomes

$$\int_{y=-1}^{1}\int_{z=0}^{1-y^2}\int_{x=y^2}^{1-z} dx\, dz\, dy$$

.

For the purpose of viewing the solid region of type 2.(b) we provide a module called viewSolid2b defined below:

```
(* Draw the solid of type 2.(b) elementary region *)

viewSolid2b[{y_,a_,b_},{z_,f_,g_},
{x_,F_,G_},opts___]:=Module[{u,v,zz,yy,surfs,pplot},

yy=a+u*(b-a);

zz=f+v*(g-f)/.y->yy;

surfs={{F,y,z},{G,y,z}}/.{z->zz,y->yy};

pplot=ParametricPlot3D[Evaluate[surfs],{u,0,1},{v,0,1},PlotStyle->
{Blue,Red}];

Show[pplot,Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1,0.05},
{v,0,1}],Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1},
{v,0,1,0.05}],AxesLabel->{"x","y","z"},opts]

]


Module[{plot,text},
```

```
plot[1]=viewRegion1[{y,-1,1},{z,0,1-y^2},AxesLabel->
{"y","z"},AspectRatio->Automatic];

text=Graphics[Text["D",{0.1,0.5}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid2b[{y,-1,1},{z,0,1-y^2},{x,y^2,1-z},ViewPoint->
{-0.5,-2,1}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]
```

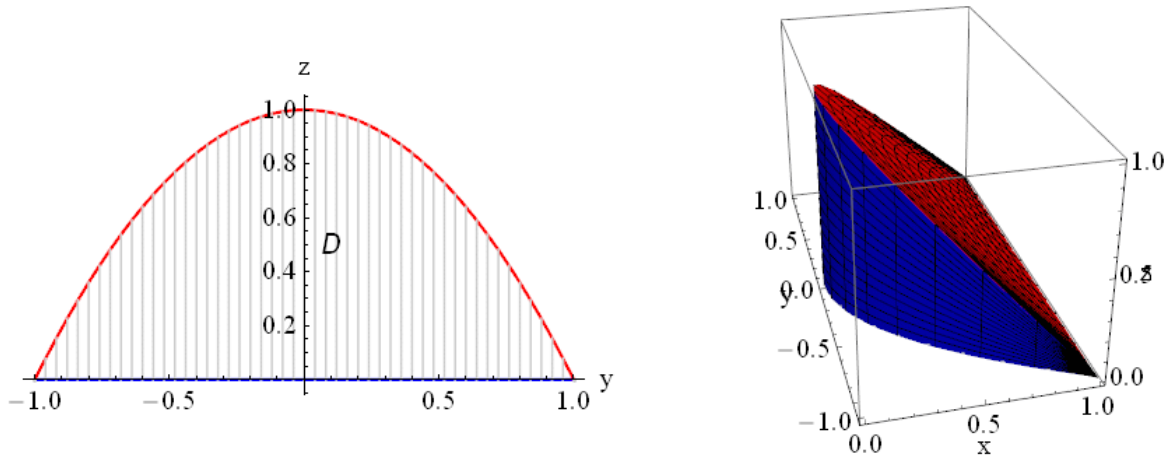


Figure 6.31. The new region description D={(y,z)|-1<=y<=1,0<=z<=1-y^2} (left) and the solid region W={(x,y,z)|y^2<=x<=1-z,-1<=y<=1,0<=z<=1-y^2} (right) of type 2.(b).

(d)

$$\int_{y=-1}^{1}\int_{z=0}^{1-y^2}\int_{x=y^2}^{1-z} dx\,dz\,dy = \frac{8}{15}$$

.

Integrate[1,{y,-1,1},{z,0,1-y^2},{x,y^2,1-z}]

Example 6.17. (a) Draw the solid whose volume is given by the iterated integral

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} dz\, dy\, dx$$

.

(b) Evaluate the integral in part (a).

(c) Rewrite the integral using the integration order $dy\, dx\, dz$ and draw the solid.

(d) Evaluate the integral in (c) and verify the result with the one obtained in (b).

Solution:

(a)

The solid region W is given by

W={(x,y,z)|0<=x<=4,0<=y<=(4 - x)/2,0<=z<=(12 - 3x - 6y)/4}.

By Definition 6.5 W is type 1.(a) elementary solid region, where D is elementary region of type 1 in the xy-plane by Definition 6.2:

D={(x,y)|0<=x<=4,0<=y<=(4 - x)/2}.

To draw the solid region we use the module viewSolid1a:

```
Module[{plot,text},

plot[1]=viewRegion1[{x,0,4},{y,0,(4-x)/2},AspectRatio->Automatic];

text=Graphics[Text["D",{1,0.7}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid1a[{x,0,4},{y,0,(4-x)/2},{z,0,(12-3x-6y)/4},ViewPoint->
{-1,-2,1}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]
```

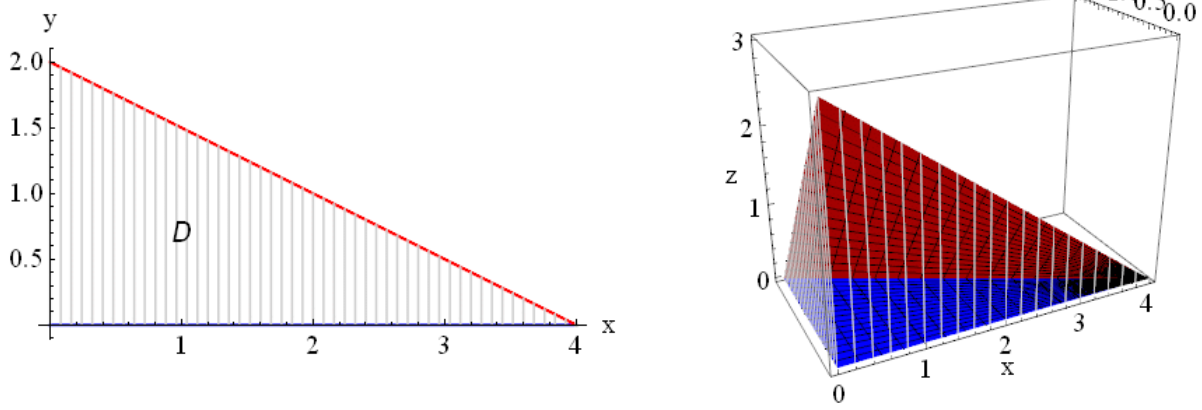


Figure 6.32. The region D={(x,y)|0<=x<=4,0<=y<=(4 - x)/2} (left) and the solid region W={(x,y,z)|0<=x<=4,0<=y<=(4 - x)/2,0<=z<=(12 - 3x - 6y)/4} (right) of type 1.(a).

(b)

$$\int_{x=0}^{4}\int_{y=0}^{(4-x)/2}\int_{z=0}^{(12-3x-6y)/4} dz\,dy\,dx = 4$$

.

Integrate[1,{x,0,4},{y,0,(4-x)/2},{z,0,(12-3x-6y)/4}]

(c)

To rewrite the integral using the integration order $dy\,dx\,dz$, which is integration over a solid region of type 3.(a) by Definition 6.5, we first need to find the new description for D in the xz-plane.

From Figure 6.32-right the variable x and z are related by setting y=0 into the function z=(12 - 3x - 6y)/4.

So, z=(12 - 3x)/4, x=(12 - 4z)/3.

Thus, the new region D is described by

D={(x,z)|0<=x<=(12 - 4z)/3,0<=z<=3}.

Next, the integration variable y in terms x and z is obtained from the function z=(12 - 3x - 6y)/4.

y=(12 - 3x - 4z)/6.

Hence, the new description of W is given by

W={(x,y,z)|0<=x<=(12 - 4z)/3,0<=y<=(12 - 3x - 4z)/6,0<=z<=3}.

The integral becomes

$$\int_{z=0}^{3}\int_{x=0}^{(12-4z)/3}\int_{y=0}^{(12-3x-4z)/6} dy\,dx\,dz$$

.

For the purpose of viewing the solid region of type 3.(a) we provide a module called viewSolid3a defined below:

```
(* Draw the solid of type 3.(a) elementary region *)

viewSolid3a[{z_,a_,b_},{x_,f_,g_},
{y_,F_,G_},opts___]:=Module[{u,v,xx,zz,surfs,pplot},

zz=a+u*(b-a);

xx=f+v*(g-f)/.z->zz;

surfs={{x,F,z},{x,G,z}}/.{x->xx,z->zz};

pplot=ParametricPlot3D[Evaluate[surfs],{u,0,1},{v,0,1},PlotStyle->
{Blue,Red}];

Show[pplot,Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1,0.05},
{v,0,1}],Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1},
{v,0,1,0.05}],AxesLabel->{"x","y","z"},opts]

]



Module[{plot,text},

plot[1]=viewRegion1[{z,0,3},{x,0,(12-4z)/3},AspectRatio-
>Automatic,AxesLabel->{"z","x"}];

text=Graphics[Text["D",{1,1.3}]];

plot[1]=Show[plot[1],text];
```

plot[2]=viewSolid3a[{z,0,3},{x,0,(12-4z)/3},{y,0,(12-3x-4z)/6},ViewPoint->{-1,-2,1}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]

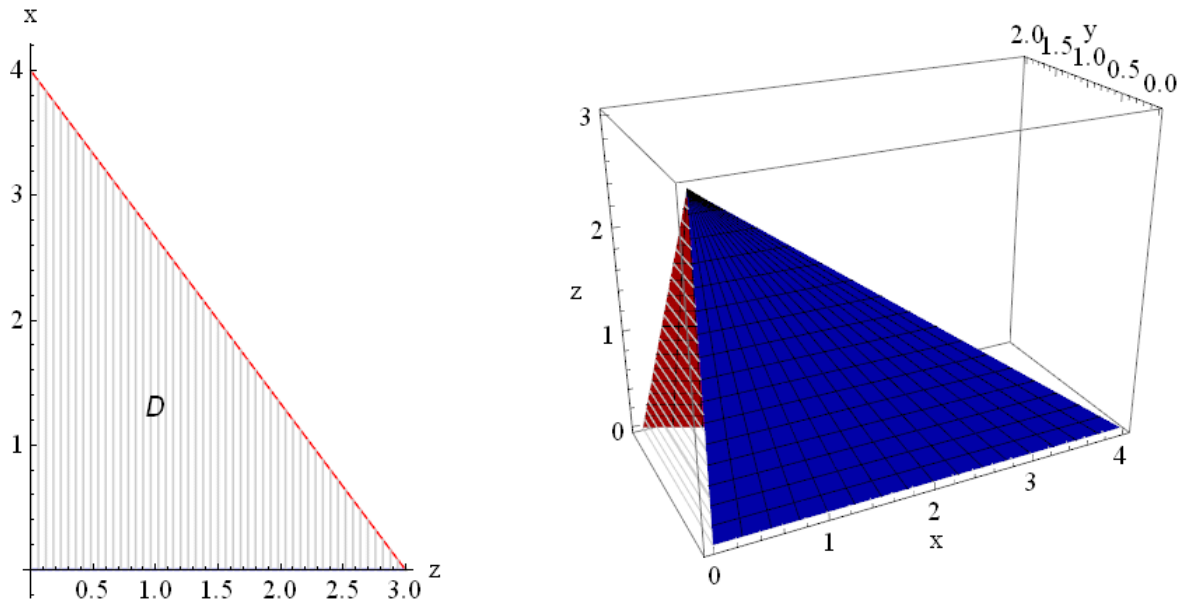


Figure 6.33. The new region description D={(x,y)|0<=x<=(12 - 4z)/3,0<=z<=3} (left) and the solid region W={(x,y,z)|0<=x<=(12 - 4z)/3,0<=y<=(12 - 3x - 4z)/6,0<=z<=3} (right) of type 3.(a).

(d)

$$\int_{z=0}^{3}\int_{x=0}^{(12-4z)/3}\int_{y=0}^{(12-3x-4z)/6} dy\,dx\,dz = 4$$

.

Integrate[1,{z,0,3},{x,0,(12-4z)/3},{y,0,(12-3x-4z)/6}]

Example 6.18. (a) Draw the solid whose volume is given by the iterated integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\, dy\, dx$$

(b) Evaluate the integral in part (a).

(c) Rewrite the integral using the integration order $dy\, dz\, dx$ and draw the solid.

(d) Evaluate the integral in (c) and verify the result with the one obtained in (b).

Solution:

(a)

The solid region W is given by

W={(x,y,z)|0<=x<=1,-1<=y<=0,0<=z<=y^2}.

By Definition 6.5 W is type 1.(a) elementary solid region, where D is elementary region of type 1 in the xy-plane by Definition 6.2:

D={(x,y)|0<=x<=1,-1<=y<=0}.

To draw the solid region we use the module viewSolid1a:

Module[{plot,text},

plot[1]=viewRegion1[{x,0,1},{y,-1,0},AspectRatio->Automatic];

```
text=Graphics[Text["D",{0.5,-0.5}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid1a[{x,0,1},{y,-1,0},{z,0,y^2}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]
```

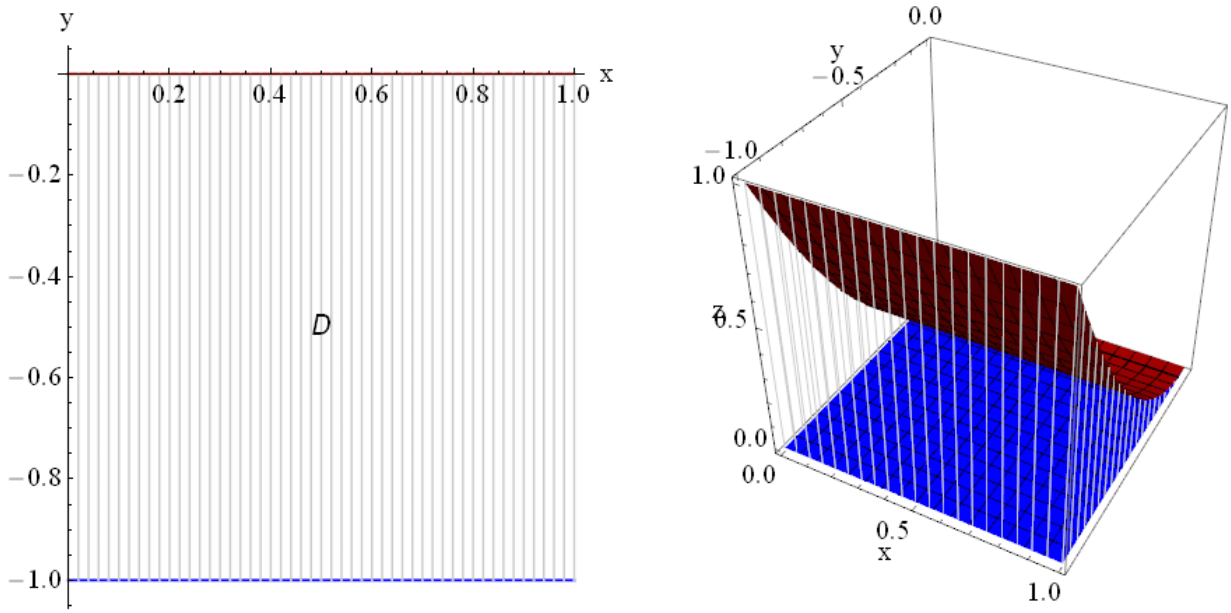


Figure 6.34. The region D={(x,y)|0<=x<=1,-1<=y<=0} (left) and the solid region W={(x,y,z)|0<=x<=1,-1<=y<=0,0<=z<=y^2} (right) of type 1.(a).

(b)

$$\int_{x=0}^{1}\int_{y=-1}^{0}\int_{z=0}^{y^2} dz\,dy\,dx = \frac{1}{3}$$

.

Integrate[1,{x,0,1},{y,-1,0},{z,0,y^2}]

(c)

To rewrite the integral using the integration order $dl\,y\,dl\,z\,dl\,x$ so that the solid becomes elementary region of type 3.(b) by Definition 6.5 we first need to describe the region D in the xz-plane.

From Figure 6.34-right the new description D is given by

D={(x,z)|0<=x<=1,0<=z<=1}.

The integration variable y in term of z is y=±Sqrt[z].

Also from Figure 6.34-right the variable y is in negative territory. So, it is bounded below and above by y=-1 and y=-Sqrt[z], respectively. So,

W={(x,y,z)|0<=x<=1,-1<=y<=-Sqrt[z],0<=z<=1}.

The integral becomes

$$\int_{x=0}^{1}\int_{z=0}^{1}\int_{y=-1}^{-\sqrt{z}} dl\,y\,dl\,z\,dl\,x$$

.

The module viewSolid3b is defined below to visualize the solid region:

(* Draw the solid of type 3.(b) elementary region *)

viewSolid3b[{x_,a_,b_},{z_,f_,g_},
{y_,F_,G_},opts___]:=Module[{u,v,xx,zz,surfs,pplot},

```
xx=a+u*(b-a);

zz=f+v*(g-f)/.x->xx;

surfs={{x,F,z},{x,G,z}}/.{x->xx,z->zz};

pplot=ParametricPlot3D[Evaluate[surfs],{u,0,1},{v,0,1},PlotStyle->
{Blue,Red}];

Show[pplot,Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1,0.05},
{v,0,1}],Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1},
{v,0,1,0.05}],AxesLabel->{"x","y","z"},opts]

]



Module[{plot,text},

plot[1]=viewRegion1[{x,0,1},{z,0,1},AspectRatio->Automatic,AxesLabel-
>{"x","z"}];

text=Graphics[Text["D",{0.5,0.5}]];

plot[1]=Show[plot[1],text];

plot[2]=viewSolid3b[{x,0,1},{z,0,1},{y,-1,-Sqrt[z]}];

GraphicsGrid[{Array[plot,2]},ImageSize->600]

]
```

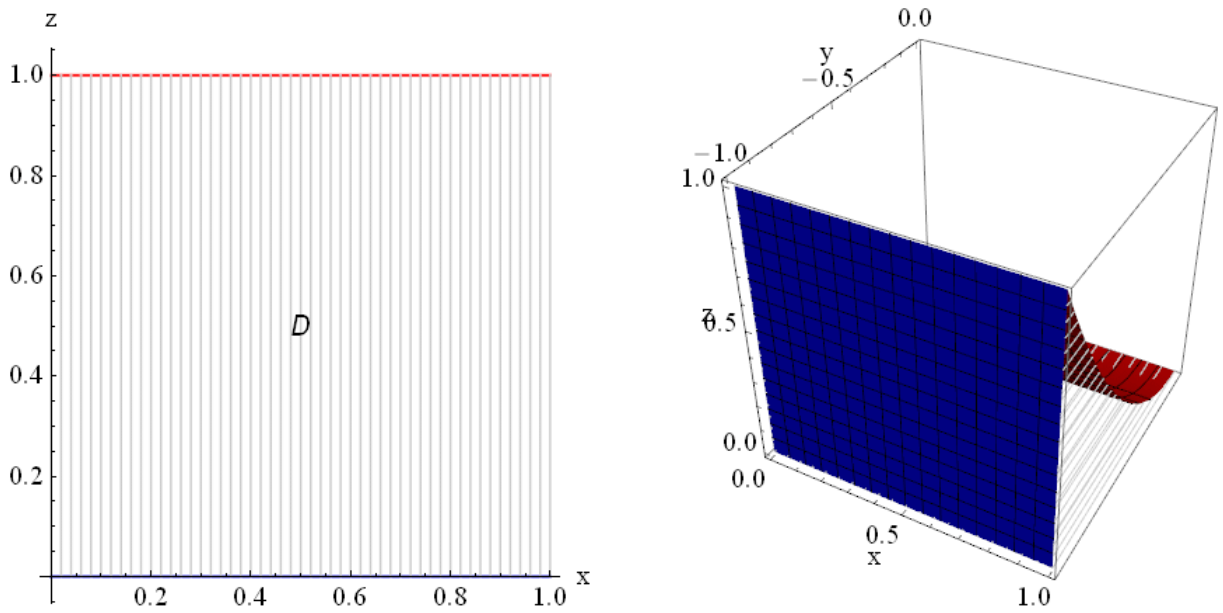


Figure 6.35. The new region description D={(x,z)|0<=x<=1,0<=z<=1} (left) and the solid region W={(x,y,z)|0<=x<=1,-1<=y<=-Sqrt[z],0<=z<=1} (right) of type 3.(b).

(d)

$$\int_{x=0}^{1}\int_{z=0}^{1}\int_{y=-1}^{-\sqrt{z}} dy\, dz\, dx = \frac{1}{3}$$

.

Integrate[1,{x,0,1},{z,0,1},{y,-1,-Sqrt[z]}]

After showing examples to find the volumes corresponding to the six elementary regions based on Definition 6.5 we shall now look at more examples with general functions of three variables.

Example 6.19. Let D be the region of the xy-plane that is bounded below by y=1+(x+1)^3 and above by y=3-x^2. Also, let W be the solid region that is bounded below by D, above by the plane z=y, and laterally by the two cylinders y=1+(x+1)^3 and y=3-x^2. Evaluate the triple integral

$$\iiint_W x^2 y z \, dV .$$

Solution:

First, we describe the region D of the xy-plane.

```
ContourPlot[{y==1+(x+1)^3,y==3-x^2},{x,-2,1},{y,0,4},Frame->False,Axes->True,AxesLabel->{"x","y"},ContourStyle->{Blue,Red},PlotLegends->"Expressions",AspectRatio->Automatic]
```

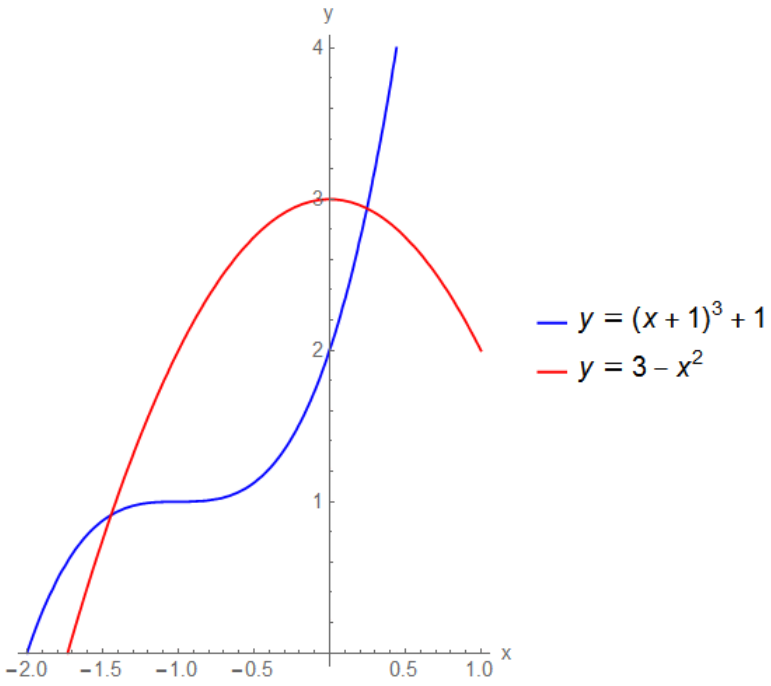


Figure 6.36. The region D of the xy-plane is bounded below by y=1+(x+1)^3 and above by y=3-x^2.

D is a elementary region of type 1 by Definition 6.2, so we need to find the limits of integration with respect to x which are the intersection points between the curve y=1+(x+1)^3 and y=3-x^2. The intersection occurs when

y=1+(x+1)^3=3-x^2.

Using FindRoot command and supplying an estimated root by using Figure 6.36 we set the left and right root to be x=-1.5 and x=0.5, respectively.

```
FindRoot[1+(x+1)^3==3-x^2,{x,-1.5}]
```

Note that FindRoot returns a list of solution with replacement, that is, x is to be replaced by -1.44504, the value of our solution. In this case, there is only one solution at the first element and the value -1.44504 is in the second position. So, we pick up this value using [[1,2]] and assign it to x1.

```
x1=FindRoot[1+(x+1)^3==3-x^2,{x,-1.5}][[1,2]]
```

Similarly to the second root,

```
x2=FindRoot[1+(x+1)^3==3-x^2,{x,0.5}][[1,2]]
```

Using viewRegion1 module we graph D:

viewRegion1[{x,x1,x2},{y,1+(x+1)^3,3-x^2},AxesOrigin->
{0,0},PlotRange->{{-2,1},{0,4}},AspectRatio->Automatic]

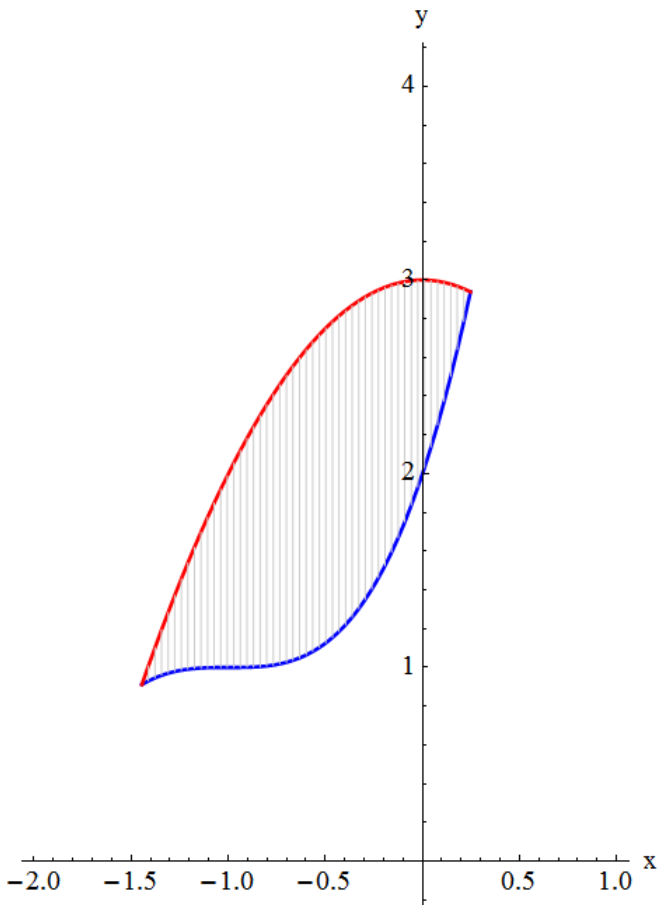


Figure 6.37. The region D={(x,y)|-1.44504<=x<=0.24698,1+
(x+1)^3<=y<=3-x^2} of Example 6.19.

Hence, the solid region W is described as

W={(x,y,z)|-1.44504<=x<=0.24698,1+(x+1)^3<=y<=3-x^2,0<=z

plot1=ContourPlot3D[{z==y,y==3-x^2,y==1+(x+1)^3},{x,-2,1},{y,0,4},
{z,0,4},Mesh->None,ContourStyle->{{Blue,Opacity[0.1]},
{Red,Opacity[0.3]},{Green,Opacity[0.3]}},PlotLegends->"Expressions"]

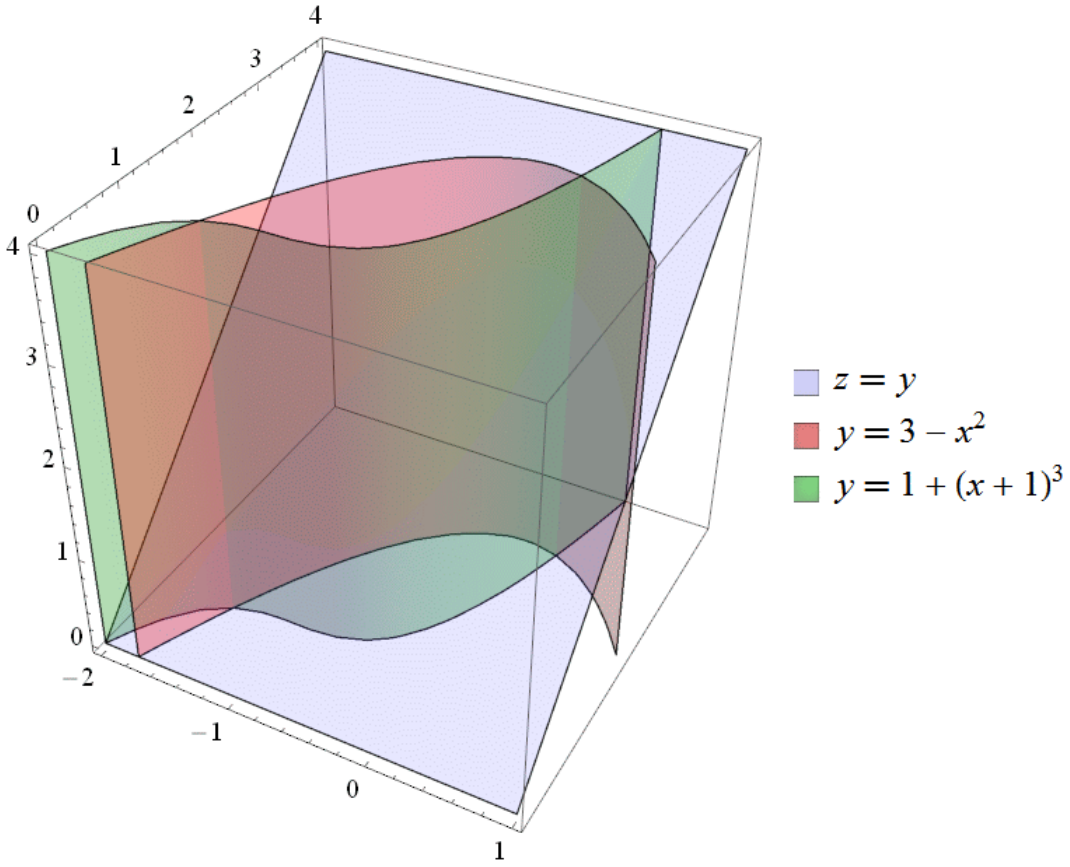


Figure 6.38. The solid region W is bounded below by D, above by the plane z=y, and laterally by the two cylinders y=1+(x+1)^3 and y=3-x^2

plot2=viewSolid1a[{x,x1,x2},{y,1+(x+1)^3,3-x^2},{z,0,y}]

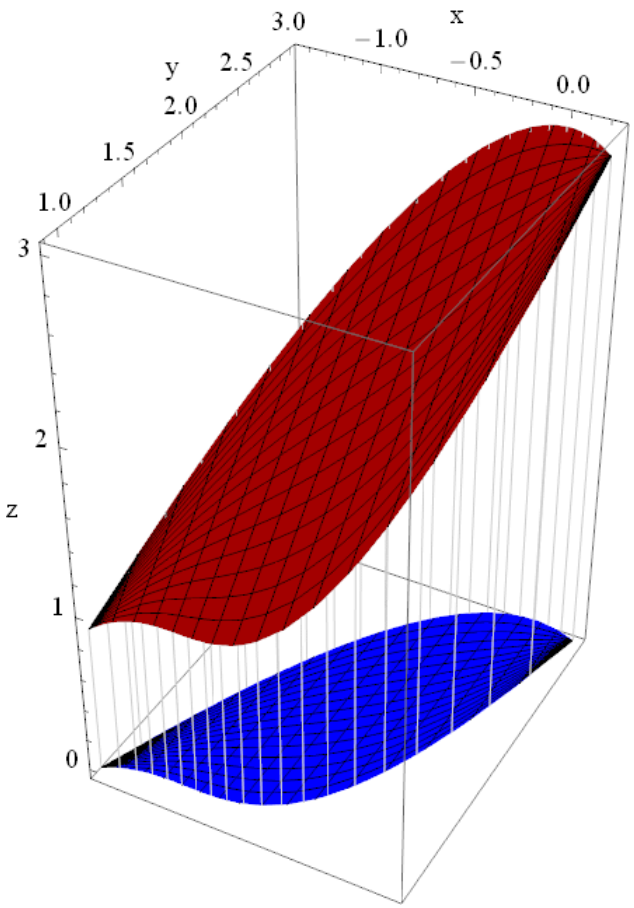


Figure 6.39. The solid region W={(x,y,z)|-1.44504<=x<=0.24698,1+(x+1)^3<=y<=3-x^2,0<=z

Show[plot1,plot2]

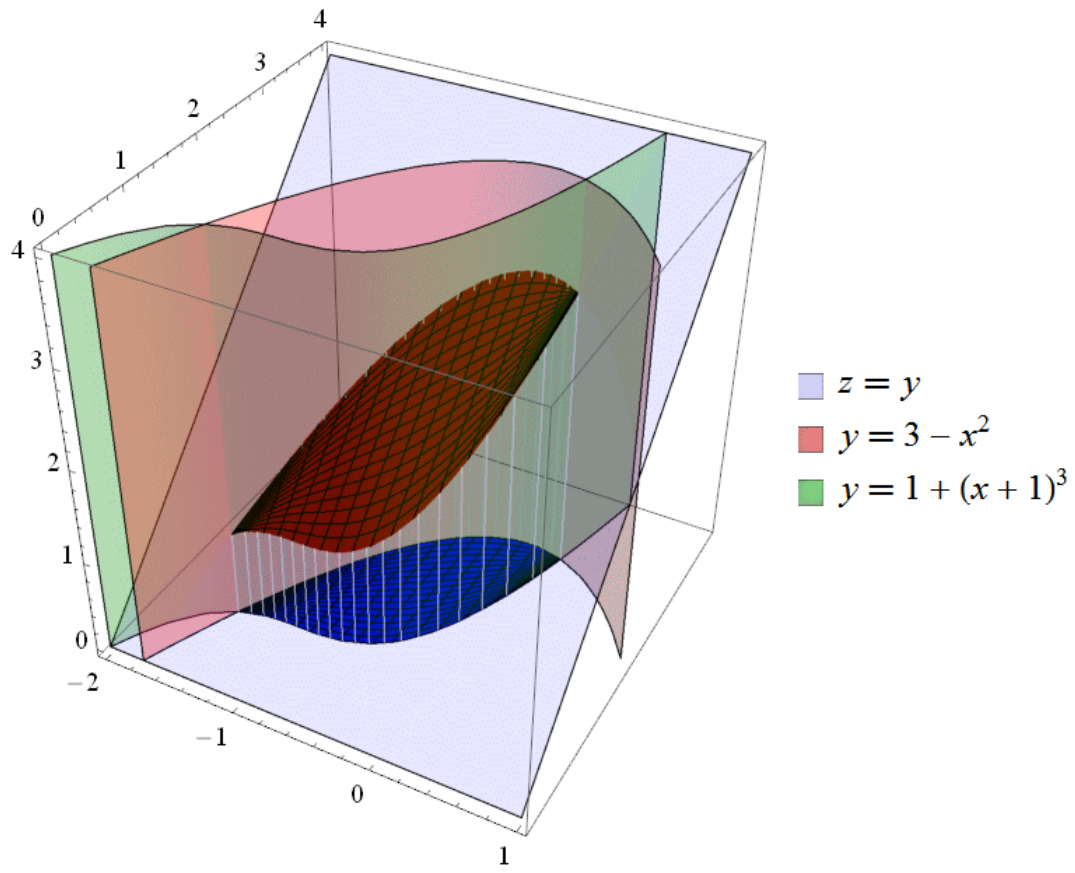


Figure 6.40. The solid region W is graphed together with boundaries y=1+(x+1)^3, y=3-x^2 and z=y.

Finally, we compute the triple integral as the iterated integrals:

$$\iiint_W x^2\, y\, z\, dV = \int_{x=-1.44504}^{0.24698} \int_{y=1+(x+1)^3}^{3-x^2} \int_{z=0}^{y} x^2\, y\, z\, dz\, dy\, dx$$

$$= 1.81756$$

Integrate[x^2 y z,{x,x1,x2},{y,1+(x+1)^3,3-x^2},{z,0,y}]

Example 6.20. Let a solid region W be bounded by x=y^2, y=z, x=y, and z=0. Integrate f(x,y,z)=4x+y over W.

Solution:

```
Clear[x,y,z]

plot1=ContourPlot3D[{x==y^2,y==z,x==y,z==0},{x,-1,1},{y,-0.5,1},
{z,0,1},Mesh->None,ContourStyle->{{Blue,Opacity[0.3]},
{Green,Opacity[0.3]},{Red,Opacity[0.3]},
{Yellow,Opacity[0.3]}},AxesLabel->{"x","y","z"},PlotLegends-
>"Expressions"]
```

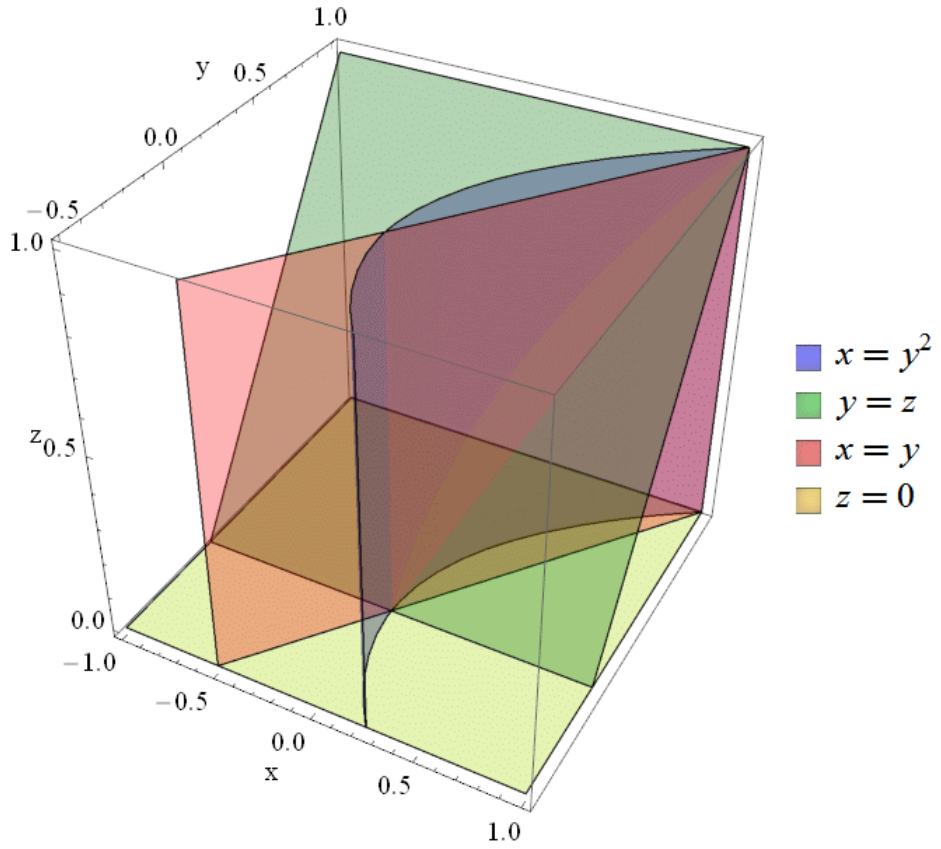


Figure 6.41. The solid region W is bounded by x=y^2, y=z, x=y, and z=0.

Using Figure 6.41 we find the description of the region D of the xy-plane. It can be considered as elementary region of type 1 by Definition 6.2 given by

D={(x,y)|0<=x<=1,x<=y<=Sqrt[x]}.

viewRegion1[{x,0,1},{y,x,Sqrt[x]},AspectRatio->Automatic]

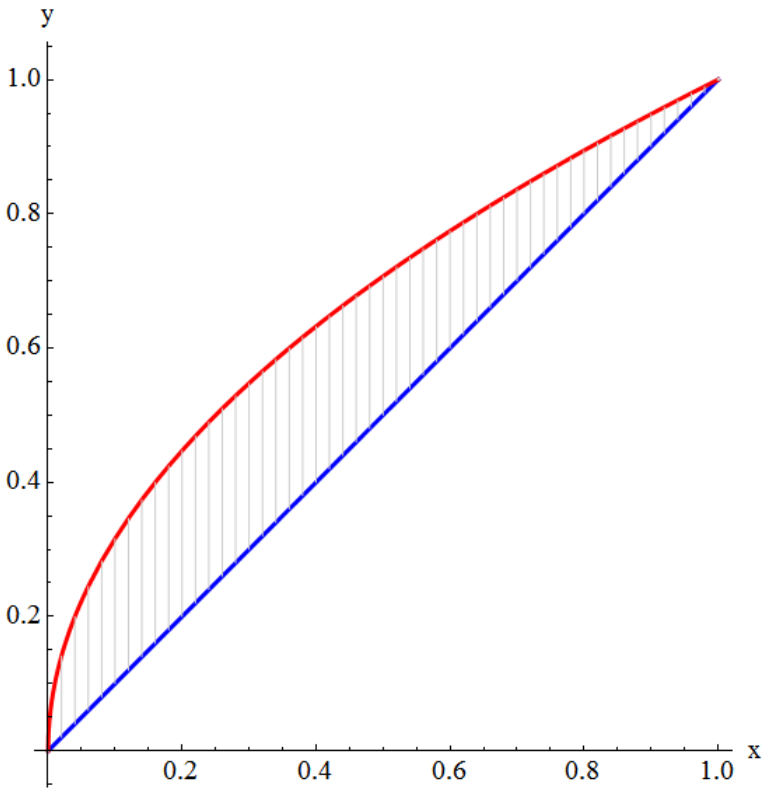


Figure 6.42. The region D is in the xy-plane given by D= {(x,y)|0<=x<=1,x<=y<=Sqrt[x]}

The solid region W is bounded below by z=0 and above by z=y, thus

W={(x,y,z)|0<=x<=1,x<=y<=Sqrt[x],0<=z<=y}.

With such description, W is a region of type 1.(a) by Definition 6.5.

plot2=viewSolid1a[{x,0,1},{y,x,Sqrt[x]},{z,0,y}]

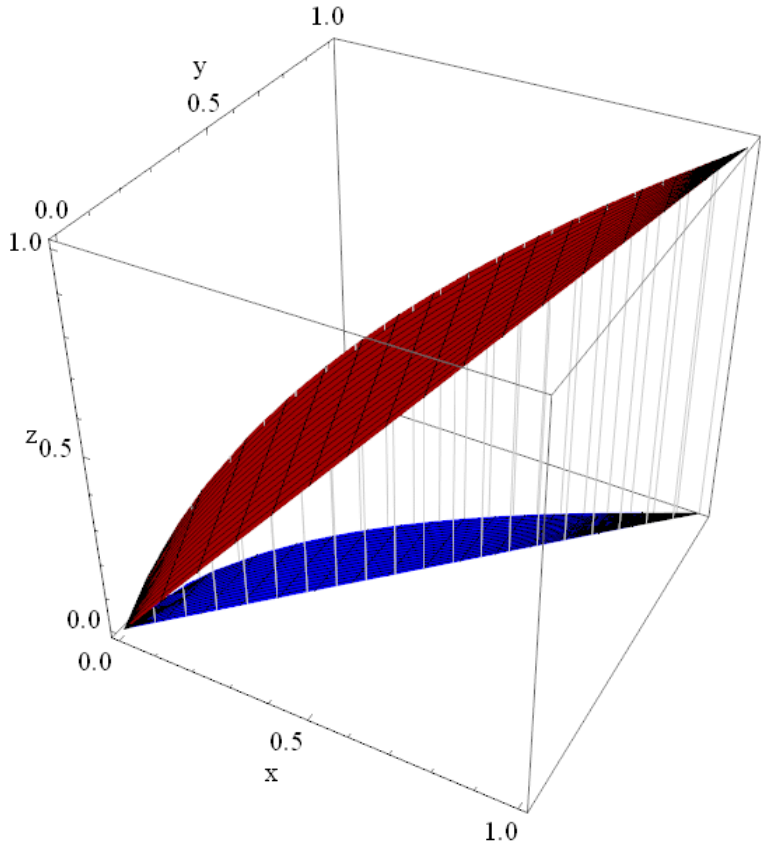


Figure 6.43. The solid region W= {(x,y,z)|0<=x<=1,x<=y<=Sqrt[x],0<=z<=y}.

Show[plot1,plot2]

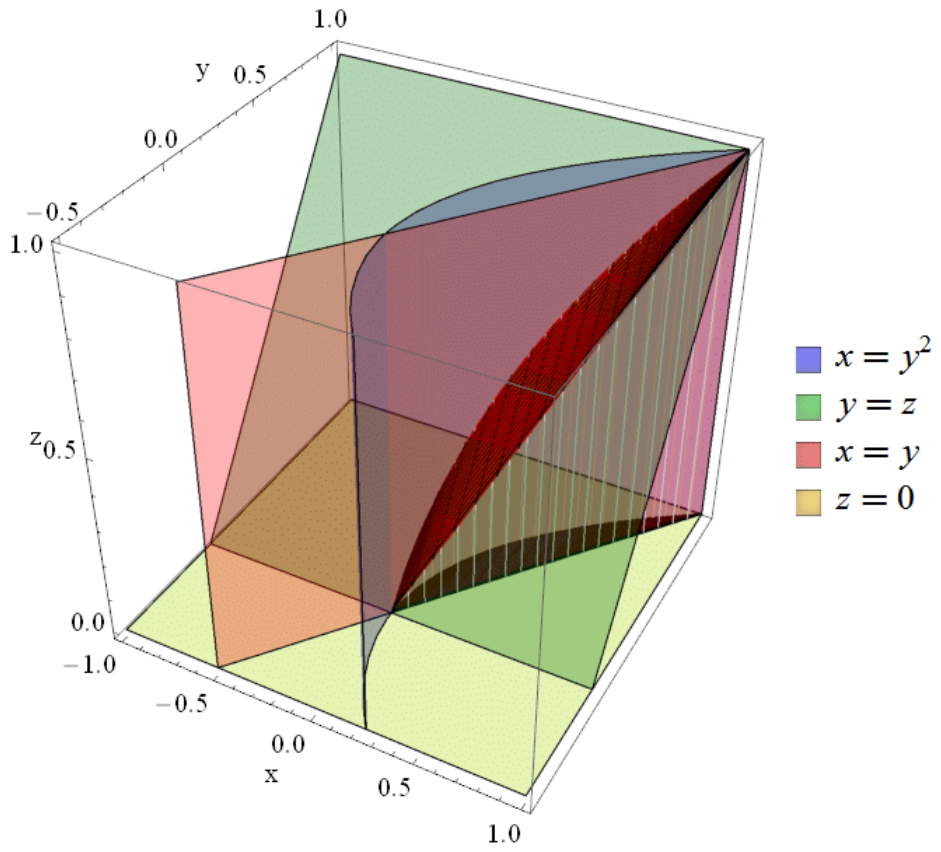


Figure 6.44. The solid region W= {(x,y,z)|0<=x<=1,x<=y<=Sqrt[x],0<=z<=y} is graphed together with the boundaries x=y^2, y=z, x=y, and z=0.

The triple integral:

$$\iiint_W f(x, y, z)\, dV = \iiint_W (4x + y)\, dV$$

$$= \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} \int_{z=0}^{y} (4x + y)\, dz\, dy\, dx$$

$$= \frac{13}{60}$$

.

```
Integrate[4x+y,{x,0,1},{y,x,Sqrt[x]},{z,0,y}]
```

6.3 Change of Variables in Double Integrals

We have seen in previous sections the process of evaluating multiple integral can be a complicated task. In many cases, the integrand and the domain of integration can be simplified if we make changes of variables. Three of the most well-known changes of variables are expressed in polar, cylindrical, and spherical coordinates.

We begin our study of coordinate transformation using polar coordinates.

6.3.1 Double Integrals in Polar Coordinates

Some double integrals are more convenient to evaluate in polar coordinates than in rectangular coordinates. This is especially true when the domain of integration is of angular sector such as circles, cardioids, and for integrands that involve x^2+y^2.

Before we delve into integration involving polar coordinates we shall define the notion of Jacobian of a general transformation from one coordinate system to another coordinate system in $\mathbb{R}^2$ which is given below:

Definition 6.7

Let T be a differentiable coordinate transformation from a uv-plane to the xy-plane such that

T:(u,v)->(x(u,v),y(u,v)), where $(u, v), (x, y) \in \mathbb{R}^2$.

The Jacobian of the transformation T, denoted $\frac{\partial(x, y)}{\partial(u, v)}$ is the determinant of the derivative matrix DT(u,v), where

$$D\mathbf{T}(u, v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} . \tag{6.13}$$

Thus, by Definition 6.7 we have

(6.14)

$$\frac{\partial(x, y)}{\partial(u, v)} = \det D\mathbf{T}(u, v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

.

Note that the Jacobian of the transformation T is a function of u and v.

The role of the Jacobian is reflected in the following theorem which we state without proof:

Theorem 6.8 (Change of Variables in Double Integrals)

Let T:D*->D be a differentiable transformation that is one-to-one on the interior of D*, where D* and D are elementary regions in the uv-plane and the xy-plane, respectively. If f(x,y) is continuous, then

(6.15)

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

.

Note that the term $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$ is the absolute value of the Jacobian of the transformation T.

Now, let us consider the polar coordinates as an example in applying Theorem 6.8.

Recall that rectangular and polar coordinates are related by

x=r cos θ,

y=r sin θ.

From which we obtain

x^2+y^2=r^2,

tan θ=y/x if x!=0.

By Definition 6.7:

T(r,θ)=(x,y)=(r cos θ,r sin θ).

Using equation (6.13) the derivative matrix of the transformation T is

$$D\mathbf{T}(u, v) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

.

Using equation (6.14) the Jacobian of the transformation T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \det D\mathbf{T}(u, v) = r\cos^2\theta + r\sin^2\theta = r$$

.

Hence, equation (6.15) for the transformation T becomes

(6.16)

$$\int\int_D f(x, y)\, dx\, dy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

,

where the domain of integration D*={ (r,θ)| θ1<=θ<=θ2, r(θ1)<=r<=r(θ2) }.

Example 6.21. Given an integral

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} e^{x^2+y^2}\, dx\, dy$$

.

(a) Compute directly using Integrate command.

(b) Transform the integral to one in polar coordinates and evaluate the polar integral.

Solution:

(a)

The region D is a circle of radius a of type 2 elementary region by Definition 6.2, confined by x>=0. It covers the first and fourth quadrant, described by

D={(x,y)|0<=x<=Sqrt[a^2-y^2],-a<=y<=a}.

Integrate[E^(x^2+y^2),{y,-a,a},{x,0,Sqrt[a^2-y^2]}]

Integrate[E^(x^2+y^2),{y,-a,a},{x,0,Sqrt[a^2-y^2]}]/.{a->1}//N

Note that Mathematica does not yield a closed form result for this integral in Cartesian coordinates.

(b)

Changing to polar coordinates using the relation:

$x=r\cos\theta$, $y=r\sin\theta$,

we obtain the new description D*

$D^*=\{(r,\theta)|0<=r<=a,-(\pi/2)<=\theta<=\pi/2\}$.

The Jacobian for this change of variables is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

.

Using equation (6.16), we have

$$\int_{y=-a}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} e^{x^2+y^2}\,dx\,dy = \int_{r=0}^{a}\int_{\theta=-\pi/2}^{\pi/2} e^{r^2}\,r\,d\theta\,dr$$

$$= \frac{1}{2}\pi\left(e^{a^2}-1\right)$$

Integrate[E^r^2 r,{θ,-(π/2),π/2},{r,0,a}]

Integrate[E^r^2 r,{θ,-(π/2),π/2},{r,0,a}]/.{a->1}//N

Example 6.22. Find the area of the region inside the cardioid r=1-cosθ and outside the circle r=1.

Solution:

In polar coordinates, a circle with radius 1 centered at the origin is represented by

r=1, 0<=θ<=2π.

Using PolarPlot command we plot the two plane curves to determine the description of the region D*.

plot=PolarPlot[{1-Cos[θ],1},{θ,0,2π},PlotStyle->{Blue,Red},PlotLegends->"Expressions"]

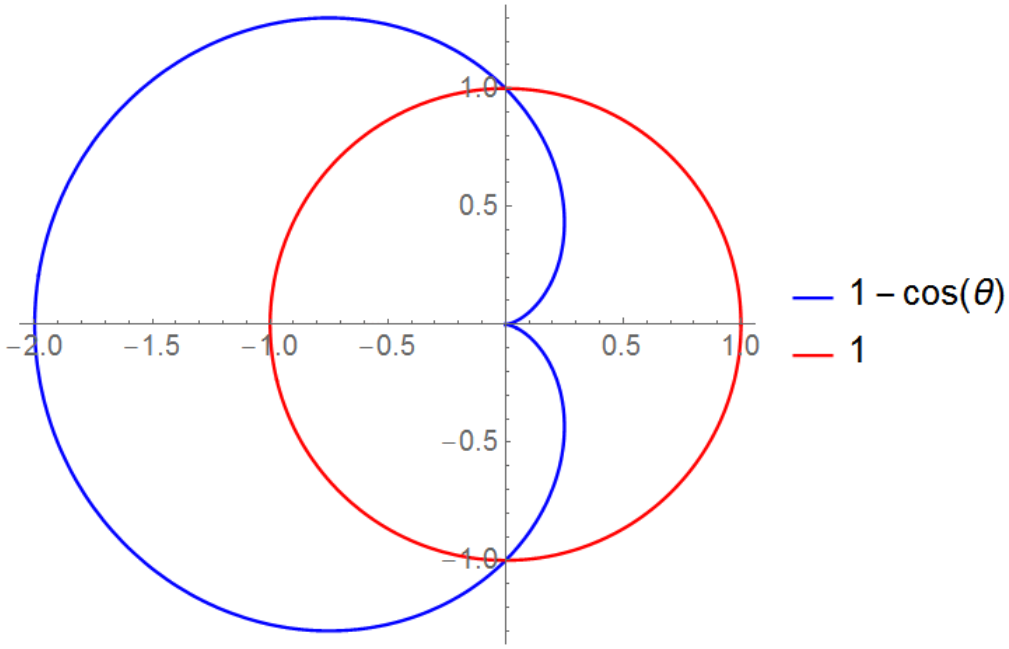


Figure 6.45. The cardioid r=1-cosθ and the circle r=1.

The region D* of interest can be described as

D*={(r,θ)|1<=r<=1-cos θ,π/2<=θ<=(3π)/2}.

To plot the area being examined we provide a module called polarRegion. Note that the parameters passed to polarRegion are the same as those entered into the Integrate command.

(* Graph the shaded area using polar coordinates *)

polarRegion[{θ_,a_,b_},{r_,R1_,R2_},opts___]:=

Module[{twocurves},

```
xp[rr_]:=rr Cos[θ];

yp[rr_]:=rr Sin[θ];

twocurves=PolarPlot[{R1,R2},{θ,a,b},PlotStyle->{{Blue},
{Red,Thickness[0.008]}},AxesLabel->{"x","y"}];

Show[twocurves,Table[Graphics[{GrayLevel[0.8],Line[{{xp[R1],yp[R1]},
{xp[R2],yp[R2]}}]}],{θ,a,b,(b-a)/50}],opts]

]


region=polarRegion[{θ,π/2,(3π)/2},{r,1,1-Cos[θ]},ImageSize->300]
```

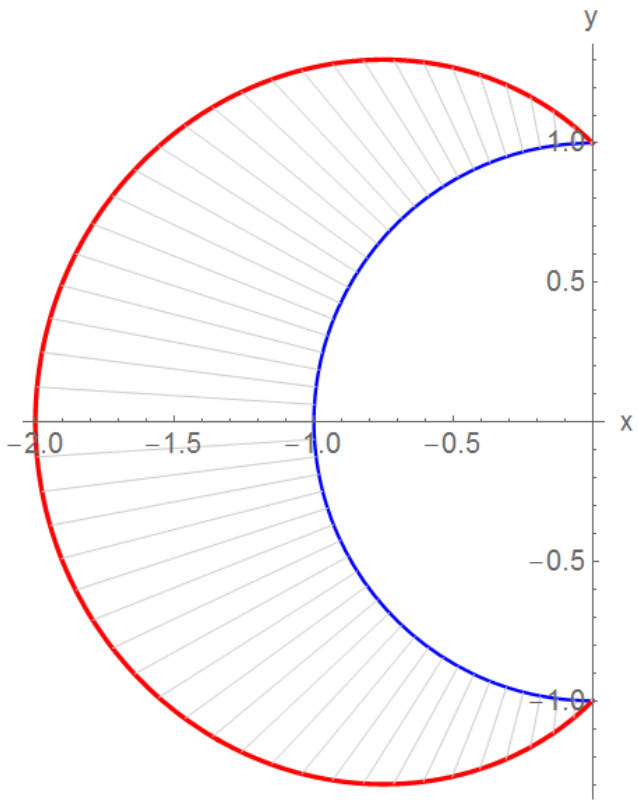


Figure 6.46. The region D*={(r,θ)|1<=r<=1-cos θ,π/2<=θ<=(3π)/2}.

Show[plot,region]

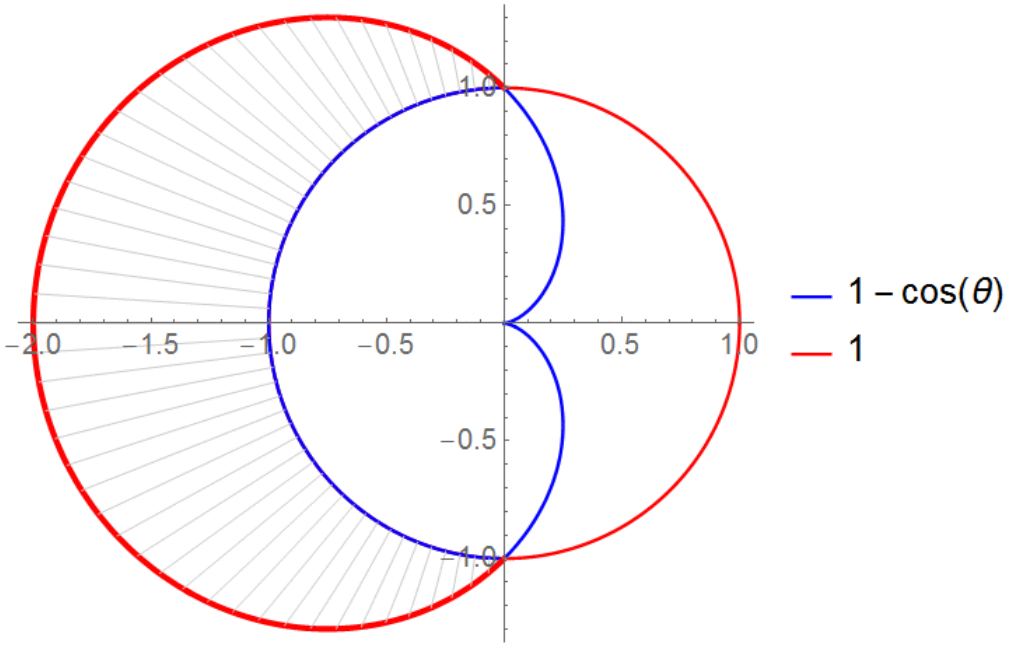


Figure 6.47. The region D* is inside the cardioid r=1-cosθ and outside the circle r=1.

To find the area of the region we use equation (6.16) with f(x,y)=1:

$$A = \int\int_D dA = \int\int_D dx\, dy = \int\int_{D^*} r\, dr\, d\theta$$

,

$$A = \int\int_D dA = \int_{\theta=\pi/2}^{3\pi/2}\int_{r=1}^{1-\cos\theta} r\, dr\, d\theta = \frac{1}{4}(8+\pi)$$

.

Integrate[r,{θ,π/2,(3π)/2},{r,1,1-Cos[θ]}]

Example 6.23. (a) Evaluate the following integral directly using Integrate command. Observe how long it takes to execute the integral by using Timing command.

$$\iint_D \frac{x}{\sqrt{x^2+y^2}}\, dA$$

,

where D is the unit square [0,1]x[0,1].

(b) Convert the integral in part (a) to polar coordinates and evaluate the integral.

Solution:

(a)

$$\int_{x=0}^{1}\int_{y=0}^{1} \frac{x}{\sqrt{x^2+y^2}}\, dA = 0.647794$$

.

```
Timing[Integrate[x/Sqrt[x^2+y^2],{x,0,1},{y,0,1}]]//N
```

(b)

First, we find the new description D* in the rθ-plane from D.

Using the change of variables:

x=r cos θ, y=r sin θ,

we obtain

r=Sqrt[x^2+y^2], θ=tan^-1 y/x.

```
Module[{vertex,plot},
vertex[1]={0,0};
vertex[2]={0,1};
vertex[3]={1,0};
vertex[4]={1,1};
plot[1]=Graphics[{Blue,PointSize[0.04],Point[Array[vertex,4]]}];
plot[2]=Graphics[{Red,Thickness[0.01],Line[{vertex[1],vertex[2],vertex[4]
,vertex[3],vertex[1]}]}];
plot[3]=Graphics[{Blue,Thickness[0.01],Dashed,Line[{vertex[1],vertex[4]}
]}];
Show[Array[plot,3],Axes->True,AxesLabel->{"x","y"},ImageSize->300]
]
```

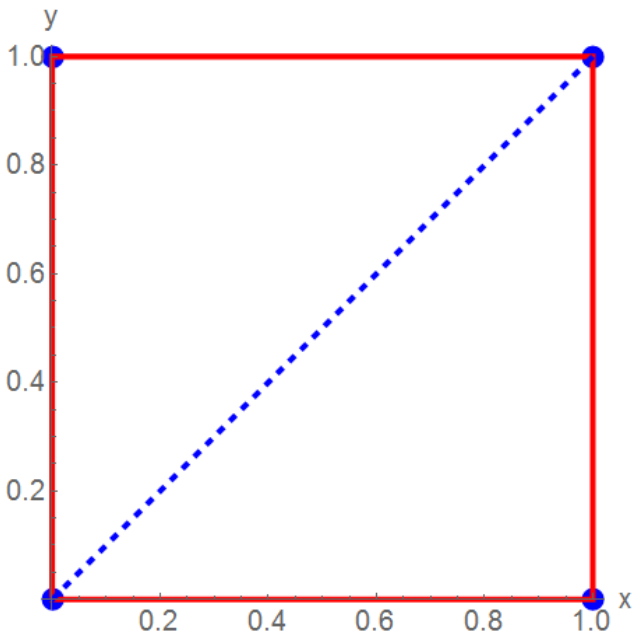


Figure 6.48. The region D={(x,y)|0<=x<=1,0<=y<=1} is divided into two subregions along the line θ=π/4.

In polar coordinates, we need to divide the region along the line θ=π/4 into two subregions.

Region 1: D1*

In this region the polar angle θ ranges from 0 to cos^-1(1/Sqrt[2])=π/4 while r depends on θ with fixed x=1.

Using the Cartesian and polar coordinates relation:

x=r cos θ, y=r sin θ,

we have

1=r cos θ,

r(θ)=sec θ.

Thus, the region D1* in polar coordinates:

D1* = {(r, θ)|0<= r<=sec θ, 0<=θ<=π/4}.

Using polarRegion module we plot D1*:

Module[{plot},

plot[1]=polarRegion[{θ,0,π/4},{r,0,Sec[θ]}];

plot[2]=Graphics[Text["D1*",{0.7,0.3}]];

Show[Array[plot,2],ImageSize->300]

]

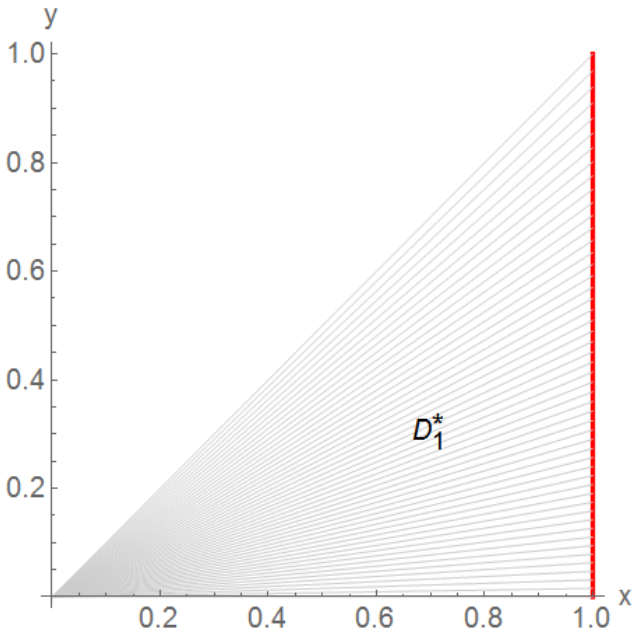


Figure 6.49. The first subregion description in polar coordinates D1* = {(r, θ)|0<= r<=sec θ, 0<=θ<=π/4}.

Region 2: D2*

In this region the polar angle θ ranges from π/4 to π/2 while r depends on θ with fixed y=1.

So,

1=r sin θ,

r(θ)=cosec θ.

Thus, the region D2* in polar coordinates is described as

D2* = {(r, θ)|0<= r<=cosec θ, π/4<=θ<=π/2}.

Using polarRegion module we plot D2*:

```
Module[{plot},
plot[1]=polarRegion[{θ,π/4,π/2},{r,0,Csc[θ]}];
plot[2]=Graphics[Text["D2*",{0.3,0.7}]];
Show[Array[plot,2],ImageSize->300]
]
```

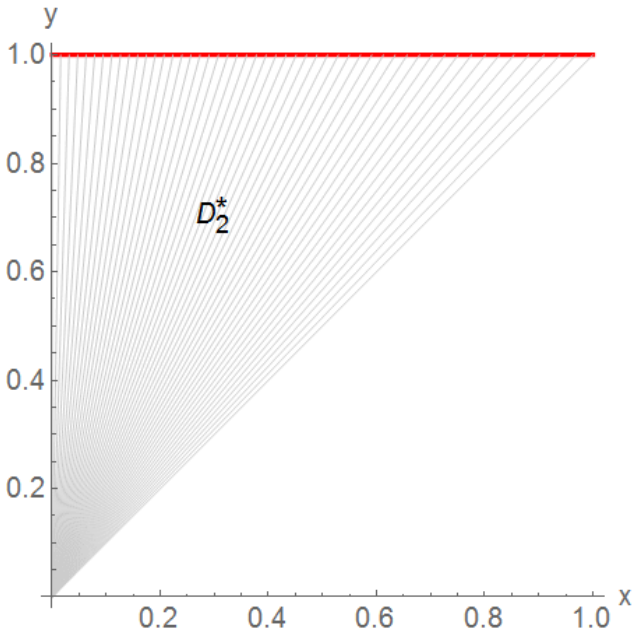


Figure 6.50. The second subregion description in polar coordinates D2* = {(r, θ)|0<= r<=cosec θ, π/4<=θ<=π/2}.

The Jacobian for this change of variables is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

.

Using equation (6.16) we have

$$\iint_D f(x, y)\, dx\, dy = \iint_{D_1^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta + \iint_{D_2^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Since x/Sqrt[x^2+y^2]=(r cos θ)/r=cosθ,

$$\int\int_D \frac{x}{\sqrt{x^2+y^2}}\,dA = \int_{\theta=0}^{\pi/4}\int_{r=0}^{\sec\theta} r\cos\theta\,dr\,d\theta$$
$$+\int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\operatorname{cosec}\theta} r\cos\theta\,dr\,d\theta$$
$$\int\int_D \frac{x}{\sqrt{x^2+y^2}}\,dA = 0.647794$$

.

Timing[Integrate[r Cos[θ],{θ,0,π/4},{r,0,Sec[θ]}]+Integrate[r Cos[θ], {θ,π/4,π/2},{r,0,Csc[θ]}]]//N

6.3.2 General Change of Variables in Double Integrals

Transformation to polar coordinates in previous section is a special type of changing variables in double integration. The need arises when we simplify the domain of integration and/or the integrand. Usually this is related to the geometry of the domain itself. In this section we shall continue our discussion about general change of variables due to a transformation.

Example 6.24. Evaluate $\int\int_D (2x+y)^2 e^{x-y}\,dA$, where D is the region enclosed by 2x+y=1, 2x+y=4, x-y=-1, and x-y=1.

Solution:

To simplify the integration we seek to transform the region D in xy-plane into a new region D* in uv-plane. This can be done due to the symmetry of the region D which we plot in the following.

ContourPlot[{2x+y==1,2x+y==4,x-y==-1,x-y==1},{x,-1,3},
{y,-1,3},Frame->False,Axes->True,AxesLabel->{"x","y"},ContourStyle->
{Blue,Red,Green,Black},AspectRatio->1, PlotLegends->"Expressions"]

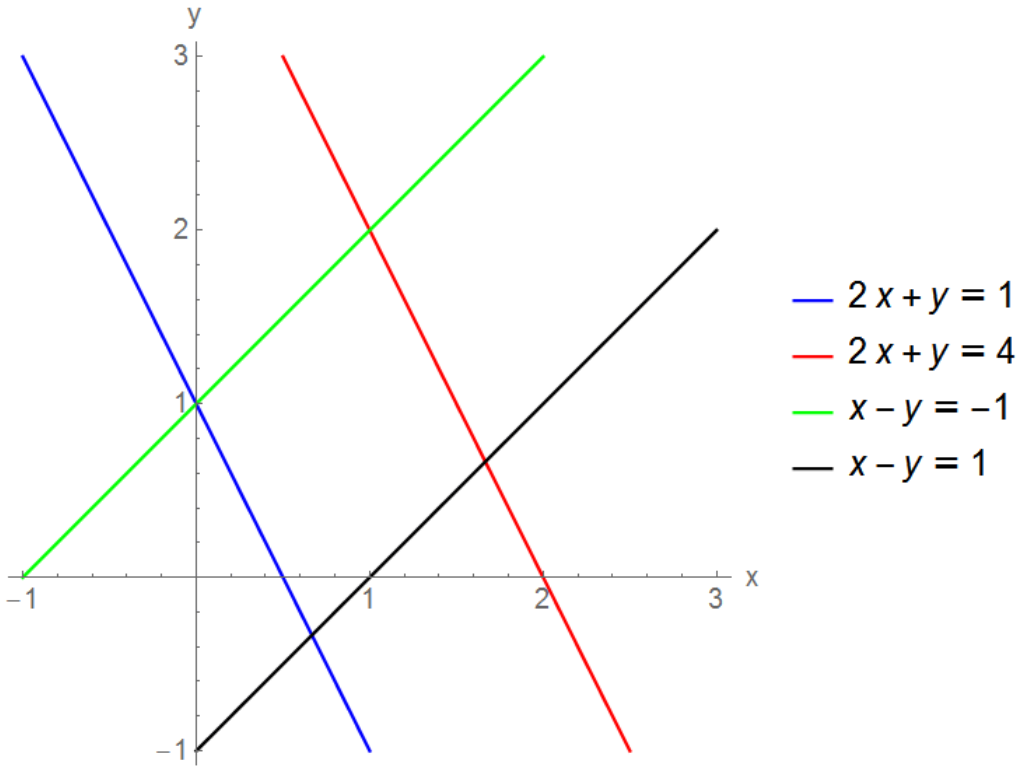


Figure 6.51. D is the region enclosed by 2x+y=1, 2x+y=4, x-y=-1, and x-y=1.

The region D is a parallelogram in the xy-plane. We want to see how the region transforms using a certain mapping that transforms it to another region in uv-plane. Let us make changes of variables by writing

$$u=2x+y, \quad v=x-y.$$

So,

$$x=1/3(u+v), \quad y=1/3(u-2v).$$

Solve[{u==2x+y,v==x-y}]

Let T be the the mapping according to Definition 6.7 that transforms the coordinates u, v to the another coordinates x, y such that

$$\mathbf{T}(u, v) = \mathbf{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(u+v) \\ \frac{1}{3}(u-2v) \end{pmatrix}$$

,

where A is the matrix representation of T.

Obviously, $\mathbf{A} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$.

Det[{{1/3,1/3},{1/3,-(2/3)}}]

Since det A=-(1/3)!=0, we know from linear algebra that matrix A is invertible. It can be checked that the inverse of A is

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$$

Inverse[{{1/3,1/3},{1/3,-(2/3)}}]//MatrixForm

It can be shown that T is one-one and onto that takes parallelograms to parallelograms and the vertices of parallelograms to the corresponding vertices.

The image of D* is D such that

D=T(D*).

The Jacobian of the transformation T can be found by equation (6.14):

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \left|-\frac{2}{9}-\frac{1}{9}\right| = \frac{1}{3}$$

.

Next, we find the vertices of the region D and then those of the new region D* using the transformation T.

The intersection point between the line 2 x+y=1 and x-y=-1 is (0,1).

Solve[{2x+y==1,x-y==-1}]

The intersection point between the line 2 x+y=4 and x-y=-1 is (1,2).

Solve[{2x+y==4,x-y==-1}]

The intersection point between the line 2 x+y=1 and x-y=1 is (2/3,-(1/3)).

Solve[{2x+y==1,x-y==1}]

The intersection point between the line 2 x+y=4 and x-y=1 is (5/3,2/3).

Solve[{2x+y==4,x-y==1}]

Thus, the vertices of the region D are

(0,1), (1,2), (2/3,-(1/3)), (5/3,2/3).

Using the transformation matrix A^-1:

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix},$$

the corresponding vertices in the new region D* can be found:

Vertices in D	Vertices in D*
x=0,y=1	u=1,v=-1
x=1,y=2	u=4,v=-1

x=2/3,y=-(1/3) u=1,v=1

x=5/3,y=(2/3) u=4,v=1

Thus, the description of D* is given by

D*={(u,v)|1<=u<=4,-1<=v<=1},

where the vertices are

(1,1),(1,-1),(4,1),(4,-1).

```
Module[{vertex,transform,plot},
vertex[1]={0,1};
vertex[2]={1,2};
vertex[3]={2/3,-(1/3)};
vertex[4]={5/3,2/3};
transform={{2,1},{1,-1}};
For[i=1,i<=4,i++,newVertex[i]=transform.vertex[i]];
plot[1]=Graphics[{Blue,PointSize[0.05],Point[Array[vertex,4]]}];
plot[2]=Graphics[{Green,Thickness[0.01],Line[{vertex[1],vertex[2],vertex[
4],vertex[3],vertex[1]}]}];
plot[3]=Show[{plot[1],plot[2]},Axes->True,AxesLabel->{"x","y"}];
plot[4]=Graphics[{Red,PointSize[0.05],Point[Array[newVertex,4]]}];
```

```
plot[5]=Graphics[{Green,Thickness[0.01],Line[{newVertex[1],newVertex[
2],newVertex[4],newVertex[3],newVertex[1]}]}];

plot[6]=Show[{plot[4],plot[5]},Axes->True,AxesLabel->
{"u","v"},AxesOrigin->{0,0}];

Show[GraphicsGrid[{{plot[3],plot[6]}},Spacings->{100,0}],ImageSize-
>500]

]
```

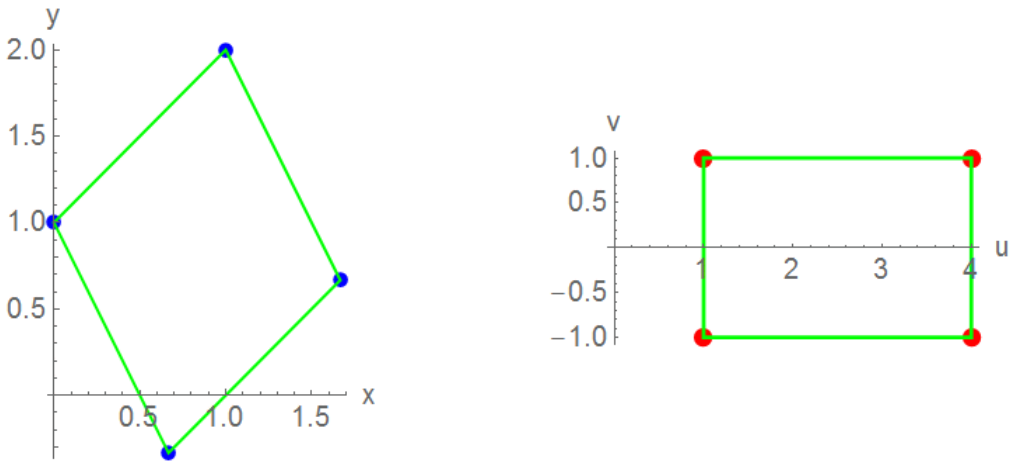


Figure 6.52. The region D in the xy-plane (left) and the transformed region D*={(u,v)|1<=u<=4,-1<=v<=1} in the uv-plane (right).

```
(* Vertices in D* *)

Array[newVertex,4]
```

Note that the region D* is geometrically simpler to integrate. Hence, the main reason for such transformation.

By Theorem 6.8 (Change of Variables in Double Integrals) equation (6.15), we have

$$\iint_D (2x+y)^2 e^{x-y}\, dA = \int_{u=1}^{4}\int_{v=-1}^{1} u^2 e^v \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$= \frac{1}{3}\int_{u=1}^{4}\int_{v=-1}^{1} u^2 e^v\, du\, dv$$

$$= \left(e - e^{-1}\right)\frac{1}{9}u^3\Big]_1^4$$

$$= 7\left(e - e^{-1}\right)$$

.

Integrate[1/3 u^2 E^v,{u,1,4},{v,-1,1}]

Example 6.25. Use Theorem 6.8 to evaluate:

$$\iint_D e^{-xy/2}\, dA$$,

where D is the region in the first quadrant bounded by the hyperbolas y=1/x, y=4/x and the lines y=1/4 x, y=2x.

Solution:

First, we plot the boundary curves:

ContourPlot[{y==1/x,y==4/x,y==1/4 x,y==2x},{x,0,5},{y,0,4},Frame->False,Axes->True,AxesLabel->{"x","y"},ContourStyle->{Blue,Red,Black,Green}, PlotLegends->"Expressions"]

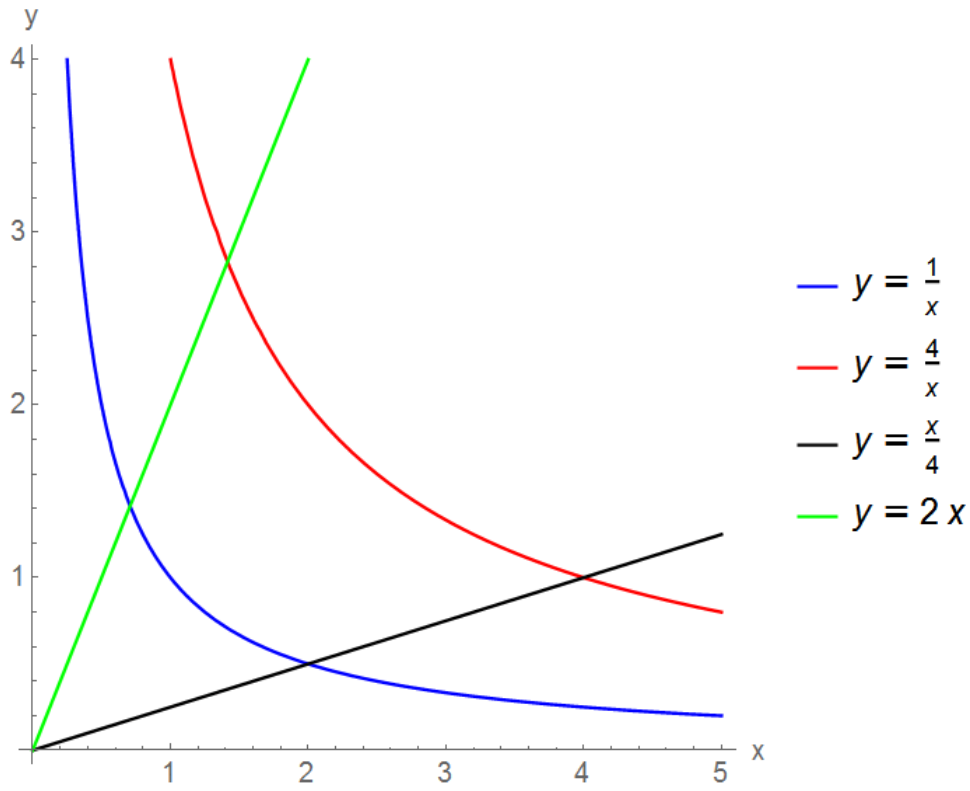


Figure 6.53. Region D is in the first quadrant bounded by the hyperbolas y=1/x, y=4/x and the lines y=1/4 x, y=2x.

Let u=y/x and v=x y.

Note that we choose these changes of variables so that the hyperbolas correspond to the lines v=1 and v=4, respectively; whereas the lines y=1/4 x and y=2x correspond to u=1/4 and u=2, respectively.

Thus, the region D* in the uv-plane can be described as

D*={(u,v)|1/4<=u<=2,1<=v<=4}

Solving for x and y in terms of u, v we get

x=Sqrt[v/u], y=Sqrt[u v],

by considering the region to be in the first quadrant.

```
Solve[{u==y/x,v==x y},{x,y}]
```

The Jacobian for this transformation by equation (6.14) is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det D\mathbf{T}(u,v) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det\begin{pmatrix} -\frac{v^{1/2}}{2u^{3/2}} & \frac{1}{2\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix} = -\frac{1}{2u}$$

.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2u}$$.

```
(* The derivative of T(u,v) *)

Outer[D,{Sqrt[v/u],Sqrt[u v]},{u,v}]//MatrixForm

(* The Jacobian of T(u,v) *)

Det[%]//PowerExpand
```

Note that Outer command is a nice tool to construct the derivative matrix of T as we have studied in Chapter 3.

By equation (6.15):

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

,

$$\iint_D e^{-xy/2}\, dA = \int_{u=1/4}^{2} \int_{v=1}^{4} \frac{1}{2u} e^{-v/2}\, du\, dv$$
$$= \left(e^{-1/2} - e^{-2}\right) \ln 8$$

.

Integrate[1/(2u) E^(-v/2),{u,1/4,2},{v,1,4}]

6.4 Change of Variables in Triple Integrals

In this section, we shall extend previous definition of the Jacobian of a transformation to the case of triple integrals. In particular we shall consider triple integrals in cylindrical coordinates and spherical coordinates.

Definition 6.9

Let T be a differentiable coordinate transformation from a uvw-space to the xyz-space such that

T:(u,v,w)->(x(u,v,w),y(u,v,w),z(u,v,w)),

where $(u, v, w), (x, y, z) \in \mathbb{R}^3$.

The Jacobian of the transformation T, denoted $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ **is the determinant of the derivative matrix DT(u,v,w), where**

$$D\mathbf{T}(u, v, w) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

.

Thus,

(6.17)

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det D\mathbf{T}(u, v, w) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

.

Note that the Jacobian of the transformation T is a function of u,v and w.

Omitting the details, the corresponding theorem is stated below:

Theorem 6.10 (Change of Variables in Triple Integrals)

Let T:W*->W be a differentiable transformation that is one-to-one on the interior of W*, where W* and W are elementary solid regions in the uvw-space and the xyz-space, respectively. If f(x,y,z) is continuous, then

(6.18)

$$\iiint_W f(x, y, z)\, dx\, dy\, dz =$$
$$\iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

6.4.1 Triple Integrals in Cylindrical Coordinates

There are many common solid regions which are difficult to integrate in rectangular coordinates, such as ellipsoids, cones, and paraboloids. When integrating over solid regions possessing an axis of rotational symmetry, cylindrical coordinates can be particularly helpful. The rectangular conversion equations for cylindrical coordinates are

$x=r\cos\theta$,

y=r sin θ,

z=z.

By Definition 6.9 the Jacobian for the cylindrical to rectangular coordinate transformation is given by

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det D\mathbf{T}(r,\theta,z) = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix}$$

$$= \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= r$$

Hence, equation (6.18) becomes

(6.19)

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

.

If f(x,y,z)=1, the triple integral is the volume of the solid region given by

(6.20)

$$V = \iiint_W dx\,dy\,dz = \iiint_{W^*} r\,dr\,d\theta\,dz$$

.

Example 6.26. Evaluate the following integral

$$\int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=\sqrt{x^2+y^2}}^{3}\frac{e^z}{\sqrt{x^2+y^2}}\,dz\,dy\,dx$$

by using cylindrical coordinates.

Solution:

First, we draw the solid region in space.

```
ContourPlot3D[{y==-Sqrt[9-x^2],y==Sqrt[9-
x^2],z==Sqrt[x^2+y^2],z==3}//Evaluate,{x,-3,3},{y,-3,3},
{z,0,4},AxesLabel->{"x","y","z"},Mesh->None,ContourStyle->
{{Yellow,Opacity[0.3]},{Yellow,Opacity[0.3]},Green,
{Blue,Opacity[0.3]}}]
```

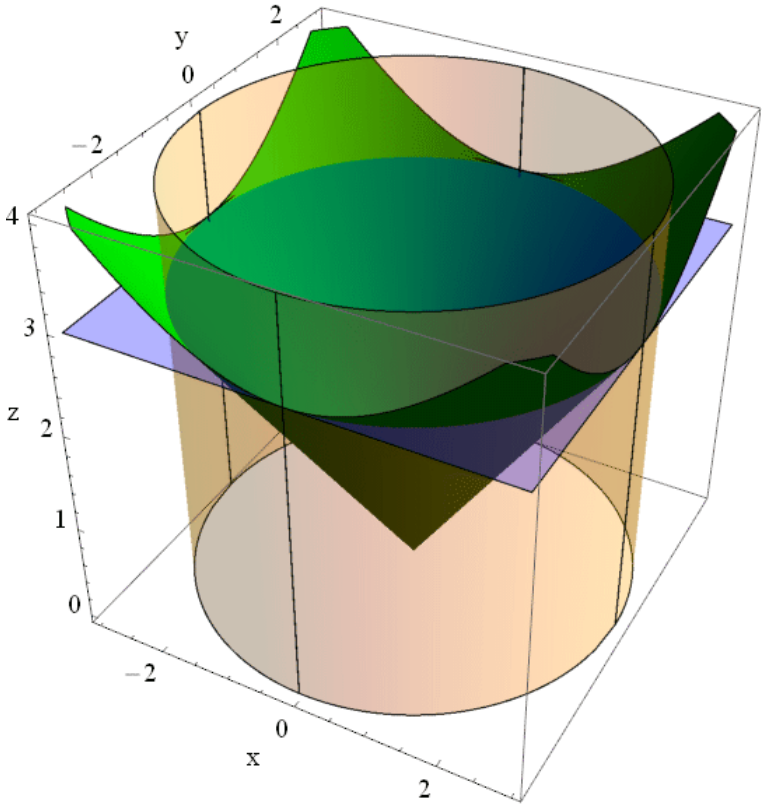


Figure 6.54. The solid region is a cone z=Sqrt[x^2+y^2] bounded by the cylinder x^2+y^2=9 and the plane z=3.

Note that the solid is a cone bounded below by the surface z=Sqrt[x^2+y^2] and above by the plane z=3. It is a elementary region of type 1.(a) by Definition 6.5:

W={(x,y,z)|-3<=x<=3,-Sqrt[9-x^2]<=y<=Sqrt[9-x^2],Sqrt[x^2+y^2] <=z<=3}.

It is rather complicated to describe the solid region and do the integration in rectangular coordinates. So, we seek to transform the integral to one in cylindrical coordinates.

The cylindrical-rectangular coordinate transformation is given by

$x=r\cos\theta$,

y=r sin θ,

z=z.

The Jacobian for this transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r$$

.

In cylindrical coordinates the cone z=Sqrt[x^2+y^2] is simply

z=r.

So, the solid region W in cylindrical coordinates can be described as

W={ (r,θ,z)|0<=r<=3,0<=θ<=2π,r<=z<=3 }

The following module view the solid region in cylindrical coordinates using the parameters in the order of integration z, r, θ as it is entered in Integrate command:

```
viewCylindrical[{θ_,α_,β_},{r_,f_,g_},
{z_,F_,G_},opts___]:=Module[{u,v,θθ,rr,surfs,pplot},

θθ=α+u*(β-α);

rr=f+v*(g-f)/.θ->θθ;

surfs={{r Cos[θ],r Sin[θ],F},{r Cos[θ],r Sin[θ],G}}/.{θ->θθ,r->rr};
```

```
pplot=ParametricPlot3D[Evaluate[surfs],{u,0,1},{v,0,1},PlotStyle->
{Blue,Red},AxesLabel->{"x","y","z"}];

Show[pplot,Table[Graphics3D[{GrayLevel[0.8],Line[surfs]}],{u,0,1,0.05},
{v,0,1,0.1}],opts]

]


viewCylindrical[{θ,0,2π},{r,0,3},{z,r,3}]
```

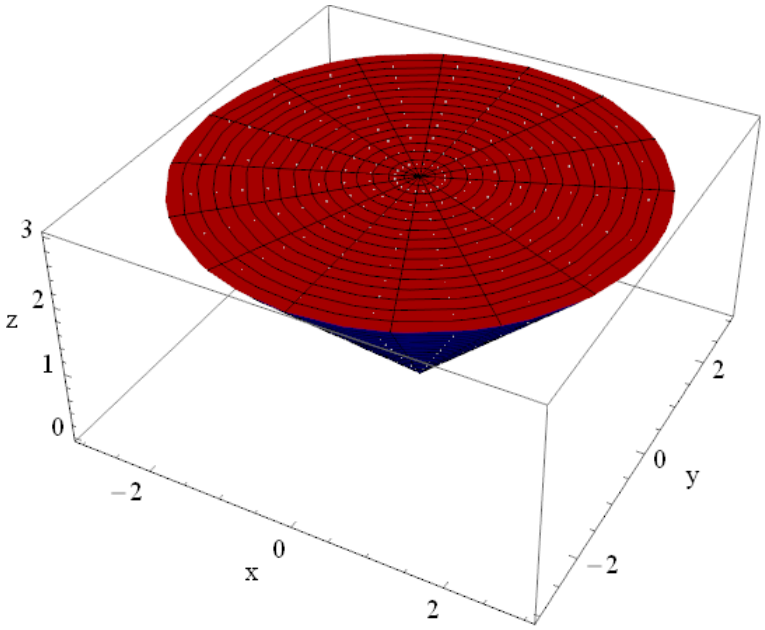


Figure 6.55. The solid region W={(r,θ,z)|0<=r<=3,0<=θ<=2π,r<=z<=3}.

Using equation (6.19):

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

we have

$$\int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{z=\sqrt{x^2+y^2}}^{3}\frac{e^z}{\sqrt{x^2+y^2}}\,dz\,dy\,dx$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=r}^{3}\frac{e^z}{r}\,r\,dz\,dr\,d\theta$$

$$=2\left(1+2e^3\right)\pi$$

Integrate[E^z,{θ,0,2π},{r,0,3},{z,r,3}]

%//N

Let us try computing the integral directly using rectangular coordinates:

Integrate[E^z/Sqrt[x^2+y^2],{x,-3,3},{y,-Sqrt[9-x^2],Sqrt[9-x^2]},
{z,Sqrt[x^2+y^2],3}]

Mathematica has no idea on how to integrate it directly using rectangular coordinates, so it does not return a correct answer. However, it can provide numerical solution using NIntegrate command as is demonstrated below:

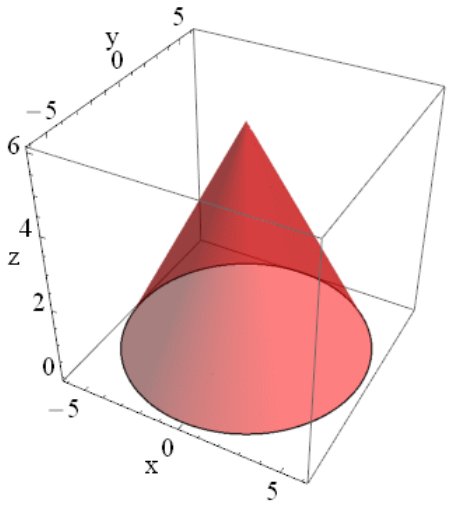
NIntegrate[E^z/Sqrt[x^2+y^2],{x,-3,3},{y,-Sqrt[9-x^2],Sqrt[9-x^2]},
{z,Sqrt[x^2+y^2],3}]

Example 6.27. Find the volume of the region under the cone z=6-r whose base is bounded by the cardioid r=3(1+cos θ).

Solution:

Let us first draw the cone z=6-r and the cardioid r=3(1+cos θ).

```
Module[{plot},

(* Plot the cone z=6-r

using ContourPlot3D in rectangular coordinates *)

plot[1]=ContourPlot3D[z==6-Sqrt[x^2+y^2]//Evaluate,{x,-6,6},{y,-6,6},
{z,0,6},BoxRatios->{1, 1, 1},ContourStyle->{Red,Opacity[0.5]},Mesh-
>None,AxesLabel->{"x","y","z"}];

(* The base of the region is bounded by the cardioid r=3(1+cosθ) *)

plot[2]=PolarPlot[3(1+Cos[θ]),{θ,0,2π},AxesLabel->{"x","y"},PlotStyle-
>AbsoluteThickness[2]];

Show[GraphicsGrid[{Array[plot,2]}],ImageSize->500]

]
```



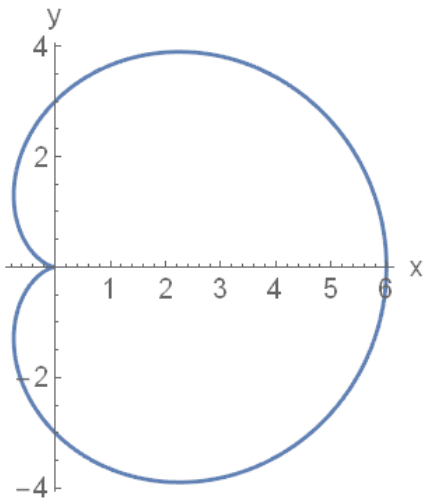


Figure 6.56. The cone z=6-r and the base region bounded by the cardioid r=3(1+cos θ).

Using Figure 6.56 we can describe the solid region as

W*={(r,θ,z)|0<=θ<=2π,0<=r<=3(1+cos θ),0<=z<=6-r}.

viewCylindrical[{θ,0,2π},{r,0,3(1+Cos[θ])},{z,0,6-r},ViewPoint->
{-3,-4,6}]

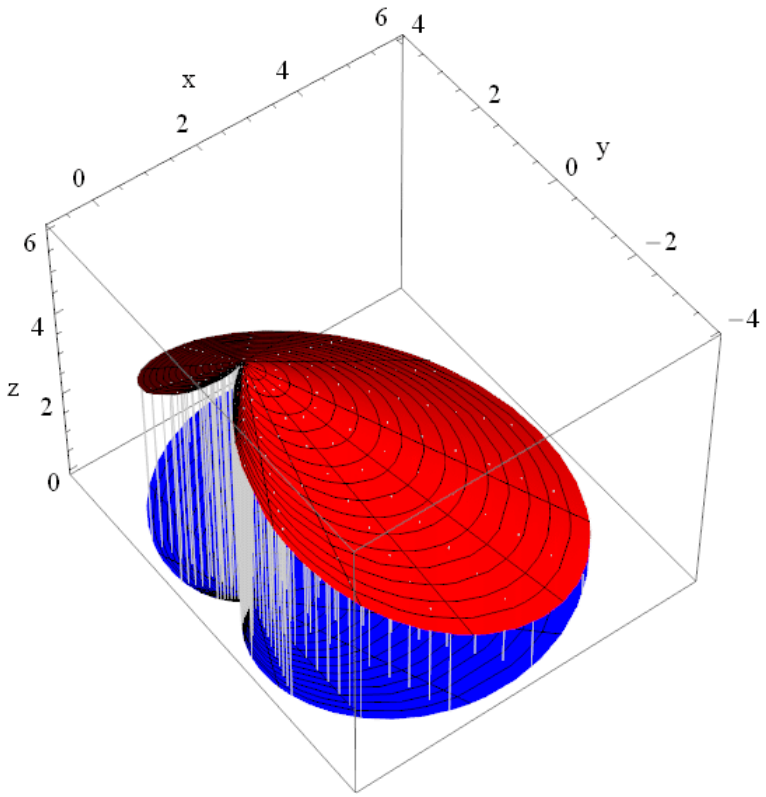


Figure 6.57. The solid region W*={(r,θ,z)|0<=θ<=2π,0<=r<=3(1+cos θ),0<=z<=6-r}.

The volume of the solid by (6.20) is

$$V = \iiint_{W^*} r\,dr\,d\theta\,dz = \int_{\theta=0}^{2\pi} \int_{r=0}^{3(1+\cos\theta)} \int_{z=0}^{6-r} r\,dz\,dr\,d\theta$$
$$= 36\pi$$

Integrate[r,{θ,0,2π},{r,0,3(1+Cos[θ])},{z,0,6-r}]

6.4.2 Triple Integrals in Spherical Coordinates

If a solid region is symmetric with respect to the origin, then the description using spherical coordinates are often most convenient. The rectangular conversion equations for spherical coordinates are given by equation (1.28):

x=r sin θ cos ϕ,

y=r sin θ sin ϕ,

z=r cos θ,

where r is the distance from a point P in space to the origin; θ is the angle between the positive z-axis and the ray through the origin and P; and ϕ is the angle between the positive x-axis and the ray made by dropping a perpendicular line from P to the xy-plane (see Section 1.8.3, Figure 1.21).

Note that r-coordinate and θ-coordinate have different meanings from those used in the cylindrical coordinates.

By Definition 6.9 the Jacobian for the spherical to rectangular coordinate transformation is given by

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \det DT(r, \theta, \phi) = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$= r^2 \sin\theta$$

Det[{{Sin[θ]Cos[ϕ],r Cos[θ]Cos[ϕ],-r Sin[θ]Sin[ϕ]},{Sin[θ]Sin[ϕ],r Cos[θ]Sin[ϕ],r Sin[θ]Cos[ϕ]},{Cos[θ],-r Sin[θ],0}}]//Simplify

Hence, the equation (6.18) becomes

(6.21)

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \iiint_{W^*} f(x(r, \theta, \phi), y(r, \theta, \phi), z(r, \theta, \phi))\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

If f(x,y,z)=1, the triple integral is the volume of the solid region given by

(6.22)

$$V = \iiint_W dx\, dy\, dz = \iiint_{W^*} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

.

Example 6.28. Evaluate

$$\iiint_W z^2\, dV$$

,

where W is the solid region lying above the cone z=Sqrt[3x^2+3y^2] and inside the sphere x^2+y^2+z^2=6z.

Solution:

Let us plot the cone z=Sqrt[3x^2+3y^2] and the sphere x^2+y^2+z^2=6z.

```
ContourPlot3D[{z==Sqrt[3x^2+3y^2],6z==x^2+y^2+z^2}//Evaluate,
{x,-3,3},{y,-3,3},{z,0,6},Mesh->None,ContourStyle->
{{Yellow,Opacity[0.8]},{Blue,Opacity[0.3]}},PlotLegends-
>"Expressions"]
```

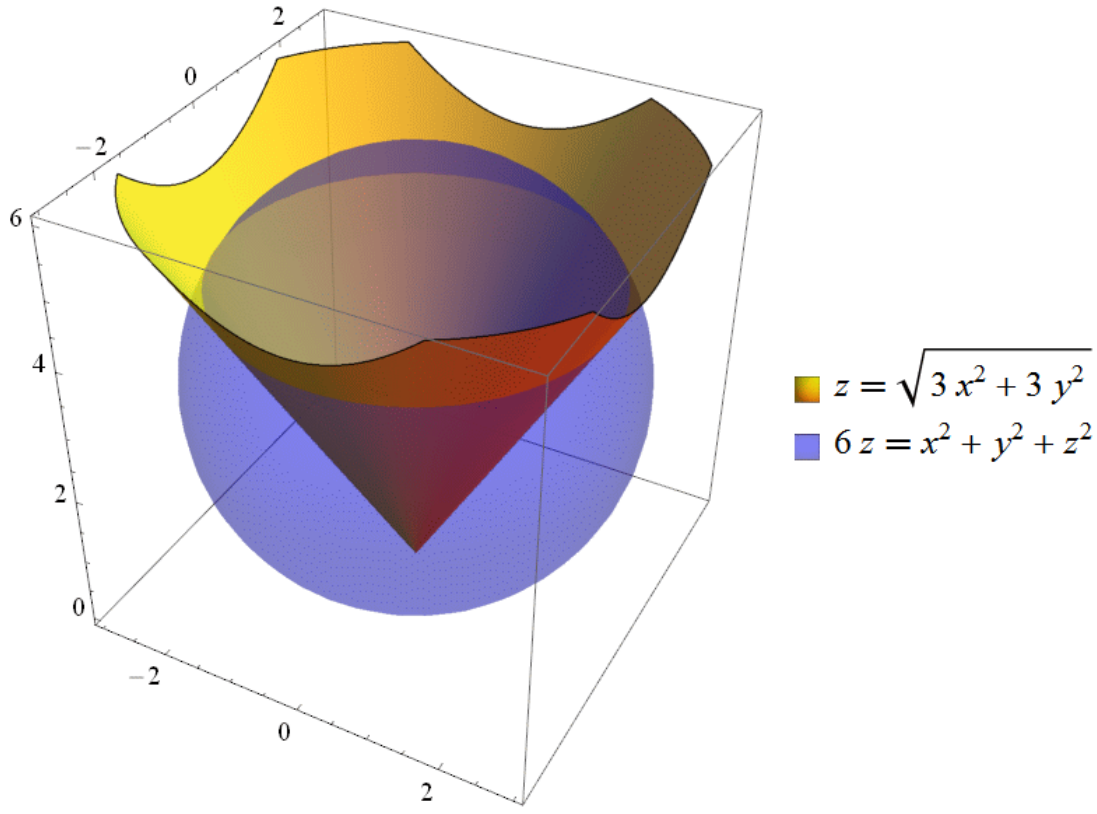


Figure 6.58. W is the solid region lying above the cone z=Sqrt[3x^2+3y^2] and inside the sphere x^2+y^2+z^2=6z.

In spherical coordinates, the equation of the cone z=Sqrt[3x^2+3y^2] becomes

r cos θ=Sqrt[3(x^2+y^2)]

=Sqrt[3(r^2 sin^2 θ cos^2 ϕ+r^2 sin^2 θ sin^2 ϕ)]

=Sqrt[3r^2 sin^2 θ]

r cos θ=Sqrt[3]r sin θ, tan θ=1/Sqrt[3].

θ=tan^-1 1/Sqrt[3]=π/6.

and the sphere x^2+y^2+z^2=6z becomes

r^2=6r cos θ

r=6cos θ.

Thus, the solid region is described in spherical coordinates by

W={ (r,θ,ϕ) |0<=r<=6cos θ,0<=θ<=π/6,0<=ϕ<=2π }.

The following module can be used to graph the solid region in spherical coordinates. The parameters passed to it are in the order of integration r, θ, ϕ as it is entered in Integrate command.

```
viewSpherical[{ϕ_,a_,b_},{θ_,f_,g_},{r_,F_,G_},opts___]:=

Module[{twosurfaces,xp,yp,zp},

xp[r1_]:=r1 Sin[θ]Cos[ϕ];

yp[r1_]:=r1 Sin[θ] Sin[ϕ];

zp[r1_]:=r1 Cos[θ];

twosurfaces=SphericalPlot3D[{F,G},{θ,f,g},{ϕ,a,b},Mesh->None,PlotStyle->{{Blue,Opacity[0.5]},{Red,Opacity[0.5]}},AxesLabel->{"x","y","z"}];

Show[twosurfaces,Table[Graphics3D[{GrayLevel[0.8],Line[{{xp[F],yp[F],zp[F]},{xp[G],yp[G],zp[G]}}]}],{ϕ,a,b,(b-a)/10},{θ,f,g,(g-f)/10}],opts]

]



viewSpherical[{ϕ,0,2π},{θ,0,π/6},{r,0,6 Cos[θ]},PlotRange->All]
```

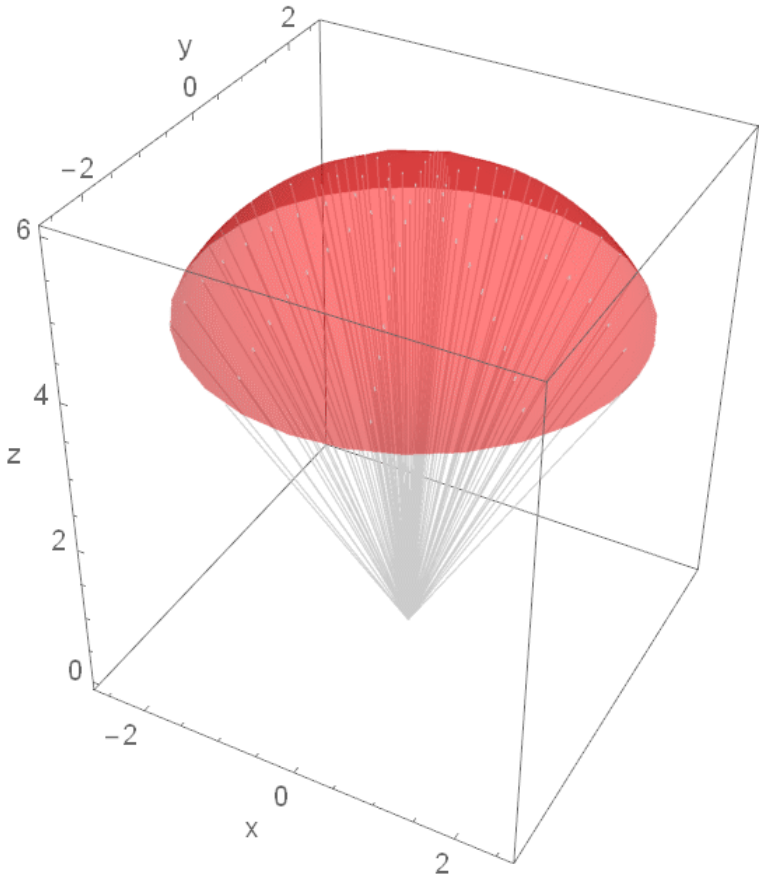


Figure 6.59. The solid region W={(r,θ,ϕ) |0<=r<=6cos θ,0<=θ<=π/6,0<=ϕ<=2π}.

So, the triple integral in spherical coordinates is given by

$$\iiint_W z^2 \, dV = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/6} \int_{r=0}^{6\cos\theta} (r\cos\theta)^2 \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/6} \int_{r=0}^{6\cos\theta} r^4 \cos^2\theta \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{8505\,\pi}{32}$$

Integrate[r^4 Cos[θ]^2 Sin[θ],{ϕ,0,2π},{θ,0,π/6},{r,0,6 Cos[θ]}]

Example 6.29. Evaluate

$$\iiint_W (2 + x^2 + y^2)\, dV,$$

where W is the region inside the sphere x^2+y^2+z^2=25 and above the plane z=3.

Solution:

```
ContourPlot3D[{x^2+y^2+z^2==25,z==3}//Evaluate,{x,-5,5},{y,-5,5},
{z,-5,5},Mesh->None,ContourStyle->{Automatic,{Green,Opacity[0.7]}}]
```

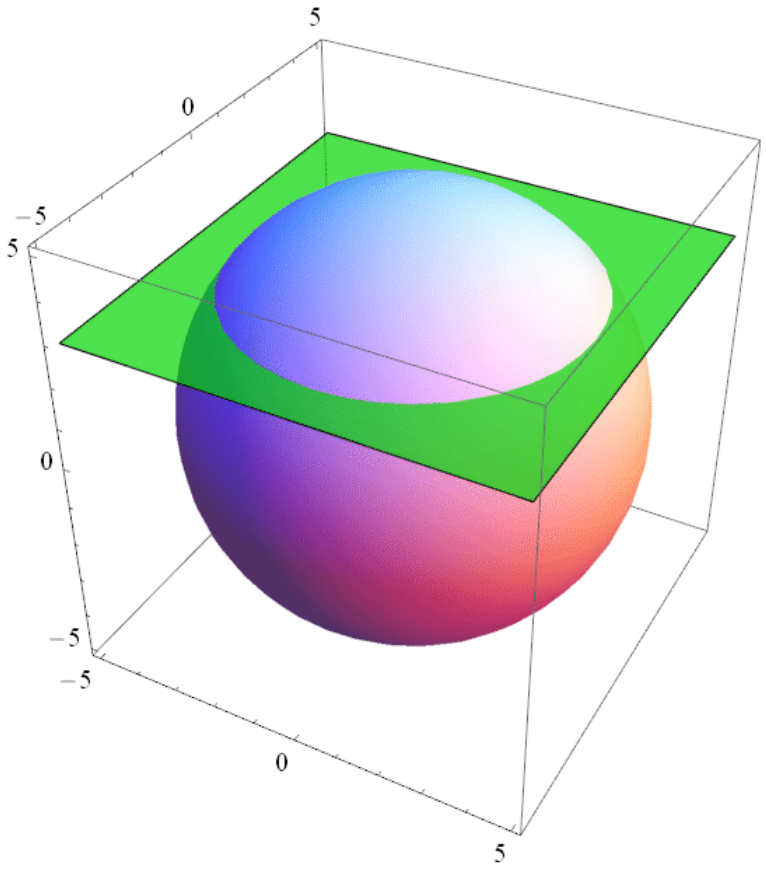


Figure 6.60. The solid region is inside the sphere x^2+y^2+z^2=25 and above the plane z=3.

The equation of the sphere x^2+y^2+z^2=25 in spherical coordinates is simply

r=5,

and the plane z=3 in spherical coordinates is

z=3, r cos θ=3.

r=3/(cos θ)=3 sec θ.

The sphere and the plane intersect when

5 cos θ=3.

Thus, the sphere and the plane intersect at θ=cos^-1 3/5.

The solid region is bounded below by the plane r=3 sec θ and above by the sphere of radius 5, so the description in spherical coordinates is given by

W={(r,θ,ϕ)|3 sec θ<=r<=5,0<=θ<=cos^-1 3/5,0<=ϕ<=2π}.

viewSpherical[{ϕ,0,2π},{θ,0,ArcCos[3/5]},{r,3Sec[θ],5},PlotRange->
{{-5,5},{-5,5},{-5,5}},ViewPoint->{6,-6,-1}]

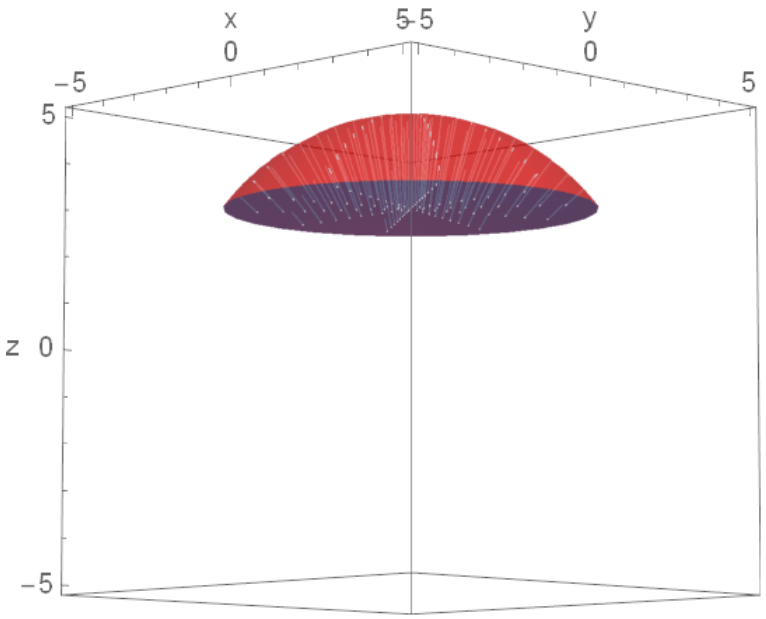


Figure 6.61. The solid region W={(r,θ,ϕ)|3 sec θ<=r<=5,0<=θ<=cos^-1 3/5,0<=ϕ<=2π}.

So, the triple integral in spherical coordinates is given by

$$\iiint_W \left(2 + x^2 + y^2\right) dV$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\cos^{-1} 3/5} \int_{r=3 \sec\theta}^{5} \left(2 + r^2 \sin^2\theta\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{656\pi}{5}$$

.

Integrate[(2+r^2 Sin[θ]^2)r^2 Sin[θ],{ϕ,0,2π},{θ,0,ArcCos[3/5]},
{r,3Sec[θ],5}]

Chapter 7. Line Integrals and Green's Theorem

Initialization

```
Clear["Global`*"]

(* The norm of vector v *)
L[v_]:=PowerExpand[Simplify[Sqrt[v.v]]]

(* The arc length of the curve α(t) from t=a to t=b *)
arclength[a_,b_][α_]:=Integrate[L[α'[tt]],{tt,a,b},
GenerateConditions->False]

SetOptions[Plot3D,PlotTheme->"Classic"];
SetOptions[ContourPlot3D,PlotTheme->"Classic"];
SetOptions[Plot,PlotTheme->"Classic",PlotStyle->AbsoluteThickness[2]];
SetOptions[ParametricPlot3D,PlotTheme->"Classic",PlotStyle-
>AbsoluteThickness[2]];
```

7.1 Line Integrals

In this section we are going to study some constructions that have found considerable utility in science and engineering. These are the scalar and vector line integrals, namely the operation of integrating scalar and vector field along a curve, respectively. The curve may be in plane or in space. Some textbooks refer to such integrals as "curve" or "path" integrals.

7.1.1 Scalar Line Integrals

Definition 7.1. Scalar Line Integral

Let c(t) be a parametrization of the curve C for a<=t<=b. If c'(t) is continuous and f(x,y,z) is a continuous scalar function whose domain includes C, then the scalar line integral of f along the path c is

(7.1)

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{c}(t)) \,\|\, \mathbf{c}'(t) \,\|\, dt$$

.

ds is referred to as the line element or arc length differential of C.

Note that we implicitly assume that the curve is three-dimensional, however, the definition is valid for two-dimensional plane as well. To evaluate the scalar line integral using Mathematica we provide two modules for this purpose: scalarLineInt2D and scalarLineInt3D corresponding to the two-dimensional case and the three-dimensional case, respectively.

```
(* Scalar line integral as defined in Definition 7.1 *)
(* for two-dimensional case:
c is given parametrically by (x(t),y(t))
*)
scalarLineInt2D[c_,f_,{a_,b_}]:=
Module[{x,y,v},
x=c[t][[1]];
y=c[t][[2]];
v=L[c'[t]];
Integrate[f[x,y]*v,{t,a,b}]
]

(* Scalar line integral as defined in Definition 7.1 *)
(* for three-dimensional case:
c is given parametrically by (x(t),y(t),z(t))
*)
scalarLineInt3D[c_,f_,{a_,b_}]:=
Module[{x,y,z,v},
x=c[t][[1]];
```

y=c[t][[2]];

z=c[t][[3]];

v=L[c'[t]];

Integrate[f[x,y,z]*v,{t,a,b}]

]

Example 7.1. Evaluate the scalar line integral of f(x,y,z)=z/(x^2 + y^2) along the path

c(t)=(E^t cos 2t,E^t sin 2t,E^t), 0<=t<=5.

Solution:

Step 1. Compute ds.

c'(t)=(E^t cos 2 t-2 E^t sin 2 t,E^t sin 2 t+2 E^t cos 2 t,E^t),

||c'(t)||=Sqrt[6] E^t,

$$ds = \| \mathbf{c}'(t) \| \, dt = \sqrt{6}\, e^t \, dt$$

.

Clear[c,f]

c[t_]:={E^t Cos[2t],E^t Sin[2t],E^t}

f[x_,y_,z_]:=z/(x^2+y^2)

c'[t]

L[c'[t]]

Step 2. Write out the integrand and evaluate

We have f(x,y,z)=z/(x^2 + y^2), so

f(c(t))=f(E^t cos 2t,E^t sin 2t,E^t)=E^t/E^(2t).

By equation (7.1):

$$\int_C f(x, y, z)\, ds = \int_0^5 f(\mathbf{c}(t)) \,\|\, \mathbf{c}'(t) \,\|\, dt$$
$$= \int_0^5 \frac{e^t}{e^{2t}} \sqrt{6}\, e^t\, dt$$
$$= \int_0^5 \sqrt{6}\, dt = 5\sqrt{6}$$

We may verify this using scalarLineInt3D module:

scalarLineInt3D[c,f,{0,5}]

Example 7.2. Let c:[0,1]-> $\mathbb{R}^2$ be the wire c(t)=(t^2,2t) in the plane with mass density ρ(x,y)=3/4 y. Find the total mass of the wire.

Solution:

Step 1. Compute ds.

c'(t)=(2t,2),

||c'(t)||=Sqrt[4t^2+4]=2Sqrt[1+t^2],

$$ds = \| \mathbf{c}'(t) \| \, dt = 2\sqrt{1+t^2}\, dt$$

.

Step 2. Write out the integrand and evaluate

We have the density ρ(x,y)=3/4 y, so

ρ(c(t))=ρ(t^2,2t)=3/4 (2t)=3/2 t.

By equation (7.1):

$$\text{Total mass} = \int_C \rho(x, y, z)\, ds = \int_a^b \rho(\mathbf{c}(t)) \| \mathbf{c}'(t) \| \, dt$$
$$= \int_0^1 3t\sqrt{1+t^2}\, dt$$
$$= 2\sqrt{2} - 1$$

Integrate[3t Sqrt[1+t^2],{t,0,1}]

Clear[c,ρ]

c[t_]:={t^2,2t}

ρ[x_,y_]:=3/4 y

scalarLineInt2D[c,ρ,{0,1}]

If f=1, then equation (7.1) reduces to

(7.2) $$\int_C ds = \int_a^b \| \mathbf{c}'(t) \| dt$$.

Equation (7.2) is the arc length of the curve C from t=a to t=b.

Example 7.3. Find the arc length of the curve in Example 7.1:

c(t)=(E^t cos 2t,E^t sin 2t,E^t), 0<=t<=5.

Solution:

By equation (7.2), the arc length of the curve is

$$\int_C ds = \int_0^5 \| \mathbf{c}'(t) \| dt$$
$$= \int_0^5 \sqrt{6}\, e^t dt = \sqrt{6}\,(-1 + e^5)$$

Integrate[Sqrt[6] E^t,{t,0,5}]

Clear[c,f]

c[t_]:={E^t Cos[2t],E^t Sin[2t],E^t}

f[x_,y_,z_]:=1

scalarLineInt3D[c,f,{0,5}]

Or using arclength module we get the same result:

arclength[0,5][c]

ParametricPlot3D[c[t],{t,0,5}]

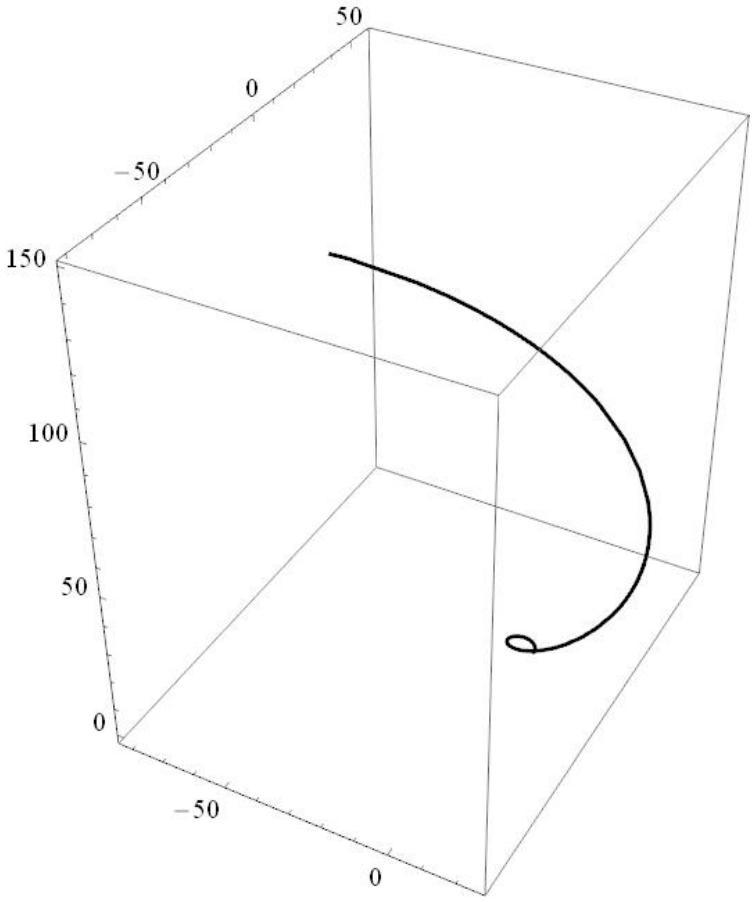


Figure 7.1. The curve c(t)=(E^t cos 2t,E^t sin 2t,E^t).

Suppose we wish to find the average value of a continuous function f along a curve C of finite length. Then we have the following definition:

Definition 7.2. Average value of a continuous function f along a curve C

Let C be the curve that has continuous derivative. If a scalar function f whose domain includes C is continuous, then the quantity

$$[f]_{\text{avg}} = \frac{\int_C f\, ds}{\int_C ds} = \frac{\int_C f\, ds}{\text{length of } C}$$

is the average value of f along C.

Example 7.4. Find the average y-coordinate of points on the path described in Figure 7.2.

Solution:

```
origin={0,0,0};

point[1]={2,0,0};

point[2]={2,1,0};

point[3]={0,1,1};

point[4]={1,0,2};

plotPoints=Graphics3D[{PointSize[0.02],Red,Point[origin],Black,Point[Arr
ay[point,4]]},Boxed->True,Axes->True,AxesLabel->{"x","y","z"}];

plotArrows=Graphics3D[{Blue,Arrow[{origin,point[1]}],Arrow[{point[1],p
oint[2]}],Arrow[{point[2],point[3]}],Arrow[{point[3],point[4]}]}];

plotTexts=Graphics3D[{Text["(2,0,0)",{2,0,0.3}],Text["(2,1,0)",
{2,1.3,0.4}],Text["(2,0,0)",{2,0,0.3}],Text["(0,1,1)",{0,1,1.3}],Text["
(1,0,2)",{1.3,0,1.9}]}];
```

Show[{plotPoints,plotArrows,plotTexts},PlotRange->{{-2,2},{-2,2},{-2,2}},ViewPoint->{2,2,1.5}]

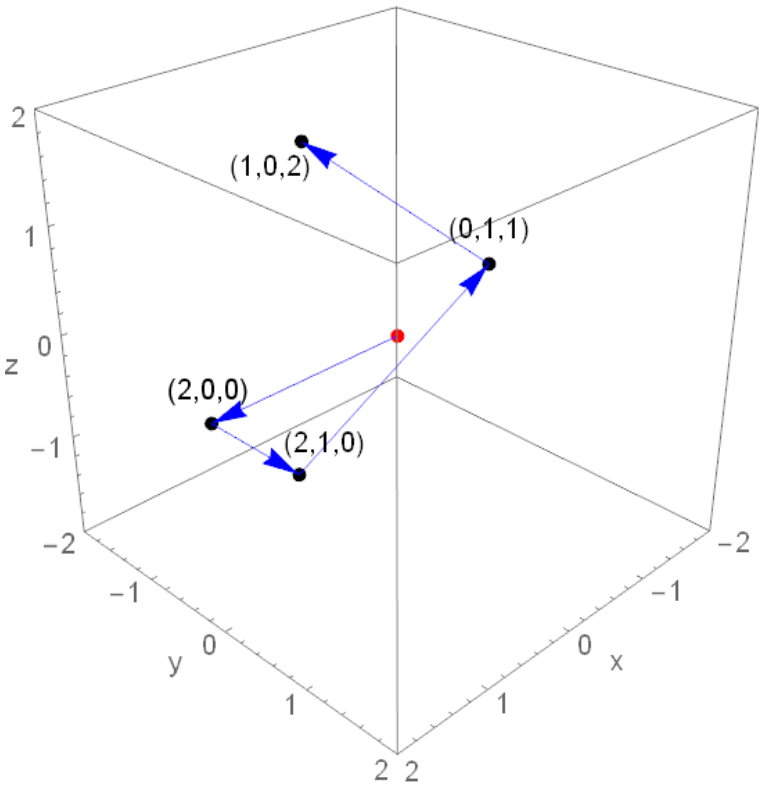


Figure 7.2. The path of Example 7.4.

In this case, the average y-coordinate of points is defined by

$$[y]_{\text{avg}} = \frac{\int_C y\, ds}{\int_C ds} = \frac{\int_C y\, ds}{\text{length of } C}.$$

The length of C can be computed from Figure 7.2:

$$\text{length of } C = 2 + 1 + \sqrt{(0-2)^2 + (1-1)^2 + (1-0)^2} + \sqrt{(1-0)^2 + (0-1)^2 + (2-1)^2}$$
$$= 3 + \sqrt{4+1} + \sqrt{1+1+1} = 3 + \sqrt{5} + \sqrt{3}$$

The scalar line integral consists of 4 line segments:

$$\int_C y\,ds = \int_{C_1} y\,ds + \int_{C_2} y\,ds + \int_{C_3} y\,ds + \int_{C_4} y\,ds$$

,

where

C1 is the line segment from (0,0,0) to (2,0,0),

C2 is the line segment from (2,0,0) to (2,1,0),

C3 is the line segment from (2,1,0) to (0,1,1),

C4 is the line segment from (0,1,1) to (1,0,2).

The parametric equations for C1, C2, C3 and C4 (respectively) are

C1: c1(t)=[(2,0,0)-(0,0,0)]t+(0,0,0)

=(2,0,0)t

=(2t,0,0), 0<=t<=1.

C2: c2(t)=[(2,1,0)-(2,0,0)]t+(2,0,0)

=(0,1,0)t+(2,0,0)

=(2,t,0), 0<=t<=1.

C3: c3(t)=[(0,1,1)-(2,1,0)]t+(2,1,0)

=(-2,0,1)t+(2,1,0)

=(-2t+2,1,t), 0<=t<=1.

C4: c4(t)=[(1,0,2)-(0,1,1)]t+(0,1,1)

=(1,-1,1)t+(0,1,1)

=(t,-t+1,t+1), 0<=t<=1.

The line integral becomes

$$\begin{aligned}
\int_C y\,ds &= \int_{C_1} y\,ds + \int_{C_2} y\,ds + \int_{C_3} y\,ds + \int_{C_4} y\,ds \\
&= \int_{t=0}^{1} 0 \cdot \|\mathbf{c}_1'(t)\|\,ds + \int_{t=0}^{1} t\,\|\mathbf{c}_2'(t)\|\,ds \\
&\quad + \int_{t=0}^{1} 1 \cdot \|\mathbf{c}_3'(t)\|\,dt + \int_{t=0}^{1} (-t+1)\,\|\mathbf{c}_4'(t)\|\,dt \\
&= 0 + \int_{t=0}^{1} t\,\|(0,\,1,\,0)\|\,dt \\
&\quad + \int_{t=0}^{1} \|(-2,\,0,\,1)\|\,dt + \int_{t=0}^{1} (-t+1)\,\|(1,\,-1,\,1)\|\,dt \\
&= \int_{t=0}^{1} t\,dt + \sqrt{5}\int_{t=0}^{1} dt + \sqrt{3}\int_{t=0}^{1} (-t+1)\,dt \\
&= \left[\frac{1}{2}t^2\right]_0^1 + \sqrt{5} + \sqrt{3}\left[-\frac{1}{2}t^2 + t\right]_0^1 \\
&= \frac{1}{2} + \sqrt{5} + \frac{1}{2}\sqrt{3}
\end{aligned}$$

So,

$$[y]_{avg} = \frac{\int_C y\,ds}{\int_C ds} = \frac{\int_C y\,ds}{\text{length of } C} = \frac{\frac{1}{2}+\sqrt{5}+\frac{1}{2}\sqrt{3}}{3+\sqrt{5}+\sqrt{3}}$$

$$\simeq 0.51694$$

```
Clear[c1,c2,c3,c4,f]

c1[t_]:={2t,0,0}

c2[t_]:={2,t,0}

c3[t_]:={-2t+2,1,t}

c4[t_]:={t,-t+1,t+1}

f[x_,y_,z_]:=y


1/(3+Sqrt[5]+Sqrt[3])*(scalarLineInt3D[c1,f,{0,1}]+scalarLineInt3D[c2,f,
{0,1}]+scalarLineInt3D[c3,f,{0,1}]+scalarLineInt3D[c4,f,{0,1}])//N
```

Example 7.5. Suppose C is the curve obtained by intersection the elliptic cylinder x^2/5+y^2=1 by the plane z=2y. Find the average value of f(x,y,z)=y E^x+z^4 on C.

Solution:

The parametric equation of the elliptic cylinder x^2/5+y^2=1 is

t-> (Sqrt[5]cos t,sin t).

The curve C obtained by intersecting the elliptic cylinder x^2/5+y^2=1 and the plane z=2y may be parametrized as

C: c(t)=(Sqrt[5]cos t,sin t,2sin t), 0<=t<=2π.

```
Clear[c,f]

c[t_]:={Sqrt[5]Cos[t],Sin[t],2Sin[t]}

f[x_,y_,z_]:=y E^x+z^4


plot[1]=ContourPlot3D[{x^2/5+y^2==1,z==2y},{x,-3,3},{y,-3,3},
{z,-3,3},Mesh->None,ContourStyle->{{Blue,Opacity[0.2]},
{Green,Opacity[0.4]}}];

plot[2]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{AbsoluteThickness[2],Dashed,Red}];Show[Array[plot,2]]
```

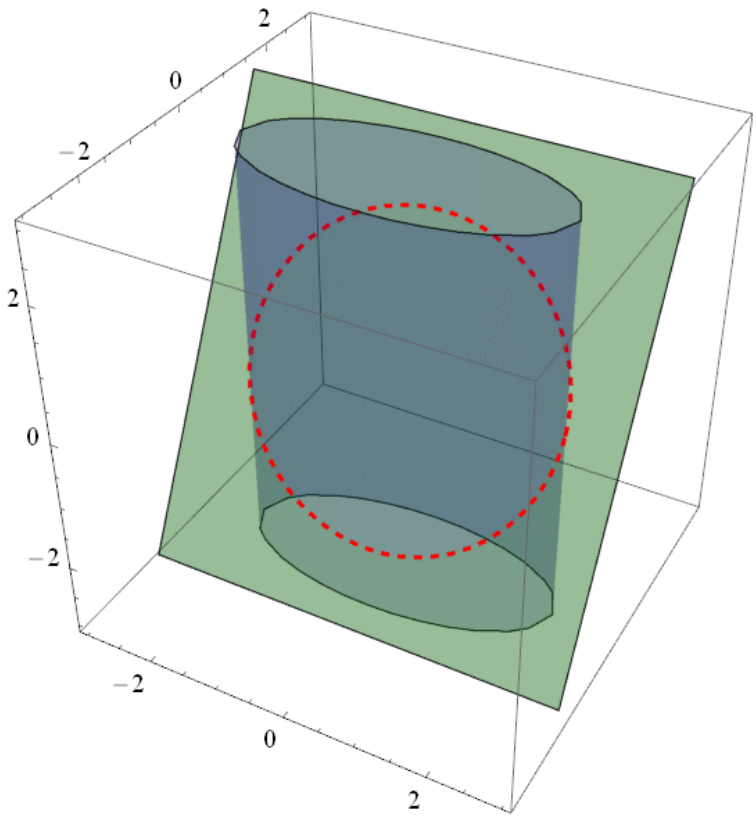


Figure 7.3. The curve C is the intersection the elliptic cylinder x^2/5+y^2=1 by the plane z=2y.

The length of C:

$$\int_C ds = \int_0^{2\pi} \| \mathbf{c}'(t) \| dt$$
$$= \int_0^{2\pi} \| -\sqrt{5} \sin t, \cos t, 2\cos t \| dt$$
$$= 2\sqrt{5}\,\pi$$
$$\int_C f\, ds = \int_0^{2\pi} f(\mathbf{c}(t)) \| \mathbf{c}'(t) \| dt$$
$$= \int_0^{2\pi} \left(\sin t\, e^{\sqrt{5}\cos t} + 16 \sin^4 t \right) \| -\sqrt{5} \sin t, \cos t, 2\cos t \| dt$$
$$= \int_0^{2\pi} \sqrt{5} \left(\sin t\, e^{\sqrt{5}\cos t} + 16 \sin^4 t \right) dt$$
$$= 12\sqrt{5}\,\pi$$

Integrate[Sqrt[5](Sin[t]E^(Sqrt[5]Cos[t])+16 Sin[t]^4),{t,0,2π}]

The average value of f(x,y,z)=y E^x+z^4 on the curve C:

$$[f]_{\text{avg}} = \frac{\int_C f\, ds}{\int_C ds} = \frac{\int_C f\, ds}{\text{length of } C} = \frac{12\sqrt{5}\,\pi}{2\sqrt{5}\,\pi} = 6$$

.

scalarLineInt3D[c,f,{0,2π}]

arclength[0,2π][c]

Definition 7.3. Continuous center of mass in $\mathbb{R}^3$

Let C be the curve that has continuous derivative. Suppose that a wire is bent in the shape of C and the mass density at each point (x,y,z) of the wire is δ(x,y,z). Then the center of mass of the wire is defined by

$$\bar{\mathbf{r}} = (\bar{x}, \bar{y}, \bar{z}) ,$$

where

$$\begin{cases} \bar{x} = \dfrac{\int_C x\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds}, \\ \bar{y} = \dfrac{\int_C y\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds}, \\ \bar{z} = \dfrac{\int_C z\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds}. \end{cases}$$

The integrals: $\int_C x\,\delta(x,y,z)\,ds$, $\int_C y\,\delta(x,y,z)\,ds$ **and** $\int_C z\,\delta(x,y,z)\,ds$ **are called the first moments of the wire about the x-, y- and z-axis, respectively. The coordinates of the center of mass** $\bar{\mathbf{r}}$ **is also called the centroid of the wire.**

Though Definition 7.3 is defined for continuous center of mass in $\mathbb{R}^3$, it is valid for the case in $\mathbb{R}^2$ as well.

Example 7.6. Find the center of mass of a spring in the shape of the circular helix:

c(t)=1/Sqrt[2] (cos t,sin t,t),

where 0<=t<=6π and the density of the spring is δ(x,y,z)=1+z as shown in Figure 7.4.

Solution:

The center of mass of the spring written as $(\bar{x}, \bar{y}, \bar{z})$ are given by

$$\bar{x} = \frac{\int_C x\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds}, \quad \bar{y} = \frac{\int_C y\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds}, \quad \bar{z} = \frac{\int_C z\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds}.$$

First we compute the total mass:

$$\begin{aligned}
\text{total mass} &= \int_C \delta(x,\ y,\ z)\,ds \\
&= \int_0^{6\pi} \delta(\mathbf{c}(t)) \,\|\mathbf{c}'(t)\|\,dt \\
&= \frac{1}{\sqrt{2}} \int_0^{6\pi} \left(1 + \frac{1}{\sqrt{2}}\,t\right) \|(-\sin t,\ \cos t,\ 1)\|\,dt \\
&= \int_0^{6\pi} \left(1 + \frac{1}{\sqrt{2}}\,t\right) dt \\
&= 6\pi + 9\sqrt{2}\,\pi^2
\end{aligned}$$

Next, we compute the first moments:

$$\begin{aligned}
&\int_C x\,\delta(x,\ y,\ z)\,ds \\
&\quad = \int_0^{6\pi} \frac{1}{\sqrt{2}} \cos t\,\delta(\mathbf{c}(t)) \,\|\mathbf{c}'(t)\|\,dt \\
&\quad = \int_0^{6\pi} \frac{1}{\sqrt{2}} \cos t \left(1 + \frac{1}{\sqrt{2}}\,t\right) \|\frac{1}{\sqrt{2}}(-\sin t,\ \cos t,\ 1)\|\,dt \\
&\quad = \int_0^{6\pi} \frac{1}{\sqrt{2}} \cos t \left(1 + \frac{1}{\sqrt{2}}\,t\right) dt \\
&\quad = 0
\end{aligned}$$

(* First moment about the x-axis *)

Integrate[1/Sqrt[2] Cos[t](1+1/Sqrt[2] t),{t,0,6π}]

$$\int_C y\,\delta(x,\ y,\ z)\,ds$$

$$= \int_0^{6\pi} \frac{1}{\sqrt{2}} \sin t\ \delta(\mathbf{c}(t))\,\|\,\mathbf{c}'(t)\,\|\,dt$$

$$= \int_0^{6\pi} \frac{1}{\sqrt{2}} \sin t \left(1 + \frac{1}{\sqrt{2}}\,t\right) \|\, \frac{1}{\sqrt{2}}\,(-\sin t,\ \cos t,\ 1)\,\|\,dt$$

$$= \int_0^{6\pi} \frac{1}{\sqrt{2}} \sin t \left(1 + \frac{1}{\sqrt{2}}\,t\right) dt$$

$$= -3\pi$$

(* First moment about the y-axis *)

Integrate[1/Sqrt[2] Sin[t](1+1/Sqrt[2] t),{t,0,6π}]

$$\int_C z\,\delta(x,\ y,\ z)\,ds$$

$$= \int_0^{6\pi} \frac{1}{\sqrt{2}}\,t\ \delta(\mathbf{c}(t))\,\|\,\mathbf{c}'(t)\,\|\,dt$$

$$= \int_0^{6\pi} \frac{1}{\sqrt{2}}\,t \left(1 + \frac{1}{\sqrt{2}}\,t\right) \|\, \frac{1}{\sqrt{2}}\,(-\sin t,\ \cos t,\ 1)\,\|\,dt$$

$$= \int_0^{6\pi} \frac{1}{\sqrt{2}}\,t \left(1 + \frac{1}{\sqrt{2}}\,t\right) dt$$

$$= 9\sqrt{2}\,\pi^2 + 36\,\pi^3$$

(* First moment about the z-axiz *)

Integrate[1/Sqrt[2] t(1+1/Sqrt[2] t),{t,0,6π}]

So,

$$\bar{x} = \frac{\int_C x\,\delta(x, y, z)\,ds}{\int_C \delta(x, y, z)\,ds} = 0$$

,

$$\bar{y} = \frac{\int_C y\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds} = \frac{-3\pi}{6\pi+9\sqrt{2}\,\pi^2} = -\frac{1}{2+3\sqrt{2}\,\pi}$$

,

$$\bar{z} = \frac{\int_C z\,\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds} = \frac{9\sqrt{2}\,\pi^2+36\,\pi^3}{6\pi+9\sqrt{2}\,\pi^2} = \frac{3\pi\left(\sqrt{2}+4\pi\right)}{2+3\sqrt{2}\,\pi}$$

.

```
Clear[c,δ,xδ,yδ,zδ]

c[t_]:=1/Sqrt[2] {Cos[t],Sin[t],t}

δ[x_,y_,z_]:=1+z

xδ[x_,y_,z_]:=x*δ[x,y,z]

yδ[x_,y_,z_]:=y*δ[x,y,z]

zδ[x_,y_,z_]:=z*δ[x,y,z]


(* Total mass *)

scalarLineInt3D[c,δ,{0,6π}]


(* Compute the center of mass x-coordinate *)

xc=scalarLineInt3D[c,xδ,{0,6π}]/scalarLineInt3D[c,δ,{0,6π}]//Simplify


(* Compute the center of mass y-coordinate *)
```

```
yc=scalarLineInt3D[c,yδ,{0,6π}]/scalarLineInt3D[c,δ,{0,6π}]//Simplify


(* Compute the center of mass z-coordinate *)

zc=scalarLineInt3D[c,zδ,{0,6π}]/scalarLineInt3D[c,δ,{0,6π}]//Simplify


plot[1]=ParametricPlot3D[c[t],{t,0,6π},BoxRatios->{1, 1, 1},AxesLabel->
{"x","y","z"}];

plot[2]=Graphics3D[{Red,PointSize[0.02],Point[{xc,yc,zc}]}];

Show[Array[plot,2]]
```

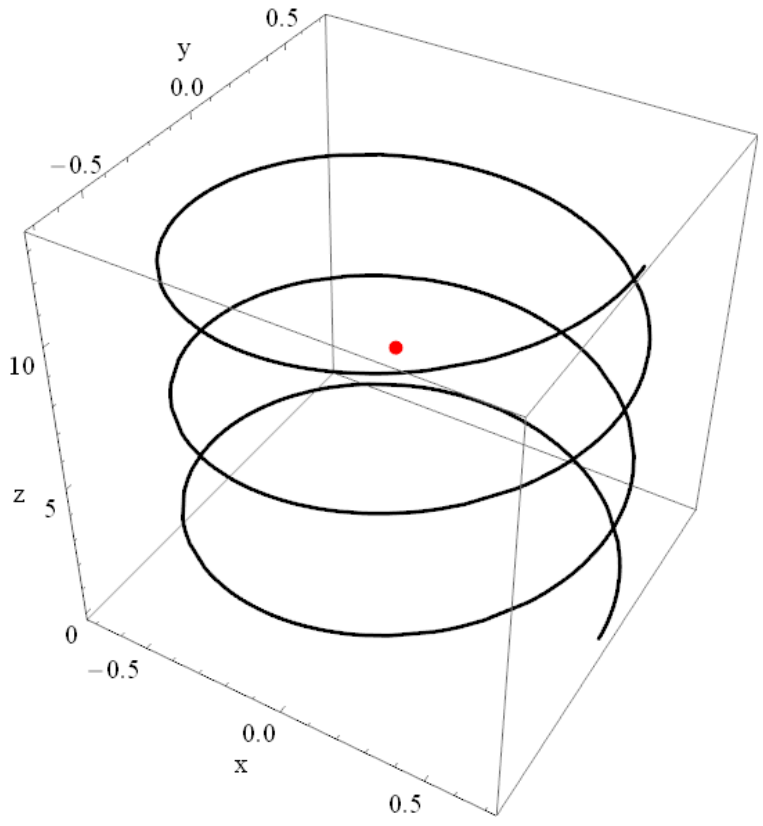


Figure 7.4. The wire with density δ(x,y,z)=1+z in the shape of the circular helix c(t)=1/Sqrt[2] (cos t,sin t,t) and its center of mass (red point).

7.1.2 Vector Line Integrals

Definition 7.4. Vector Line Integral

Let c(t) be a parametrization of the curve C for a<=t<=b. If c'(t) is continuous and F(x,y,z) is a continuous vector function whose domain includes C, then the vector line integral of F along the path c is

(7.3)

$$\int_{\mathbf{c}} \mathbf{F}(x, y, z) \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

.

$d\mathbf{s}$ is referred to as the differential of position along C.

Note that Definition 7.4 is valid for two-dimensional plane as well. To evaluate the vector line integral using Mathematica we provide two modules for this purpose: vectorLineInt2D and vectorLineInt3D corresponding to the two-dimensional case and the three-dimensional case, respectively.

(* Vector line integral as defined in Definition 7.4 *)

(* for two-dimensional case:

c is given parametrically by (x(t),y(t))

*)

vectorLineInt2D[c_,f_,{a_,b_}]:=

```
Module[{x,y,v},
x=c[t][[1]];
y=c[t][[2]];
v=c'[t];
Integrate[f[x,y].v,{t,a,b}]
]


(* Vector line integral as defined in Definition 7.4 *)
(* for three-dimensional case:
c is given parametrically by (x(t),y(t),z(t))
*)
vectorLineInt3D[c_,f_,{a_,b_}]:=
Module[{x,y,z,v},
x=c[t][[1]];
y=c[t][[2]];
z=c[t][[3]];
v=c'[t];
Integrate[f[x,y,z].v,{t,a,b}]
]
```

Example 7.7. Evaluate vector line integral of F=x i-y j+z k along the path c(t)=(t,2t^2,3t^3) for 0<=t<=1.

Solution:

Using equation (7.3) we have

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F}(x, y, z)\cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{c}(t))\cdot \mathbf{c}'(t)\,dt \\ &= \int_{-1}^1 \mathbf{F}\left(t, 2t^2, 3t^3\right)\cdot \mathbf{c}'(t)\,dt \\ &= \int_0^1 \left(t, -2t^2, 3t^3\right)\cdot\left(1, 4t, 9t^2\right)dt \\ &= \int_0^1 \left(t - 8t^3 + 27t^5\right)dt = 3\end{aligned}$$

Integrate[t-8t^3+27t^5,{t,0,1}]

Clear[c,F]

c[t_]:={t,2t^2,3t^3}

F[x_,y_,z_]:={x,-y,z}

vectorLineInt3D[c,F,{0,1}]

Figure 7.5 shows the curve c(t)=(t,2t^2,3t^3) in the vector field F=x i-y j+z k. The magnitude and the sign of a vector line integral depends on the angles between F and tangential vector c'(t). In this example most of the dot products F(c(t)).c'(t) are positive because the angles between the vectors are acute.

```
(* Plot the curve, some points on the curve and the vector fields

  at these points *)

a=0;b=1;

nn=5;

Δ=(b-a)/nn;

plot[1]=ParametricPlot3D[c[t]//Evaluate,{t,a,b},AxesLabel->
{"x","y","z"},BoxRatios->{1, 1, 1},PlotRange->{{0,1.8},Automatic,
{0,3.2}}];

plot[2]=VectorPlot3D[F[x,y,z],{x,0,3},{y,0,3},{z,0,3},VectorPoints-
>10,VectorScale->{0.08, 0.25, Automatic},VectorStyle->Gray];

plot[3]=Graphics3D[{Red,PointSize[0.02],Point[Table[c[(k-1) Δ],
{k,2,nn}]]}];

plot[4]=Graphics3D[{Blue,Arrowheads[0.02],Table[Arrow[{c[(k-1) Δ],c[(k-
1) Δ]+F[c[(k-1) Δ][[1]],c[(k-1) Δ][[2]],c[(k-1) Δ][[3]]]}],{k,2,nn}]}];

plot[5]=Graphics3D[{Red,Arrowheads[0.03],Table[Arrow[{c[(k-1) Δ],c[(k-
1) Δ]+0.2c'[(k-1) Δ]}],{k,2,nn}]}];

Show[Array[plot,5]]
```

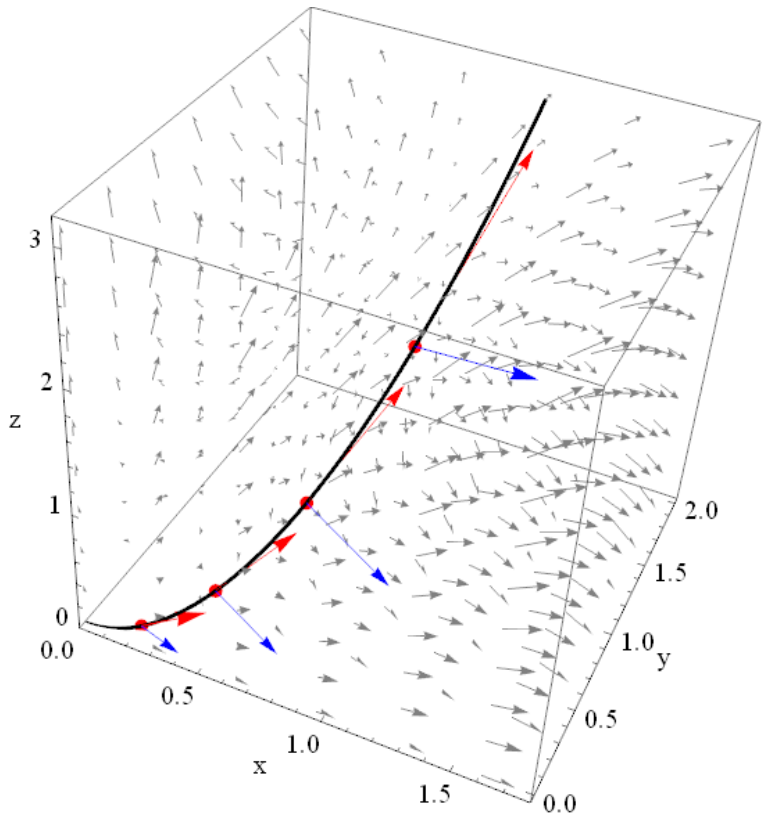


Figure 7.5. The curve c(t)=(t,2t^2,3t^3) in the vector field F=x i-y j+z k.

Example 7.8. Determine the line integral of F=2y i-3j along the ellipse parametrized by c(t)=(3+4cos t,2+2 sin t) for 0<=t<=2π.

Solution:

Using equation (7.3) we have

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$
$$= \int_0^{2\pi} \mathbf{F}(3 + 4\cos t,\ 2 + 2\sin t) \cdot \mathbf{c}'(t)\, dt$$
$$= \int_0^{2\pi} (4 + 4\sin t,\ -3) \cdot (-4\sin t,\ 2\cos t)\, dt$$
$$= \int_0^{2\pi} (-16\sin t - 16\sin^2 t - 6\cos t)\, dt$$
$$= -16\pi$$

Integrate[-16Sin[t]-16Sin[t]^2-6Cos[t],{t,0,2π}]

Clear[c,F]

c[t_]:={3+4Cos[t],2+2Sin[t]}

F[x_,y_]:={2y,-3}

vectorLineInt2D[c,F,{0,2π}]

The magnitude of F is larger in the top half of the ellipse (see Figure 7.6). The dot products F(c(t)).c'(t) are negative because the angles between the vectors are obtuse. So the negative contribution from the top half appears to dominate the positive contribution of the bottom half. Hence, the total line integral is negative.

(* Plot the curve, some points on the curve and the vector fields

at these points *)

a=0;b=2π;

nn=8;

```
Δ=(b-a)/nn;

plot[1]=ParametricPlot[c[t]//Evaluate,{t,a,b},AxesLabel->
{"x","y"},PlotRange->{{-2,9},{-2,6}}];

plot[2]=VectorPlot[F[x,y]//Evaluate,{x,-2,9},{y,-2,6},VectorPoints-
>10,VectorScale->{0.08, 0.25, Automatic},VectorStyle->Gray];

plot[3]=Graphics[{Red,PointSize[0.02],Point[Table[c[(k-1) Δ],{k,1,nn}]]}];

plot[4]=Graphics[{Blue,Arrowheads[0.02],Table[Arrow[{c[(k-1) Δ],c[(k-1)
Δ]+0.3F[c[(k-1) Δ][[1]],c[(k-1) Δ][[2]]]}],{k,1,nn}]}];

plot[5]=Graphics[{Red,Arrowheads[0.03],Table[Arrow[{c[(k-1) Δ],c[(k-1)
Δ]+0.3c'[(k-1) Δ]}],{k,2,nn}]}];

Show[Array[plot,5]]
```

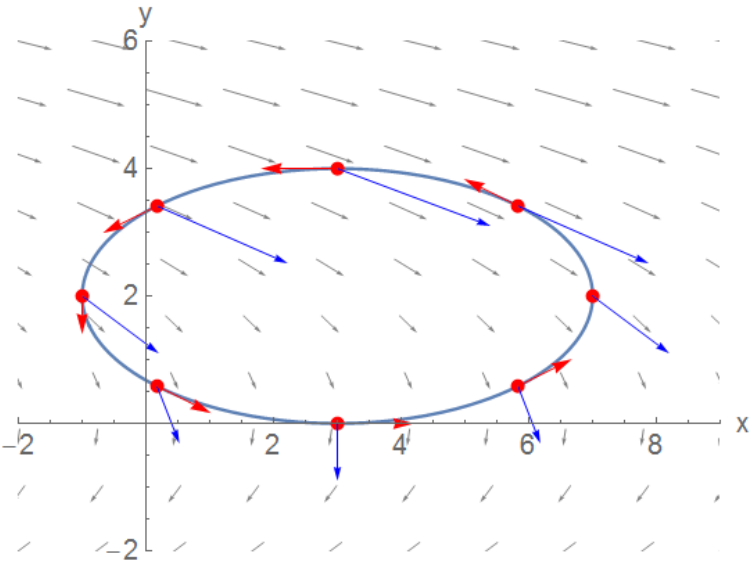


Figure 7.6. The curve c(t)=(3+4cos t,2+2 sin t) in the vector field F=2y i-3j. Note that the magnitude of F is larger in the top half of the ellipse than the bottom half.

Consider F to be a force field in space. Then the line integral $\int_c \mathbf{F} \cdot d\mathbf{s}$ represents the work done by F on a particle as it moves along the path c.

Suppose c(t)=(x(t),y(t),z(t)) has continuous derivative and

F(x,y,z)=M(x,y,z)i+N(x,y,z)j+P(x,y,z)k.

is a continuous vector field. Then, by equation (7.3) of Definition 7.4 we have

$$\begin{aligned}\int_c \mathbf{F}(x, y, z) \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\ &= \int_a^b (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \cdot (x'(t), y'(t), z'(t))\, dt \\ &= \int_a^b (M x'(t) + N y'(t) + P z'(t))\, dt\end{aligned}$$

Since $dx = x'(t)\, dt$, $dy = y'(t)\, dt$, $dz = z'(t)\, dt$, we obtain the differential form of the vector line integral:

$$\int_c M\, dx + N\, dy + P\, dz$$.

Hence,

$$\begin{aligned}\int_c \mathbf{F}(x, y, z) \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\ &= \int_c M\, dx + N\, dy + P\, dz\end{aligned}$$

Example 7.9. Find the work done by the force field F=(x y-z^2)i+y^2 j+x z^2 k on a particle as the particle moves around the perimeter of the square with vertices (1,1,3), (-1,1,3), (-1,-1,3), (1,-1,3), oriented counterclockwise about the z-axis (Figure 7.7).

Solution:

```
Clear[F]

F[x_,y_,z_]:={x y-z^2,y^2,x z^2}


p[1]={1,1,3};

p[2]={-1,1,3};

p[3]={-1,-1,3};

p[4]={1,-1,3};

plot[1]=VectorPlot3D[F[x,y,z]//Evaluate,{x,-2,2},{y,-2,2},
{z,2,4},VectorPoints->5,VectorScale->{0.1, 0.2, Automatic},VectorStyle-
>Gray];

plot[2]=Graphics3D[{PointSize[0.02],Red,Point[Array[p,4]]}];

plot[3]=Graphics3D[{AbsoluteThickness[1.5],Line[{{p[1],p[2]},
{p[2],p[3]},{p[3],p[4]},{p[4],p[1]}}]}];

Show[Array[plot,3],Axes->True,AxesLabel->{"x","y","z"},PlotRange->
{{-2,2},{-2,2},{2,4}}]
```

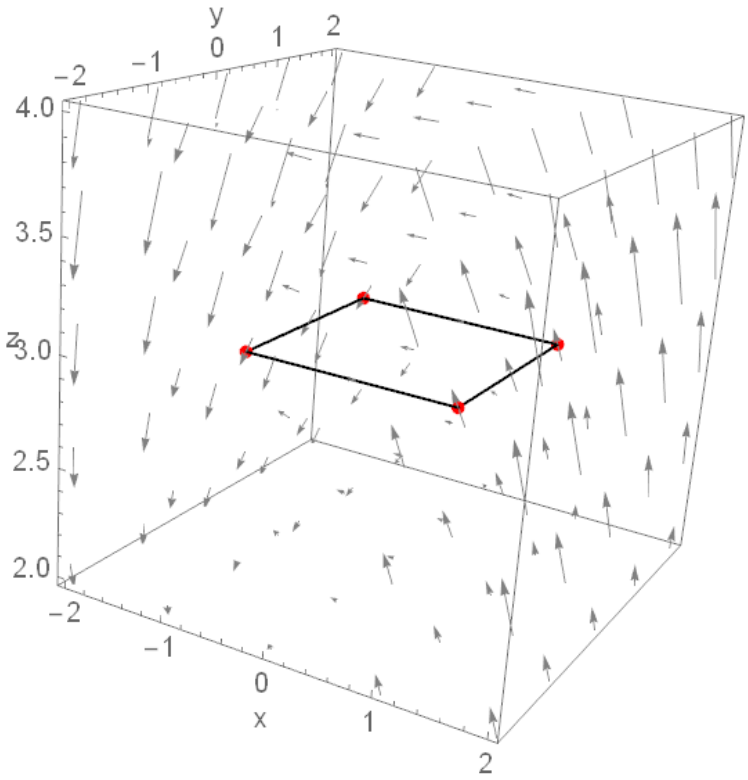


Figure 7.7. The square with vertices (1,1,3), (-1,1,3), (-1,-1,3), (1,-1,3) in the vector field F=(x y-z^2)i+y^2 j+x z^2 k.

There are four line segments to consider. The line equations are given (respectively) by orienting counterclockwise about the z-axis:

c1(t)=(1-2t,1,3), for 0<=t<=1,

c2(t)=(-1,1-2t,3), for 0<=t<=1,

c3(t)=(-1+2t,-1,3), for 0<=t<=1,

c4(t)=(1,-1+2t,3), for 0<=t<=1.

Clear[c1,c2,c3,c4]

c1[t_]=p[1]+(p[2]-p[1])t

c2[t_]=p[2]+(p[3]-p[2])t

c3[t_]=p[3]+(p[4]-p[3])t

c4[t_]=p[4]+(p[1]-p[4])t

{1-2 t,1,3}

{-1,1-2 t,3}

{-1+2 t,-1,3}

{1,-1+2 t,3}

Total work done:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{1} \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt + \int_{t=0}^{1} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt + \int_{t=0}^{1} \mathbf{F}(\mathbf{c}_3(t)) \cdot \mathbf{c}_3'(t)\, dt + \int_{t=0}^{1} \mathbf{F}(\mathbf{c}_4(t)) \cdot \mathbf{c}_4'(t)\, dt$$

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_{t=0}^{1} \mathbf{F}(1-2t,\, 1,\, 3) \cdot (-2,\, 0,\, 0)\, dt \\ &\quad + \int_{t=0}^{1} \mathbf{F}(-1,\, 1-2t,\, 3) \cdot (0,\, -2,\, 0)\, dt \\ &\quad + \int_{t=0}^{1} \mathbf{F}(-1+2t,\, -1,\, 3) \cdot (2,\, 0,\, 0)\, dt \\ &\quad + \int_{t=0}^{1} \mathbf{F}(1,\, -1+2t,\, 3) \cdot (0,\, 2,\, 0)\, dt \\ &= 18 - \frac{2}{3} - 18 + \frac{2}{3} \\ &= 0\end{aligned}$$

vectorLineInt3D[c1,F,{0,1}]

vectorLineInt3D[c2,F,{0,1}]

vectorLineInt3D[c3,F,{0,1}]

vectorLineInt3D[c4,F,{0,1}]

Example 7.10. Evaluate $\int_C z\,dx + x\,dy + y\,dz$, where C is the curve obtained by intersecting the surfaces z=x^2 and x^2+y^2=4 and oriented counterclockwise around the z-axis.

Solution:

The line integral $\int_C z\,dx + x\,dy + y\,dz$ may be written as vector line integral:

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_C z\,dx + x\,dy + y\,dz$$

,

where F=z i+x j+y k.

Since the surface x^2+y^2=4 is a cylinder of radius 2, the parametric equation is given by

(s,t)->(2cos s,2sin s,t).

So, we may parametrize the curve of intersection as

c(t)=(2cos s,2sin s,4cos^2 s),

where s ranges from 0 to 2π which orientation is counterclockwise around the positive z-axis.

Clear[F,c]

```
c[s_]:={2 Cos[s],2Sin[s],4Cos[s]^2}

F[x_,y_,z_]:={z,x,y}


plot[1]=ContourPlot3D[z==x^2,{x,-3,3},{y,-3,3},{z,-1,5},AxesLabel->
{"x","y","z"},Mesh->None,ContourStyle->Opacity[0.2],PlotTheme-
>"Classic"];

plot[2]=ContourPlot3D[x^2+y^2==4,{x,-3,3},{y,-3,3},{z,-1,5},AxesLabel->
{"x","y","z"},Mesh->None,ContourStyle->Opacity[0.2],PlotTheme-
>"Classic"];

plot[3]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Dashed,Blue,Thickness[0.01]}];

Show[Array[plot,3]]
```

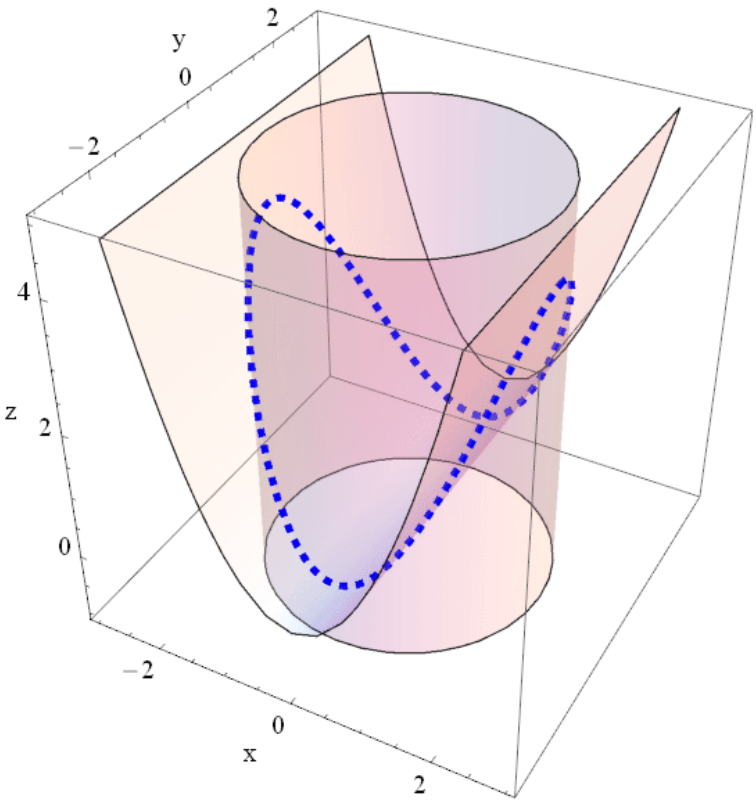


Figure 7.8. The curve of intersection by the surfaces z=x^2 and x^2+y^2=4.

So,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C z\,dx + x\,dy + y\,dz$$
$$= \int_0^{2\pi} \left[(4\cos^2 s)(-2\sin s)\,ds + (2\cos s)(2\cos s)\,ds + (2\sin s)(-8\cos s \sin s)\,ds \right]$$
$$= \int_0^{2\pi} (-8\cos^2 s \sin s + 4\cos^2 s - 16\cos s \sin^2 s)\,ds$$
$$= 4\pi$$

Integrate[-8 Cos[s]^2 Sin[s]+4Cos[s]^2-16Cos[s]Sin[s]^2,{s,0,2π}]

The result checks with the one using the vectorLineInt3D module:

vectorLineInt3D[c,F,{0,2π}]

```
plot[1]=ParametricPlot3D[c[t]//Evaluate,{t,0,2π},AxesLabel->
{"x","y","z"},PlotRange->All];

plot[2]=VectorPlot3D[F[x,y,z]//Evaluate,{x,-3,3},{y,-3,3},
{z,-1,5},VectorPoints->5,VectorScale->{0.15, 0.2, Automatic},VectorStyle-
>Gray,AxesLabel->{"x","y","z"}];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->650]
```

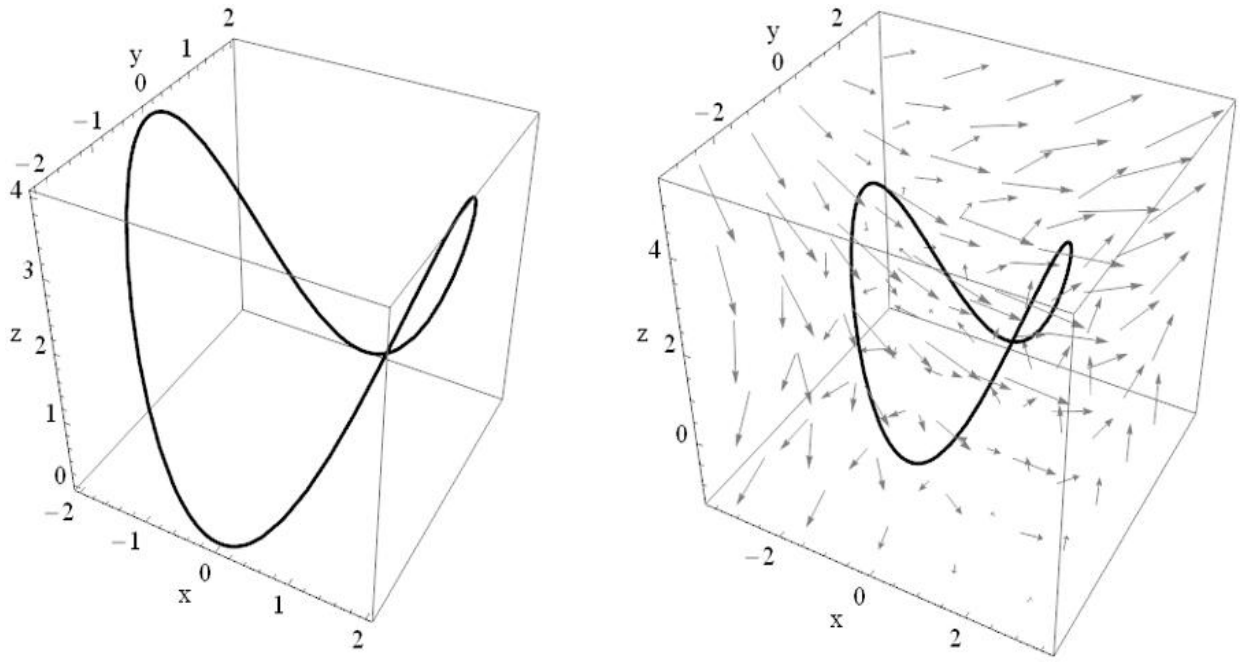


Figure 7.9. The curve of intersection c(t)=(2cos s,2sin s,4cos^2 s) in the vector field F=z i+x j+y k.

Example 7.11. Find the work done by the force field F=x i+x^2 j on a particle as the particle moves from (-1,0) to (1,0) in the xy-plane as shown in Figure 7.10;

(a) along the x-axis,

(b) along the semicircle y=Sqrt[1-x^2],

(c) along the dotted polygonal path.

```
Clear[c,F]

c[t_]:={Cos[t],Sin[t]}

F[x_,y_]:={x,x^2}
```

```
p[1]={-1,0};

p[2]={0,1};

p[3]={1,1};

p[4]={1,0};

plotSemicircle=ParametricPlot[c[t],{t,π,0},AxesLabel->{"x","y"}];

plotPoints=Graphics[{PointSize[0.02],Red,Point[Array[p,4]]}];

plotArrows=Graphics[{Dashed,Arrow[{{p[1],p[2]},{p[2],p[3]},
{p[3],p[4]}}],Text["c1",{-0.5,0.4}],Text["c2",{0.5,0.95}],Text["c3",
{0.95,0.5}]}];

plot[1]=Show[plotSemicircle,plotPoints,plotArrows];

plot[2]=VectorPlot[F[x,y]//Evaluate,{x,-1,1},{y,0,1},VectorPoints-
>8,VectorScale->{0.1, 0.5, Automatic},VectorStyle->Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->650]
```

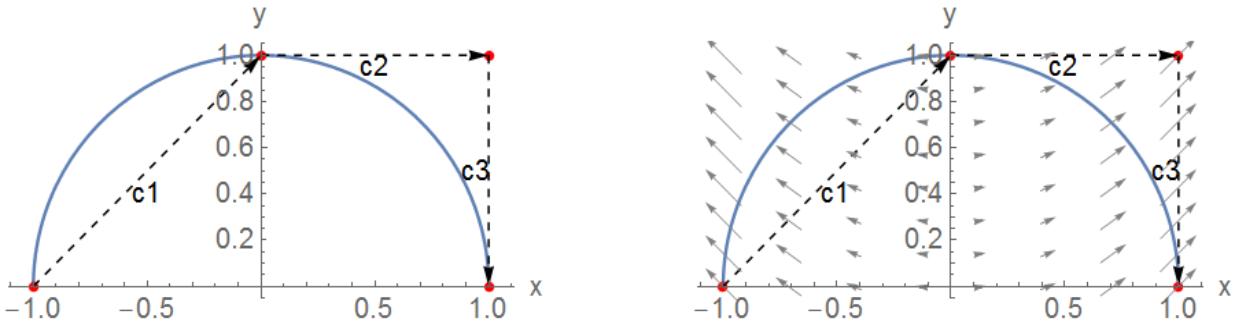


Figure 7.10. A particle moves from (-1,0) to (1,0) in the force field F=x i+x^2 j.

Solution (a)

Since $d\mathbf{s} = dx\,\mathbf{i} + dy\,\mathbf{j}$, using equation (7.3) the work done along the x-axis from x=-1 to x=1 is

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{x=-1}^{1} (x\,\mathbf{i} + x^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$
$$= \int_{x=-1}^{1} (x\,dx + x^2\,dy)$$

Since y=0, so $dy = 0$. Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{x=0}^{1} x\,dx = \left[\frac{1}{2}x^2\right]_{-1}^{1} = 0$$

.

Solution (b)

The semicircle y=Sqrt[1-x^2] may be parametrized by

c(t)=(cos t,sin t), for 0<=t<=π.

The work done along the semicircle from x=-1 to x=1 is

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)\,dt$$
$$= \int_{\pi}^{0} \mathbf{F}(\cos t,\ \sin t)\cdot(-\sin t,\ \cos t)\,dt$$
$$= \int_{\pi}^{0} (\cos t,\ \cos^2 t)\cdot(-\sin t,\ \cos t)\,dt$$
$$= \int_{\pi}^{0} (-\cos t\sin t + \cos^3 t)\,dt = 0$$

Integrate[-Cos[t]Sin[t]+Cos[t]^3,{t,π,0}]

vectorLineInt2D[c,F,{π,0}]

Solution (c)

The polygonal path consists of three line segments:

c1(t)=(-1+t,t), for 0<=t<=1,

c2(t)=(t,1), for 0<=t<=1,

c3(t)=(1,1-t), for 0<=t<=1.

Total work done along the polygonal path:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{1} \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt + \int_{t=0}^{1} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt + \int_{t=0}^{1} \mathbf{F}(\mathbf{c}_3(t)) \cdot \mathbf{c}_3'(t)\, dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{1} \mathbf{F}(-1+t,\, t) \cdot (1,\, 1)\, dt + \int_{t=0}^{1} \mathbf{F}(t,\, 1) \cdot (1,\, 0)\, dt + \int_{t=0}^{1} \mathbf{F}(1,\, 1-t) \cdot (0,\, -1)\, dt = -\frac{1}{6} + \frac{1}{2} - 1 = -\frac{2}{3}$$

Clear[c1,c2,c3]

c1[t_]:={-1+t,t}

c2[t_]:={t,1}

c3[t_]:={1,1-t}

(* Total work done along the polygonal path *)

```
vectorLineInt2D[c1,F,{0,1}]+vectorLineInt2D[c2,F,
{0,1}]+vectorLineInt2D[c3,F,{0,1}]
```

We see that the work done by the force field F=x i+x^2 j from (-1,0) to (1,0) depends on the path chosen. We say that F does not have path-independent line integrals.

Example 7.12. Redo Example 7.11 but with the force field F=2x sin y i+x^2 cos yj.

```
Clear[c,F]

c[t_]:={Cos[t],Sin[t]}

F[x_,y_]:={2x Sin[y],x^2 Cos[y]}


p[1]={-1,0};

p[2]={0,1};

p[3]={1,1};

p[4]={1,0};

plotSemicircle=ParametricPlot[c[t],{t,π,0},AxesLabel->{"x","y"}];

plotPoints=Graphics[{PointSize[0.02],Red,Point[Array[p,4]]}];

plotArrows=Graphics[{Dashed,Arrow[{{p[1],p[2]},{p[2],p[3]},
{p[3],p[4]}}],Text["c1",{-0.5,0.4}],Text["c2",{0.5,0.95}],Text["c3",
{0.95,0.5}]}];
```

```
plot[1]=Show[plotSemicircle,plotPoints,plotArrows];

plot[2]=VectorPlot[F[x,y]//Evaluate,{x,-1,1},{y,0,1},VectorPoints->8,VectorScale->{0.1, 0.5, Automatic},VectorScale->Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->650]
```

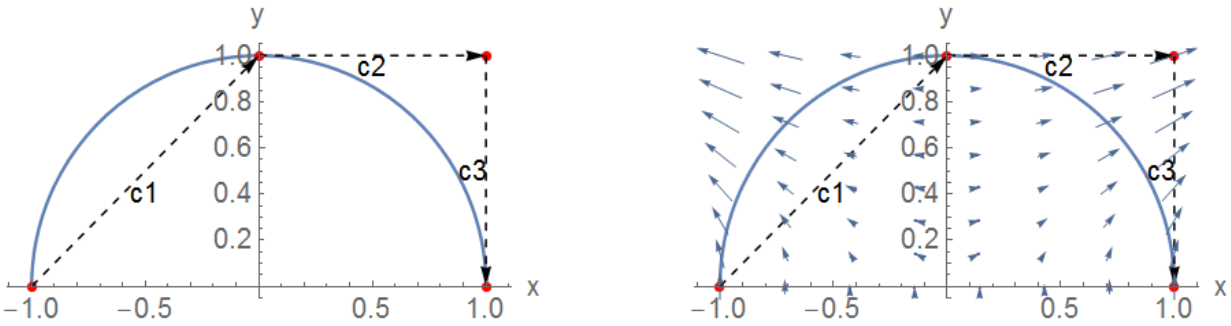


Figure 7.11. A particle moves from (-1,0) to (1,0) in the force field F=2x sin y i+x^2 cos yj.

Solution (a) along the x-axis.

Since $d\mathbf{s} = dx\,\mathbf{i} + dy\,\mathbf{j}$, using equation (7.3) the work done along the x-axis from x=-1 to x=1 is

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{x=-1}^{1} (2x\sin y\,\mathbf{i} + x^2\cos y\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$
$$= \int_{x=-1}^{1} (2x\sin y\,dx + x^2\cos y\,dy)$$

Since y=0, so $dy = 0$. Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{x=-1}^{1} 0\,dx = 0$$

.

Solution (b) along the semicircle y=Sqrt[1-x^2].

```
vectorLineInt2D[c,F,{π,0}]
```

Solution (c) along the dotted polygonal path.

```
Clear[c1,c2,c3]
c1[t_]:={-1+t,t}
c2[t_]:={t,1}
c3[t_]:={1,1-t}

vectorLineInt2D[c1,F,{0,1}]
vectorLineInt2D[c2,F,{0,1}]
vectorLineInt2D[c3,F,{0,1}]

(* Total work done along the polygonal path *)
vectorLineInt2D[c1,F,{0,1}]+vectorLineInt2D[c2,F,
{0,1}]+vectorLineInt2D[c3,F,{0,1}]
```

In this example, the work done by the force field F=2x sin y i+x^2 cos yj along the three different paths give the same result. However, how do we know if there is no other path that gives different result? As a matter of fact,

we shall see later that this vector field has indeed path-independent line integrals. First, we shall give the definition of what we mean by path-independent line integrals.

Definition 7.5. Path-independent line integrals

Let C1 and C2 be any two simple, oriented curves that have continuous derivatives from A to B. A continuous vector field F has path-independent line integrals if

(7.4) $$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$.

Definition 7.5 implies that the line integral of F along a path from A to B depends only on the endpoints A and B, not on the particular path followed.

From equation (7.4) we obtain

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 0$$.

Since C1 and C2 are oriented curves, the second line integral on the left-hand side implies that the integral is from B to A. Thus, we may consider a closed curve C such that the line integral is the circulation of F around C and denote it with the symbol $\oint$ that satisfies

(7.5)

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 0$$

.

This fact leads to the following theorem:

Theorem 7.6. Closed integrals

Let F be a continuous vector field. Then F has path-independent line integrals if and only if $\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for all simple, closed curves C (that has continuous derivatives) in the domain of F.

Theorem 7.7. Conservative vector field

Let F be defined and continuous on a domain $D \subseteq \mathbb{R}^3$. Suppose there exists a function V such that $\mathbf{F} = \nabla V$. Then $\mathbf{F} = \nabla V$ if and only if F has path-independent line integrals over curves in D. If c is a path from A to B in D, then

(7.6) $$\int_c \mathbf{F} \cdot d\mathbf{s} = V(B) - V(A)$$.

We call F a conservative vector field and V a potential function for F.

Theorem 7.7 implies that a vector field F has path-independent line integrals when it is a conservative vector field and that the line integral of F along any path is determined by the values of the potential function V at the endpoints of the path. In practice there is a simple criterion we can use to find out if a vector field is conservative, which is stated in the following theorem:

Theorem 7.8. Criterion for conservative vector fields

Let F be a vector field that has continuous derivative whose domain is a simply-connected region D in $\mathbb{R}^3$. Then $\mathbf{F} = \nabla V$ for some potential function V on D if and only if $\nabla \times \mathbf{F} = 0$ at all points of D.

A region D is said to be connected if any two points in D can be joined by a path whose image lies in D. Thus, a simply-connected region is simple and connected.

As an example, consider the vortex field (Figure 7.12) given by F=(-y/(x^2 + y^2),x/(x^2 + y^2)). It can be shown that $\nabla \times \mathbf{F} = 0$. However, F is not defined at (x,y)=(0,0). Therefore, the domain of F must exclude this point in order to be simply-connected (see also Example 7.17).

```
Curl[{-y/(x^2+y^2),x/(x^2+y^2),0},{x,y,z},"Cartesian"]//Simplify
```

```
VectorPlot[{-y/(x^2+y^2),x/(x^2+y^2)},{x,-1,1},{y,-1,1},VectorPoints->10,VectorScale->{0.08,0.4,Automatic}]
```

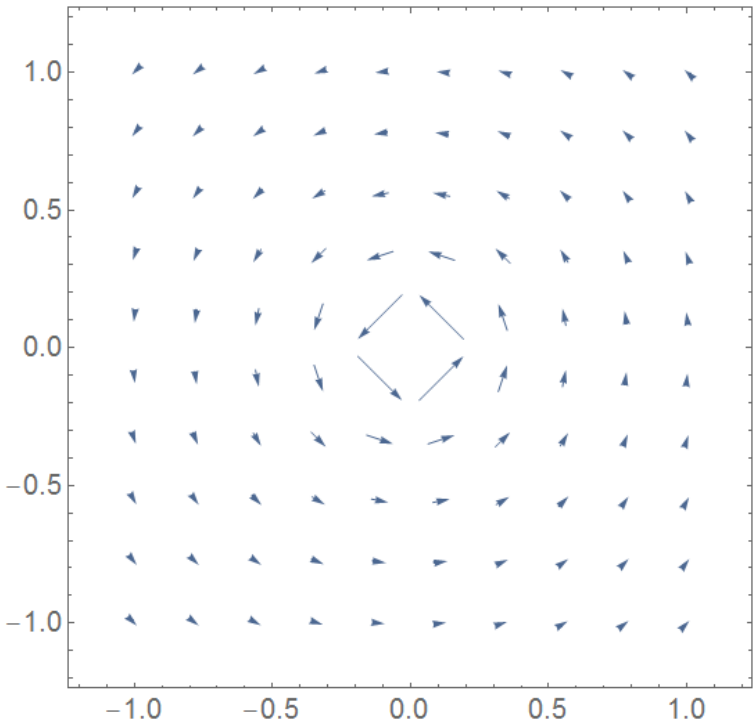


Figure 7.12. The vortex field F=(- y/(x^2 + y^2),x/(x^2 + y^2)).

These definitions and theorems provide theoretical foundations for conservative vector field which plays important role in physics and engineering. The readers are invited to consult advanced textbooks for proofs. Next, we give more examples to solidify our understanding of line integrals and particularly conservative force fields.

Example 7.13. (a) Verify that the force field F in Example 7.12 is conservative.

(b) Find a scalar-valued potential function V for F such that $\mathbf{F} = \nabla V$.

Solution (a)

Recall the force field F in Example 7.12:

F=2x sin y i+x^2 cos yj.

To evaluate the curl of F we set it to be three-dimensional by adding a zero z-component. So,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\sin y & x^2\cos y & 0 \end{vmatrix}$$

$$= 0\,\mathbf{i} - 0\,\mathbf{j} + (2x\cos y - 2x\cos y)\,\mathbf{k}$$

$$= \mathbf{0}$$

By Theorem 7.8, F is conservative.

Clear[F]

F[x_,y_]:={2x Sin[y],x^2 Cos[y]}

Curl[Join[F[x,y],{0}],{x,y,z},"Cartesian"]

Note that we have used Join command to set F to be three-dimensional by adding a zero z-component.

Solution (b)

Let $\mathbf{F} = \nabla V$. Then

$$\mathbf{F} = 2x \sin y\,\mathbf{i} + x^2 \cos y\,\mathbf{j} = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j}$$

.

So,

(7.7) $$\frac{\partial V}{\partial x} = 2x \sin y \quad , \quad \frac{\partial V}{\partial y} = x^2 \cos y \quad .$$

Integrating $\frac{\partial V}{\partial x}$ with respect to x gives

(7.8)

$$V(x, y) = \int \frac{\partial V}{\partial x}\, dx = x^2 \sin y + P(y)$$

,

where P(y) is a function of y.

Differentiating equation (7.8) with respect to y and compare it with the second equation in (7.7), we see that P(y) must be a constant. Hence,

V(x,y)=x^2 sin y+C,

where C is constant of integration.

Here, we provide a Mathematica module called findScalarPotential2D to find the scalar-valued potential function if given a two-dimensional vector field F:

(* Find the scalar-valued potential function for F

Requirement: F is conservative and two-dimensional given by F(x,y)

```
*)
findScalarPotential2D[F_]:=
Module[{Fx,dFy,dPy,Py},
Clear[P];
Fx=Integrate[F[x,y][[1]],x]+P[y];
dFy=D[Fx,y];
dPy=Solve[dFy==F[x,y][[2]],D[P[y],y]]/.Rule->Equal//Flatten[#,1]&;
Py=DSolve[dPy,P[y],y];
Fx/.Py
]
```

Now, we can check our result using findScalarPotential2D module:

```
findScalarPotential2D[F]
```

Example 7.14. Consider the line integral

$$\int_C 2\,x\,y\,dx + \left(x^2 + 2\,y\,z\right)dy + y^2\,dz$$

.

(a) Evaluate this integral, where C is the line segment from (0,0,0) to (1,1,1).

(b) Evaluate this integral, where C is the path from (0,0,0) to (1,1,1) parametrized by t->(t,t^2,t^3), for 0<=t<=1.

Solution (a)

We may write F=2x y i+(x^2+2 y z) j+y^2 k.

Since $d\mathbf{s} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$, so

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_C 2xy\,dx + \left(x^2 + 2yz\right)dy + y^2\,dz$$

.

In this case, the line segment may be parametrized by

C1: c1(t)=(t,t,t), for 0<=t<=1.

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{c_1}(t))\cdot \mathbf{c_1}'(t)\,dt = \int_0^1 \mathbf{F}(t,t,t)\Big)\cdot(1,1,1)\,dt \\ &= \int_0^1 \left(2t^2, 3t^2, t^2\right)\cdot(1,1,1)\,dt \\ &= \int_0^1 6t^2\,dt \\ &= 2\end{aligned}$$

Clear[c1,F]

c1[t_]:={t,t,t}

F[x_,y_,z_]:={2x y,x^2+2y z,y^2}

vectorLineInt3D[c1,F,{0,1}]

Solution (b)

In this case, the path c2 is parametrized by

C2: c2(t)=(t,t^2,t^3), for 0<=t<=1.

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt = \int_0^1 \mathbf{F}\left(t, t^2, t^3\right) \cdot \left(1, 2t, 3t^2\right) dt$$
$$= \int_0^1 \left(2t^3, t^2 + 2t^5, t^4\right) \cdot \left(1, 2t, 3t^2\right) dt$$
$$= \int_0^1 \left[2t^3 + 2t\left(t^2 + 2t^5\right) + 3t^6\right] dt$$
$$= 2$$

Integrate[2 t^3+2 t (t^2+2 t^5)+3 t^6,{t,0,1}]

Clear[c2]

c2[t_]:={t,t^2,t^3}

vectorLineInt3D[c2,F,{0,1}]

Note that the line integral

$$\int_C 2xy\, dx + \left(x^2 + 2yz\right) dy + y^2\, dz$$

gives the same result for both paths. We can verify if the force F=2x y i+ (x^2+2 y z) j+y^2 k is conservative by evaluating

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 + 2yz & y^2 \end{vmatrix}$$

$$= (2y - 2y)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (2x - 2x)\,\mathbf{k}$$

$$= \mathbf{0}$$

By Theorem 7.8, F is conservative.

```
Curl[F[x,y,z],{x,y,z},"Cartesian"]
```

```
plot[1]=ParametricPlot3D[{c1[t],c2[t]}//Evaluate,{t,0,1},AxesLabel->
{"x","y","z"},PlotStyle->{{AbsoluteThickness[2],Green},
{AbsoluteThickness[2],Red}},PlotRange->All];

plot[2]=VectorPlot3D[F[x,y,z]//Evaluate,{x,0,1},{y,0,1},
{z,0,1},VectorPoints->5,VectorScale->{0.15, 0.2, Automatic},VectorStyle-
>Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->650]
```

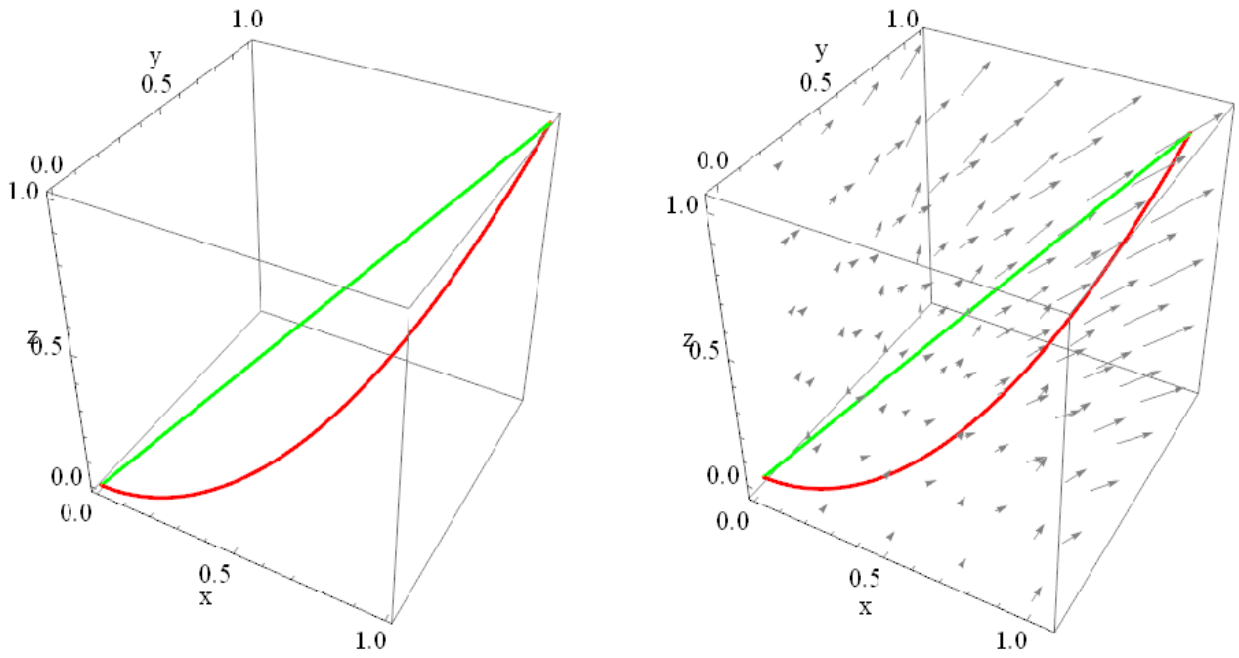


Figure 7.13. Two different paths with the same endpoints in the conservative vector field F=2x y i+(x^2+2 y z) j+y^2 k.

Example 7.15. Find the scalar-valued potential function V for F in Example 7.14.

Solution:

Recall the vector field F given in Example 7.14:

F=2x y i+(x^2+2 y z) j+y^2 k.

Let $\mathbf{F} = \nabla V$. Then

$$\mathbf{F} = 2xy\,\mathbf{i} + \left(x^2 + 2yz\right)\mathbf{j} + y^2\,\mathbf{k} = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}$$

.

So, we have

(7.9) $\frac{\partial V}{\partial x} = 2xy$, $\frac{\partial V}{\partial y} = x^2 + 2yz$, $\frac{\partial V}{\partial z} = y^2$.

Integrating $\frac{\partial V}{\partial x}$ with respect to x gives

(7.10)

$$V(x, y, z) = \int \frac{\partial V}{\partial x} dx = x^2 y + P(y, z)$$

,

where P(y,z) is any function of y and z.

Differentiating equation (7.10) with respect to y and compare it with the second equation in (7.9), we see that

$$\frac{\partial V}{\partial y} = x^2 + \frac{\partial P}{\partial y} = x^2 + 2yz$$.

So,

$$\frac{\partial P}{\partial y} = 2yz$$.

Integrating $\frac{\partial P}{\partial y}$ with respect to y gives

P(y,z)=y^2 z+Q(z),

where Q(z) is a function of z alone. So,

(7.11) V(x,y,z)=x^2 y+P(y,z)

=x^2 y+y^2 z+Q(z)

Finally, differentiating equation (7.11) with respect to z and compare it with the third equation of (7.9):

$$\frac{\partial V}{\partial z} = y^2 + Q'(z) = y^2$$.

Therefore, Q(z)=C is a constant.

Thus, a scalar potential for the original vector field F is given by

V(x,y,z)=x^2 y+y^2 z+C.

Similar to findScalarPotential2D module here is one called findScalarPotential3D to find the scalar-valued potential function for three-dimensional vector field F:

```
(* Find the scalar-valued potential function for F

Requirement:

F is conservative and three-dimensional given by F(x,y,z)

*)

findScalarPotential3D[F_]:=

Module[{dFy,dPy,dFz,dQz,Qz,P,Q},

Fx=Integrate[F[x,y,z][[1]],x];

dFy=D[Fx+P[y,z],y];

dPy=Solve[dFy==F[x,y,z][[2]],D[P[y,z],y]][[1,1,2]];

Fy=Integrate[dPy,y];

dFz=D[Fx+Fy+Q[z],z];

dQz=Solve[dFz==F[x,y,z][[3]],D[Q[z],z]]/.Rule->Equal//Flatten[#,1]&;
```

```
Qz=DSolve[dQz,Q[z],z];

Fz=Q[z]/.Qz;

Fx+Fy+Fz

]
```

Let us obtain the potential function for F using findScalarPotential3D:

```
Clear[F]

F[x_,y_,z_]:={2x y,x^2+2y z,y^2}

findScalarPotential3D[F]
```

We might want to verify our potential function V such that $\mathbf{F} = \nabla V$:

```
Grad[x^2 y+y^2 z+C[1],{x,y,z},"Cartesian"]==F[x,y,z]
```

Example 7.16. Suppose a vector field is given by

F=(2x y z-y^2 z^3) i+N(x,y,z)j+(x^2 y-3x y^2 z^2)k.

(a) Find N(x,y,z) such that F is conservative.

(b) Find a scalar potential function for F.

Solution (a)

By Theorem 7.8 the criterion for a vector field to be conservative is the curl of F vanishes,

$$\nabla \times \mathbf{F} = \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xyz - y^2 z^3) & N(x, y, z) & (x^2 y - 3xy^2 z^2) \end{vmatrix}$$

So, we have

$$\left(x^2 - 6xyz^2 - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left[(2xy - 3y^2 z^2) - (2xy - 3y^2 z^2)\right]\mathbf{j} + \left(\frac{\partial N}{\partial x} - 2xz + 2yz^3\right)\mathbf{k} = \mathbf{0}$$

$$x^2 - 6xyz^2 - \frac{\partial N}{\partial z} = 0,$$

$$\frac{\partial N}{\partial z} = x^2 - 6xyz^2.$$

$$\frac{\partial N}{\partial x} - 2xz + 2yz^3 = 0,$$

$$\frac{\partial N}{\partial x} = 2xz - 2yz^3.$$

Integrating $\frac{\partial N}{\partial z}$ and $\frac{\partial N}{\partial x}$ with respect to z and x, respectively give

N(x,y,z)=x^2 z-2x y z^3+P(x,y),

N(x,y,z)=x^2 z-2x y z^3+Q(y,z),

where P is any function of x and y; and Q is any function of y and z.

Comparing these two equations we conclude that P(x,y)=Q(y,z) is a constant. We may set P(x,y)=Q(y,z)=0. Therefore,

N(x,y,z)=x^2 z-2x y z^3.

Thus,

F=(2x y z-y^2 z^3) i+(x^2 z-2x y z^3)j+(x^2 y-3x y^2 z^2)k

is conservative.

Solution (b)

To find a scalar potential function for F we may use the method described in Example 7.15. Alternatively, we use the findScalarPotential3D module. It is a very convenient tool. However, the reader should master the hand-calculations method given in Example 7.15.

The potential function for F=(2x y z-y^2 z^3) i+(x^2 z-2x y z^3)j+(x^2 y-3x y^2 z^2)k is

V(x,y,z)=y z (x^2-x y z^2)+C,

where C is constant of integration.

```
Clear[F]

F[x_,y_,z_]:={2x y z-y^2 z^3,x^2 z-2x y z^3,x^2 y-3x y^2 z^2}


VectorPlot3D[F[x,y,z],{x,-3,3},{y,-3,3},{z,-3,3},VectorPoints->5,VectorScale->{0.2, 0.15, Automatic},VectorStyle->Gray]
```

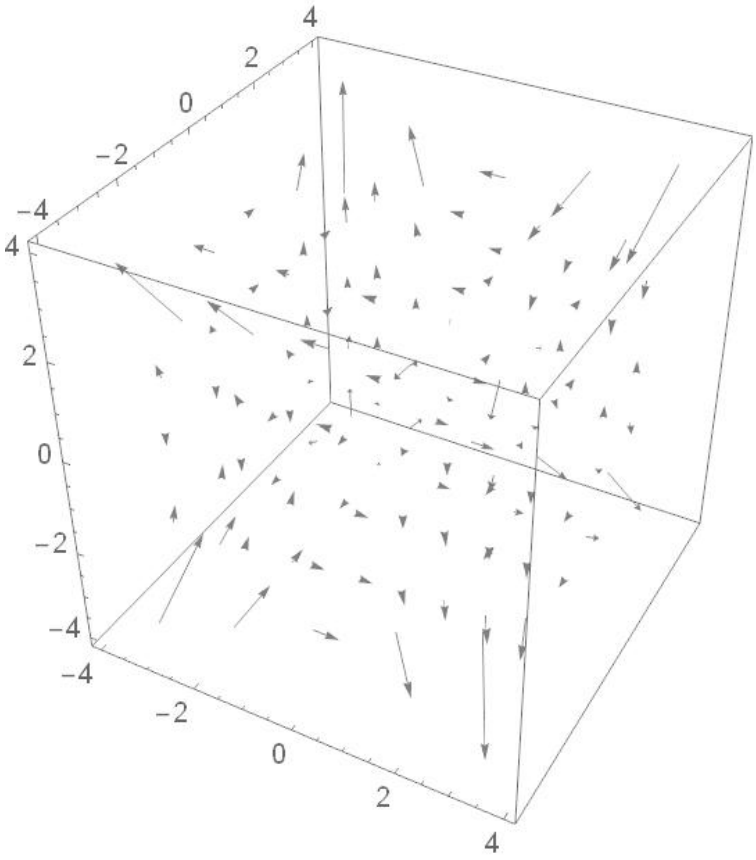


Figure 7.14. The conservative vector field F=(2x y z-y^2 z^3) i+(x^2 z-2x y z^3)j+(x^2 y-3x y^2 z^2)k.

findScalarPotential3D[F]

(* Verification *)

Grad[y z (x^2-x y z^2)+C[1],{x,y,z}]==F[x,y,z]//Simplify

Example 7.17. Given a vector field

F=x/(x^2 + y^2) i+y/(x^2 + y^2) j.

(a) Determine where it is conservative.

(b) Find a scalar potential function for F.

(c) Find the work done by F in moving a particle along the curve y=x^2+x-1 from (0,-1) to (1,1).

(d) Find the work done by F in part (c) using Theorem 7.7.

Solution (a)

It is easy to check that $\nabla \times \mathbf{F} = 0$. But F is not necessary conservative. The criterion for a vector field to be conservative by Theorem 7.8 is $\nabla \times \mathbf{F} = 0$ at all points of region D. However, F has discontinuity at x=y=0. Therefore, we have to exclude the point (0,0) from its domain. Thus, the domain D is simply-connected if it is defined by

$$D = \{ (x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0) \}$$

.

Then F is conservative on domain D.

```
Clear[F]

F[x_,y_]:={x/(x^2+y^2),y/(x^2+y^2)}


Curl[Join[F[x,y],{0}],{x,y,z},"Cartesian"]//Simplify


VectorPlot[F[x,y],{x,-1,1},{y,-1,1},VectorPoints->10,VectorScale->
{0.15,0.25,Automatic}]
```

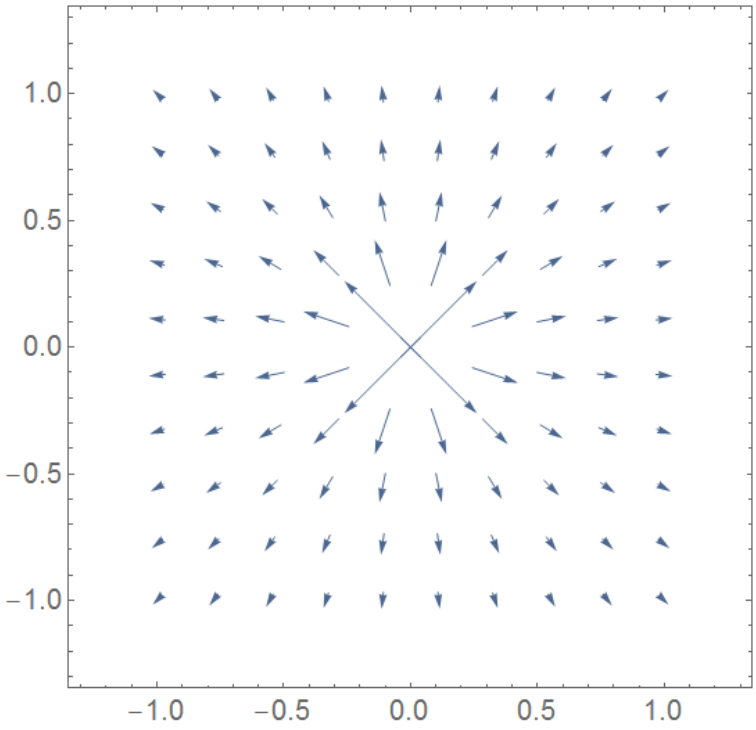


Figure 7.15. The radial vector field F=x/(x^2 + y^2) i+y/(x^2 + y^2) j.

Solution (b)

The scalar potential function for F=x/(x^2 + y^2) i+y/(x^2 + y^2) j is

V(x,y)=1/2 ln (x^2+y^2)+C,

where C is constant of integration.

findScalarPotential2D[F]

Solution (c)

The work done in moving a particle along the curve y=x^2+x-1 from (0,-1) to (1,1) is

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{x=0}^{1}\left(\frac{x}{x^2+y^2}\,\mathbf{i}+\frac{y}{x^2+y^2}\,\mathbf{j}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_{x=0}^{1}\left[\frac{x}{x^2+\left(x^2+x-1\right)^2}\,dx+\frac{x^2+x-1}{x^2+\left(x^2+x-1\right)^2}\,dy\right]$$

Since $dy=(2x+1)\,dx$, we have

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{x=0}^{1}\left[\frac{x+2x\left(x^2+x-1\right)+x^2+x-1}{x^2+\left(x^2+x-1\right)^2}\right]dx$$

$$= \int_{x=0}^{1}\left[\frac{2x^3+3x^2-1}{x^2+\left(x^2+x-1\right)^2}\right]dx$$

$$= \frac{1}{2}\ln 2$$

Integrate[(2x^3+3x^2-1)/(x^2+(x^2+x-1)^2),{x,0,1}]

Clear[c]

c[t_]:={t,t^2+t-1}

Using the vectorLineInt2D module we obtain the same result.

vectorLineInt2D[c,F,{0,1}]

plot[1]=ParametricPlot[c[t]//Evaluate,{t,-2,2},AxesLabel->
{"x","y"},PlotRange->{{-2,2},{-2,2}},Epilog->
{Red,PointSize[0.03],Point[{{0,-1},{1,1}}]}];

```
plot[2]=VectorPlot[F[x,y]//Evaluate,{x,-2,2},{y,-2,2},VectorPoints->10,VectorScale->{0.2, 0.25, Automatic},VectorStyle->Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->500]
```

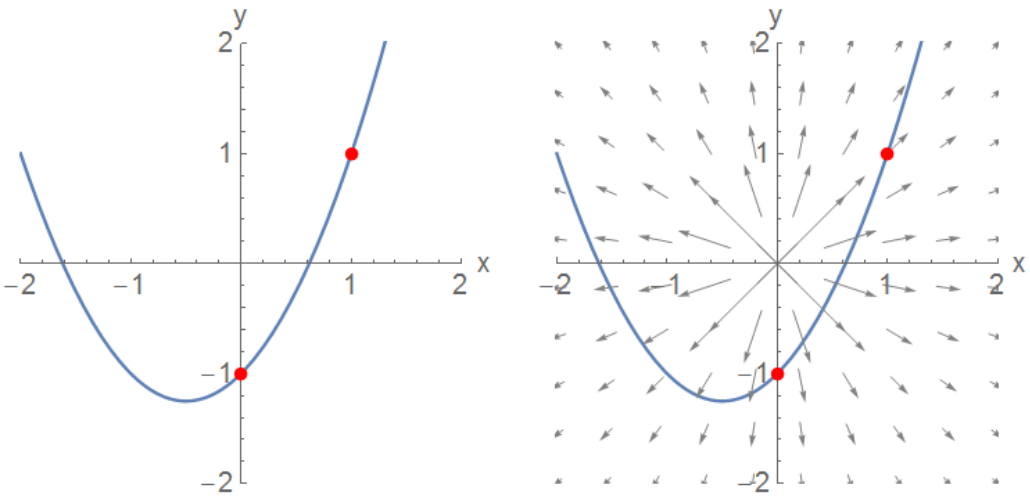


Figure 7.16. The work done by the vector field F=x/(x^2 + y^2) i+y/(x^2 + y^2) j in moving a particle along the curve y=x^2+x-1 from (0,-1) to (1,1).

Solution (d)

Since the scalar potential function is

V(x,y)=1/2 ln (x^2+y^2)+C

By equation (7.6) of Theorem 7.7:

$$\text{Work done} = \int_C \mathbf{F} \cdot d\mathbf{s} = V(1, 1) - V(0, -1)$$
$$= \frac{1}{2} \ln 2 + C - \left(\frac{1}{2} \ln 1 + C \right)$$
$$= \frac{1}{2} \ln 2$$

Note that the constant of integration C cancels out in the process. This gives us freedom to choose the reference point of the potential function or potential energy. In physics, potential energy has no absolute meaning, only differences of potential energy are physically meaningful. In the following example we show that the total energy of a particle moving under the influence of a conservative force field is constant in time.

Example 7.18. (a) Suppose that a particle obeys Newton's Second Law of Motion. Show that the work done by a force field F in moving the particle of mass m along a path c(t) from c(a) to c(b) is equal to the change in kinetic energy of the particle.

(b) If F is a conservative vector field in $\mathbb{R}^3$ with $\mathbf{F} = -\nabla V$, where V is the potential energy of a particle moving under the influence of F. Show that the total energy of the particle is constant in time.

Solution (a)

Let a=(dc'(t))/dt be the particle's acceleration. According to Newton's Second Law of Motion:

F=m a=m (dc'(t))/dt.

The work done by a force field F in moving a particle of mass m along a path c(t) from c(a) to c(b) is

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C m \frac{d\mathbf{c}'(t)}{dt} \cdot \mathbf{c}'(t)\, dt = \int_C m\, \mathbf{c}'(t) \cdot d\mathbf{c}'(t)$$

Suppose m is constant, so

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = m\int_{\mathbf{c}(a)}^{\mathbf{c}(b)} \mathbf{c}'(t)\cdot d\mathbf{c}'(t) = \frac{1}{2}m\int_{\mathbf{c}(a)}^{\mathbf{c}(b)} d(\mathbf{c}'\cdot\mathbf{c}')$$
$$= \frac{1}{2}m\int_{\mathbf{c}(a)}^{\mathbf{c}(b)} d\left(\|\mathbf{c}'\|^2\right)$$
$$= \frac{1}{2}m\left(\|\mathbf{c}'(b)\|^2 - \|\mathbf{c}'(a)\|^2\right)$$

Write the speed v(t)=||v(t)||=||c'(t)||, thus

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \frac{1}{2}m(v(b))^2 - \frac{1}{2}m(v(a))^2$$

.

The kinetic energy of the particle is given by $\frac{1}{2}m(v(t))^2$. Hence, the work done by a force field F in moving a particle of mass m along a path c(t) from c(a) to c(b) is equal to the change in kinetic energy of the particle. Note that in this case, F is not necessarily conservative. In part (b) we shall consider the case when F is conservative.

Solution (b)

Let F be a conservative vector field with $\mathbf{F} = -\nabla V$. If a particle travels along a path c, its potential energy at time t is defined to be V(c(t)). Note that the negative sign in F follows the convention used in physics.

By equation (7.6) of Theorem 7.7 we have

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = -V(B) - (-V(A))$$

,

where the particle of mass m moves from point A to point B.

However, the work done by F is also equal to the change in kinetic energy of the particle. Using the result from part (a):

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2} m(v(B))^2 - \frac{1}{2} m(v(A))^2$$

.

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = -V(B) + V(A) = \frac{1}{2} m(v(B))^2 - \frac{1}{2} m(v(A))^2$$

,

$$V(A) + \frac{1}{2} m(v(A))^2 = V(B) + \frac{1}{2} m(v(B))^2$$

.

The last equation implies that the sum of the potential and kinetic energies of the particle remains constant from A to B. Hence, the total energy of the particle is constant in time. This is known as the law of conservation of energy.

Example 7.19. Suppose an object of mass M is located at the origin. Let another object of mass m be located at (x,y,z). According to Newton's law of universal gravitation, the gravitational force field F of M on m is given by

F=-((G M m)/(x^2 + y^2 + z^2)^(3/2))(x i+y j+z k),

where G is the gravitational constant. The negative sign implies that the force is directed toward M.

(a) Show that F is conservative.

(b) Show that the work done by F as the mass m moves from $\mathbf{x}_0 = (x_0, y_0, z_0)$ to $\mathbf{x}_1 = (x_1, y_1, z_1)$ depends only on $\|\mathbf{x}_0\|$ and $\|\mathbf{x}_1\|$.

Solution (a)

Note that $\mathbf{x}_0 = (x_0, y_0, z_0)$ to $\mathbf{x}_1 = (x_1, y_1, z_1)$ are the position vectors of m with respect to the origin. Since M is located at the origin, thus $\|\mathbf{x}_0\|$ and $\|\mathbf{x}_1\|$ are the distances of m from M.

The domain of F is simply-connected except at the origin. So, we exclude the origin from the domain of F and use Theorem 7.8 to test if $\nabla \times \mathbf{F} = 0$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{GMmx}{\left(x^2+y^2+z^2\right)^{3/2}} & -\frac{GMmy}{\left(x^2+y^2+z^2\right)^{3/2}} & -\frac{GMmz}{\left(x^2+y^2+z^2\right)^{3/2}} \end{vmatrix}$$

$$= \mathbf{0}$$

```
Clear[F]

F[x_,y_,z_]:=-((G M m)/(x^2+y^2+z^2)^(3/2)){x,y,z}


Curl[F[x,y,z],{x,y,z},"Cartesian"]
```

Since F is conservative, by Theorem 7.8 there is a scalar-valued function V such that $\mathbf{F} = \nabla V$.

Using findScalarPotential3D we obtain

```
V(x,y,z)=(G M m)/Sqrt[x^2 + y^2 + z^2]+C,
```

where C is constant of integration.

findScalarPotential3D[F]

Since F is conservative, by Theorem 7.7 the work done from $\mathbf{x}_0 = (x_0, y_0, z_0)$ to $\mathbf{x}_1 = (x_1, y_1, z_1)$ is path-independent. Using equation (7.6) the work done is

$$\int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} \mathbf{F} \cdot d\mathbf{s} = V(x_1, y_1, z_1) - V(x_0, y_0, z_0)$$

$$= \frac{GmM}{\sqrt{x_1^2 + y_1^2 + z_1^2}} - \frac{GmM}{\sqrt{x_0^2 + y_0^2 + z_0^2}}$$

Since $\| \mathbf{x}_0 \| = \sqrt{x_0^2 + y_0^2 + z_0^2}$ and $\| \mathbf{x}_1 \| = \sqrt{x_1^2 + y_1^2 + z_1^2}$, we may write

$$\int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} \mathbf{F} \cdot d\mathbf{s} = \frac{GmM}{\|\mathbf{x}_1\|} - \frac{GmM}{\|\mathbf{x}_0\|}$$

.

The last equation implies that the work done by F as the object of mass m moves from $\mathbf{x}_0 = (x_0, y_0, z_0)$ to $\mathbf{x}_1 = (x_1, y_1, z_1)$ depends only on $\| \mathbf{x}_0 \|$ and $\| \mathbf{x}_1 \|$.

In equation (7.5) we have shown that if F is conservative, the circulation of F around every closed path is zero. Now, we might ask what happens when F is not conservative? For the case when the vector field is in the plane the answer is in a theorem known as Green's Theorem. In

Chapter 8 we shall study the case when the vector field is in space. The corresponding theorem is known as Stokes's Theorem.

7.2 Green's Theorem

Before presenting Green's Theorem we need some notation. Let C be a simple closed curve in $\mathbb{R}^2$ and let D be the plane region bounded by C. We say that C is a simple closed curve if it does not intersect itself. We denote the boundary curve C of the region D by ∂D , that is C= ∂D . We shall orient C so that D is on the left as one traverses C.

Let F(x,y)=M(x,y)i+N(x,y)j has continuous partial derivatives throughout D. If C is parametrized by c(t)=(x(t),y(t)) for a<=t<=b, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C M\,dx + N\,dy$$
$$= \int_a^b (M(x(t), y(t))\,\mathbf{i} + N(x(t), y(t))\,\mathbf{j}) \cdot \mathbf{c}'(t)\,dt$$

or

(7.12)

$$\int_C M\,dx + N\,dy = \int_a^b [M(x(t), y(t))\,x'(t) + N(x(t), y(t))\,y'(t)]\,dt$$

.

Theorem 7.9. Green's Theorem

Let D be a region in $\mathbb{R}^2$ whose boundary C= ∂D consists of finitely many simple, closed curves which have continuous derivatives. Orient the curves of C so that D is on the left as one traverses C. Then

(7.13)

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

.

When C consists of many closed curves, the symbol $\oint_C$ indicates that the line integral is taken over all these curves.

Green's Theorem tells us how to compute the double integral (a two-dimensional notion) over the plane region D by an appropriate vector line integral (a one-dimensional construction) around a closed curve C, the boundary of D.

Suppose that F is a conservative vector field. Then by Theorem 7.7, there is a potential function V such that $\mathbf{F} = \nabla V$. By Theorem 7.8,

$$\nabla \times \mathbf{F} = \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x, y) & N(x, y) & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

Equation (7.13) reduces to

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_C M\,dx + N\,dy = 0$$

.

Hence, Green's Theorem implies that if F(x,y)=M(x,y)i+N(x,y)j is conservative, then the closed path integral vanishes. Otherwise, the closed path integral around C is related to a double integral over the plane region D bounded by C.

Example 7.20 (a) Calculate the line integral directly

$\oint_C y\,dx + x^2 y\,dy$,

where C is the unit circle centered at (0,0).

(b) Verify the answer in part (a) by using Green's Theorem.

Solution (a)

The standard parametrization of the unit circle is

x=cos t, y=sin t.

$dx = -\sin t\,dt$, $dy = \cos t\,dt$.

So,

$$\oint_C y\,dx + x^2 y\,dy = \oint_C \sin t\,(-\sin t\,dt) + \cos^2 t \sin t\,(\cos t\,dt)$$
$$= \oint_C \left(-\sin^2 t + \cos^3 t \sin t\right) dt$$
$$= -\pi$$

Integrate[-Sin[t]^2+Cos[t]^3 Sin[t],{t,0,2π}]

We may verify the result using vectorLineInt2D module:

Clear[c,F]

c[t_]:={Cos[t],Sin[t]}

F[x_,y_]:={y,x^2 y}

vectorLineInt2D[c,F,{0,2π}]

plot[1]=ParametricPlot[c[t]//Evaluate,{t,0,2π},AxesLabel->
{"x","y"},PlotRange->{{-2,2},{-2,2}}];

plot[2]=VectorPlot[F[x,y]//Evaluate,{x,-2,2},{y,-2,2},VectorPoints->10,VectorScale->{0.2, 0.25, Automatic},VectorStyle->Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->500]

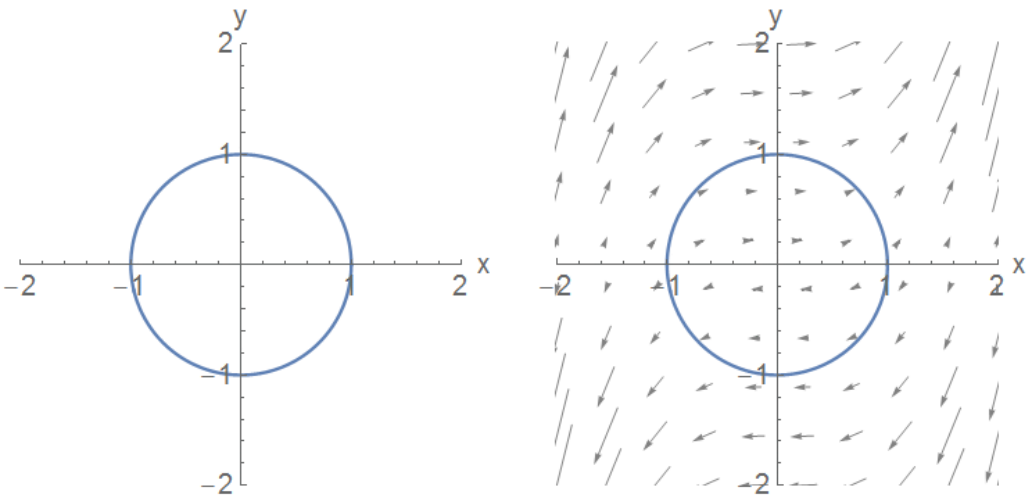


Figure 7.17. The line integral $\oint_C y\,dx + x^2\,y\,dy$ along the unit circle C.

Solution (b)

In this example, M(x,y)=y and N(x,y)=x^2 y. So,

$\frac{\partial N}{\partial x} = 2xy$, $\frac{\partial M}{\partial y} = 1$.

Using Green's Theorem (Theorem 7.9) we get

$$\oint_C y\,dx + x^2\,y\,dy = \iint_D (2xy - 1)\,dx\,dy$$

,

where D is the disk x^2+y^2<=1 enclosed by C.

So,

$$\oint_C y\,dx + x^2\,y\,dy = \iint_D (2xy - 1)\,dx\,dy$$

$$= \int_{x=-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2xy - 1)\,dy\,dx$$

$$= -\pi$$

Thus, the line integral and the double integral agree according to Green's Theorem.

Integrate[2x y-1,{x,-1,1},{y,-Sqrt[1-x^2],Sqrt[1-x^2]}]

Consider a special line integral: $\oint_C -y\,dx + x\,dy$.

Using Green's Theorem we get

$$\oint_C -y\,dx + x\,dy = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dx\,dy$$

$$= \iint_D 2\,dx\,dy$$

$$= 2 \cdot (\text{area of } D)$$

Therefore,

(7.14)

$$\text{Area enclosed by } C = \frac{1}{2}\left(\oint_C -y\,dx + x\,dy\right)$$

.

Thus, we can compute an enclosed area by using the line integral along the boundary. Usually the line integral is easier to consider than the double integral. It is the mathematical basis of the planimeter, a device that computes the area of an irregular shape when you trace the boundary with a pointer at the end of a movable arm. For more information about Green's Theorem and planimeters, see the article "As the Planimeter's Wheel Turns: Planimeter Proofs for Calculus Class" by Tanya Leise in The College Mathematics Journal, Vol. 38, No. 1 (Jan. 2007), pp. 24-31.

Example 7.21. The area of a cardioid

Let a be a positive constant. Use Green's Theorem to calculate the area enclosed by the cardioid:

cardioid[a](t)=(2a (1+cos t)cos t,2a (1+cos t)sin t), 0<=t<=2π.

(see [Tan], Section 2.3 Cardioid)

Solution:

According to Green's Theorem the area enclosed by the cardioid is

$$\text{Area} = \frac{1}{2}\left(\oint_C -y\,dx + x\,dy\right)$$

.

x=2a (1+cos t)cos t, y=2a (1+cos t)sin t,

$$dx = \{-2a\sin t\cos t - 2a(1+\cos t)\sin t\}\,dt,$$

$$dy = \{-2a\sin^2 t + 2a(1+\cos t)\cos t\}\,dt.$$

So,

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\left(\oint_C -y\,dx + x\,dy\right) \\
&= \frac{1}{2}\left[\int_0^{2\pi} -(2a(1+\cos t)\sin t)\{-2a\sin t\cos t - 2a(1+\cos t)\sin t\}\,dt\right. \\
&\qquad \left. + (2a(1+\cos t)\cos t)\{-2a\sin^2 t + 2a(1+\cos t)\cos t\}\,dt\right] \\
&= \frac{1}{2}\left[\int_0^{2\pi} 4a^2\sin^2 t\cos t(1+\cos t) + 4a^2\sin^2 t(1+\cos t)^2\right. \\
&\qquad \left. - 4a^2\sin^2 t\cos t(1+\cos t) + 4a^2\cos^2 t(1+\cos t)^2\right]dt \\
&= \frac{1}{2}\cdot\int_0^{2\pi} 4a^2(1+\cos t)^2\,dt \\
&= 6\pi a^2
\end{aligned}$$

```
Clear[a]

2*Integrate[a^2 (1+Cos[t])^2,{t,0,2π}]


cardioid[a_][t_]:={2a(1+Cos[t])Cos[t],2a(1+Cos[t])Sin[t]}


ParametricPlot[cardioid[a][t],{t,0,2π},{a,0,1},Frame->None,ImageSize->200]
```

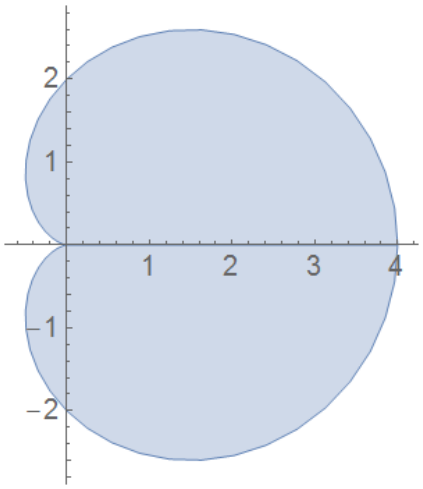


Figure 7.18. The area enclosed by the cardioid[1].

Example 7.22. Verify Green's Theorem for the following vector field:

F=E^(2x+y) i+E^-y j,

where D is the square with vertices (1,1), (-1,1), (-1,-1), and (1,-1) (see Figure 7.19).

Solution:

According to Green's Theorem (equation (7.13)):

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

.

First we compute the left-hand side by orienting the path C counterclockwise as stipulated by the statement of Green's Theorem.

The path C= ∂D consists of 4 line segments. So,

$$\oint_{\partial D} M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy + \int_{C_3} M\,dx + N\,dy + \int_{C_4} M\,dx + N\,dy$$

In this example:

$$M(x, y) = e^{2x+y}, \quad N(x, y) = e^{-y}.$$

First path C1: the line segment from vertices (1,1) to (-1,1) where y=1 is constant.

$$\int_{x=1}^{-1} M\,dx + N\,dy = \int_{x=1}^{-1} e^{2x+1}\,dx = \frac{1-e^4}{2e}$$

Integrate[E^(2x+1),{x,1,-1}]

Second path C2: the line segment from (-1,1) to (-1,-1) where x=-1 is constant.

$$\int_{y=1}^{-1} M\,dx + N\,dy = \int_{y=1}^{-1} e^{-y}\,dy = -2\sinh 1$$

Integrate[E^-y,{y,1,-1}]

Third path C3: the line segment from (-1,-1) to (1,-1) where y=-1 is constant.

$$\int_{x=-1}^{1} M\,dx + N\,dy = \int_{x=-1}^{1} e^{2x-1}\,dx$$
$$= \frac{-1+e^4}{2e^3}$$

Integrate[E^(2x-1),{x,-1,1}]

Fourth path C4: the line segment from (1,-1) to (1,1) where x=1 is constant.

$$\int_{y=-1}^{1} M\,dx + N\,dy = \int_{y=-1}^{1} e^{-y}\,dy$$
$$= 2\sinh 1$$

Integrate[E^-y,{y,-1,1}]

So,

$$\oint_{\partial D} M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$
$$+ \int_{C_3} M\,dx + N\,dy + \int_{C_4} M\,dx + N\,dy$$
$$= \frac{1-e^4}{2e} - 2\sinh 1 + \frac{-1+e^4}{2e^3} + 2\sinh 1$$
$$= \frac{1-e^4}{2e} + \frac{-1+e^4}{2e^3}$$
$$= \frac{-1+e^2+e^4-e^6}{2e^3}$$

Clear[F]

F[x_,y_]:={E^(2x+y),E^-y}

p[1]={1,1};

```
p[2]={-1,1};

p[3]={-1,-1};

p[4]={1,-1};

plot[1]=VectorPlot[F[x,y],{x,-2,2},{y,-2,2},VectorPoints->20,VectorScale->
{0.5, 0.1, Automatic},VectorStyle->Gray,Frame->False,Axes-
>True,AxesLabel->{"x","y"},PlotRange->{{-2,2},{-2,2}}];

plot[2]=Graphics[{PointSize[0.02],Red,Point[Array[p,4]]}];

plot[3]=Graphics[{Blue,Arrow[{{p[1],p[2]},{p[2],p[3]},{p[3],p[4]},
{p[4],p[1]}}],Text["C1",{0.2,0.8}],Text["C2",{-0.8,0.2}],Text["C3",
{-0.2,-0.8}],Text["C4",{0.8,-0.2}]}];

plot[4]=Show[Array[plot,3]]
```

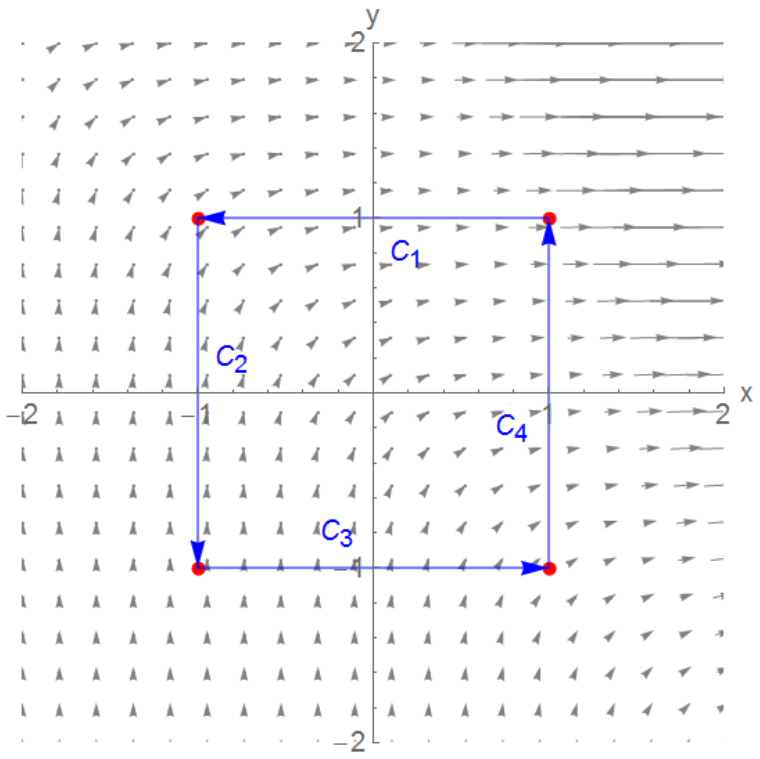


Figure 7.19. The path C around the square D in the vector field F=E^(2x+y) i+E^-y j.

Next, we compute the right-hand side of equation (7.13):

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_{x=-1}^{1}\int_{y=-1}^{1} \left(-e^{2x+y}\right) dx\, dy$$

$$= -\frac{\left(-1+e^2\right)^2\left(1+e^2\right)}{2e^3}$$

$$= \frac{-1+e^2+e^4-e^6}{2e^3}$$

which checks with the line integral:

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \frac{-1+e^2+e^4-e^6}{2e^3}$$

Integrate[-E^(2x+y),{x,-1,1},{y,-1,1}]

Example 7.23. Let D be the first quadrant region bounded by the line y=x and the parabola y=x^2. Verify Green's Theorem for the vector field F=x^3 i+x^2 j.

Solution:

Clear[F]

F[x_,y_]:={x^3,x^2}

plot[1]=Plot[{x^2,x},{x,0,1},AspectRatio->1];

```
plot[2]=VectorPlot[F[x,y],{x,-2,2},{y,-2,2},VectorPoints->40,VectorScale->
{0.2, 0.2, Automatic}];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->500]
```

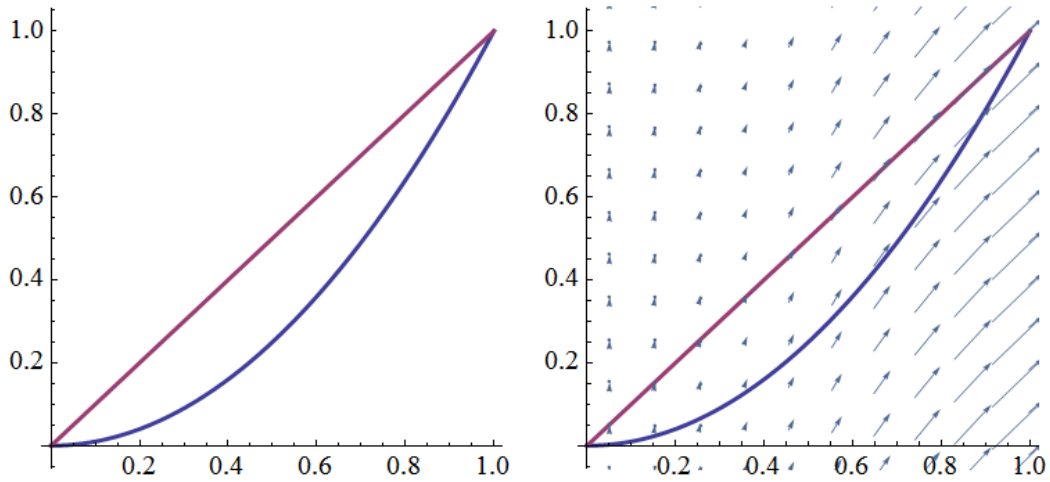


Figure 7.20. The region D, bounded by the line y=x and the parabola y=x^2 in the vector field F=x^3 i+x^2 j.

According to Green's Theorem (Theorem 7.9):

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

,

where in this example: M(x,y)=x^3 and N(x,y)=x^2.

The boundary C consists of two curves C1 and C2, parametrized (respectively) by

C1: c1(t)=(t,t^2), for 0<=t<=1.

C2: c2(t)=(t,t), for 0<=t<=1.

The region D and its boundary ∂D is oriented according to Green's Theorem, so the integrals become

$$\oint_{\partial D} M\,dx + N\,dy = \oint_C \mathbf{F}\cdot d\mathbf{s} = \int_{C_1} \mathbf{F}\cdot d\mathbf{s} + \int_{C_2} \mathbf{F}\cdot d\mathbf{s}$$
$$= \int_0^1 \left(t^3\,\mathbf{i} + t^2\,\mathbf{j}\right)\cdot \mathbf{c}_1'\,dt + \int_1^0 \left(t^3\,\mathbf{i} + t^2\,\mathbf{j}\right)\cdot \mathbf{c}_2'\,dt$$
$$= \int_0^1 \left(t^3\,\mathbf{i} + t^2\,\mathbf{j}\right)\cdot (\mathbf{i} + 2\,t\,\mathbf{j})\,dt + \int_1^0 \left(t^3\,\mathbf{i} + t^2\,\mathbf{j}\right)\cdot (\mathbf{i} + \mathbf{j})\,dt$$
$$= \int_0^1 3\,t^3\,dt + \int_1^0 \left(t^3 + t^2\right)dt$$
$$= \frac{3}{4}\left[t^4\right]_0^1 + \left[\frac{1}{4}\,t^4 + \frac{1}{3}\,t^3\right]_1^0 = \frac{3}{4} - \frac{1}{4} - \frac{1}{3}$$
$$= \frac{1}{6}$$

We might want to use the vectorLineInt2D module to check our result:

```
Clear[c1,c2]

c1[t_]:={t,t^2}

c2[t_]:={t,t}


vectorLineInt2D[c1,F,{0,1}]+vectorLineInt2D[c2,F,{1,0}]
```

Whereas, the right-hand side of equation (7.13):

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^1 \int_{y=x^2}^x 2\,x\,dy\,dx$$
$$= 2\int_0^1 x\left(x - x^2\right)dx$$
$$= 2\left[\frac{1}{3}\,x^3 - \frac{1}{4}\,x^4\right]_0^1 = \frac{1}{6}$$

The two integrals agree:

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{1}{6}$$

.

In this example, the double integral is easier to evaluate than the line integral.

Example 7.24. Find the area inside the quadrilateral using Green's Theorem whose vertices taken counterclockwise are (2,0), (4,1), (1,2) and (-1,1).

Solution:

```
point[1]={2,0};

point[2]={4,1};

point[3]={1,2};

point[4]={-1,1};

plot[1]=Graphics[{Red,PointSize[0.02],Point[Array[point,4]]}];

plot[2]=Graphics[{Arrow[{point[1],point[2]}],Arrow[{point[2],point[3]}],A
rrow[{point[3],point[4]}],Arrow[{point[4],point[1]}]}];

Show[Array[plot,2],Axes->True,AxesOrigin->{0,0}]
```

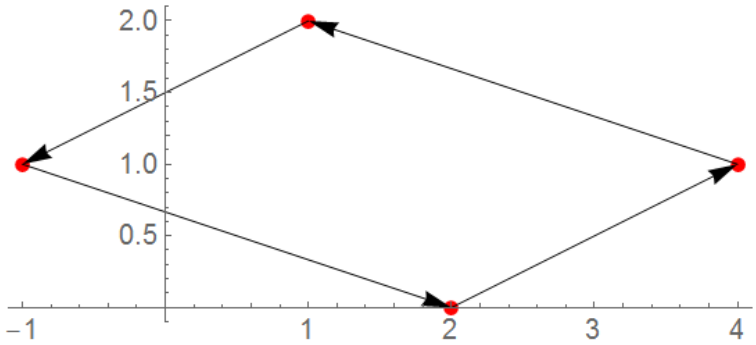


Figure 7.21. The quadrilateral whose vertices: (2,0), (4,1), (1,2) and (-1,1) are oriented counterclockwise.

The path C consists of four line segments parametrized by

C1: c1(t)=(2t+2,t), for 0<=t<=1.

C2: c2(t)=(-3t+4,t+1), for 0<=t<=1.

C3: c3(t)=(-2t+1,-t+2), for 0<=t<=1.

C4: c4(t)=(3t-1,-t+1), for 0<=t<=1.

According to Green's Theorem for area, equation (7.14):

$$\text{Area enclosed by } C = \frac{1}{2}\left(\oint_C -y\,dx + x\,dy\right)$$

.

```
Clear[c1,c2,c3,c4,F]

c1[t_]:={2t+2,t}

c2[t_]:={-3t+4,t+1}
```

```
c3[t_]:={-2t+1,-t+2}

c4[t_]:={3t-1,-t+1}

F[x_,y_]:=1/2 {-y,x}


plot[1]=ParametricPlot[{c1[t],c2[t],c3[t],c4[t]},{t,0,1},PlotStyle-
>Blue,Epilog->{Red,Text["C1",{3.2,0.3}],Text["C2",{2.3,1.8}],Text["C3",
{0.3,1.9}],Text["C4",{0.4,0.3}]}];

plot[2]=VectorPlot[F[x,y],{x,-1,4},{y,0,2},VectorPoints->8,VectorScale->
{0.3, 0.2, Automatic},VectorStyle->Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->650]
```

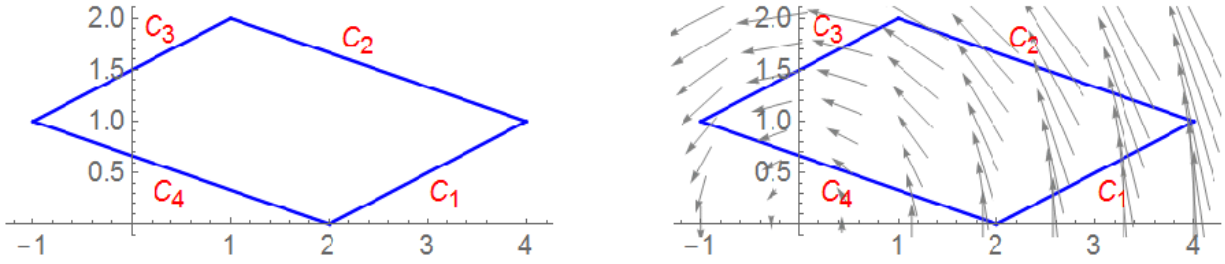


Figure 7.22. The closed path C forms a quadrilateral in the vector field F=1/2 (-y i+x j).

The area of the quadrilateral consists of four line integrals:

$$\text{Area} = \frac{1}{2}\oint_C -y\,dx + x\,dy$$
$$= \frac{1}{2}\int_{C_1} -y\,dx + x\,dy + \frac{1}{2}\int_{C_2} -y\,dx + x\,dy$$
$$+ \frac{1}{2}\int_{C_3} -y\,dx + x\,dy + \frac{1}{2}\int_{C_4} -y\,dx + x\,dy$$

Along the path C1:

$$\frac{1}{2}\int_{C_1} -y\,dx + x\,dy = \frac{1}{2}\left[\int_{t=0}^{1} -t\,(2)\,dt + \int_{t=0}^{1}(2t+2)\,dt\right]$$
$$= -\left[\frac{1}{2}t^2\right]_0^1 + \left[\frac{1}{2}t^2 + t\right]_0^1 = -\frac{1}{2} + \frac{3}{2}$$
$$= 1$$

(* Compute the line integral along C1 *)

vectorLineInt2D[c1,F,{0,1}]

Along the path C2:

$$\frac{1}{2}\int_{C_2} -y\,dx + x\,dy = \frac{1}{2}\left[\int_{t=0}^{1} -(t+1)\,(-3)\,dt + \int_{t=0}^{1}(-3t+4)\,dt\right]$$
$$= \frac{3}{2}\left[\frac{1}{2}t^2 + t\right]_0^1 + \frac{1}{2}\left[-\frac{3}{2}t^2 + 4t\right]_0^1$$
$$= \frac{3}{2}\left[\frac{3}{2}\right] + \frac{1}{2}\left[-\frac{3}{2} + 4\right] = \frac{9}{4} + \frac{5}{4} = \frac{14}{4}$$
$$= \frac{7}{2}$$

(* Compute the line integral along C2 *)

vectorLineInt2D[c2,F,{0,1}]

Along the path C3:

$$\frac{1}{2}\int_{C_3} -y\,dx + x\,dy = \frac{1}{2}\left[\int_{t=0}^{1} -(-t+2)\,(-2)\,dt + \int_{t=0}^{1}(-2t+1)\,(-1)\,dt\right]$$
$$= \left[-\frac{1}{2}t^2 + 2t\right]_0^1 - \left[-t^2 + t\right]_0^1 = \left[-\frac{1}{2} + 2\right] - 0$$
$$= \frac{3}{2}$$

(* Compute the line integral along C3 *)

vectorLineInt2D[c3,F,{0,1}]

Along the path C4:

$$\frac{1}{2}\int_{C_4} -y\,dx + x\,dy = \frac{1}{2}\left[\int_{t=0}^{1} -(-t+1)(3)\,dt + \int_{t=0}^{1}(3t-1)(-1)\,dt\right]$$
$$= -\frac{3}{2}\left[-\frac{1}{2}t^2 + t\right]_0^1 - \frac{1}{2}\left[\frac{3}{2}t^2 - t\right]_0^1$$
$$= -\frac{3}{2}\left[-\frac{1}{2}+1\right] - \frac{1}{2}\left[\frac{3}{2}-1\right] = -\frac{3}{4} - \frac{1}{4} = -\frac{4}{4}$$
$$= -1$$

(* Compute the line integral along C4 *)

vectorLineInt2D[c4,F,{0,1}]

So,

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$
$$= \frac{1}{2}\int_{C_1} -y\,dx + x\,dy + \frac{1}{2}\int_{C_2} -y\,dx + x\,dy$$
$$+ \frac{1}{2}\int_{C_3} -y\,dx + x\,dy + \frac{1}{2}\int_{C_4} -y\,dx + x\,dy$$
$$= 1 + \frac{7}{2} + \frac{3}{2} - 1$$
$$= 5$$

Example 7.25. (a) Evaluate the line integral $\oint_C 5y\,dx - 3x\,dy$ directly, where C is the cardioid with polar coordinate equation r=1+cos θ, oriented counterclockwise.

(b) Redo part (a) using Green's Theorem.

Solution (a)

As can be seen from Figure 7.23, the cardioid starts from $\theta=0$ at (2,0) and along the path counterclockwise to $\theta=2\pi$.

The parametric equation of the cardioid is

C: $c(\theta)=((1+\cos\theta)\cos\theta,(1+\cos\theta)\sin\theta), \quad 0<=\theta<=2\pi.$

Note that this is the cardioid given in Example 7.21 with $a=1/2$. Therefore, the area is $3\pi/2$.

```
Clear[c,F]

c[θ_]:={(1+Cos[θ])Cos[θ],(1+Cos[θ])Sin[θ]}

F[x_,y_]:={5y,-3x}

ParametricPlot[c[t],{t,0,2π},ImageSize->Small]
```

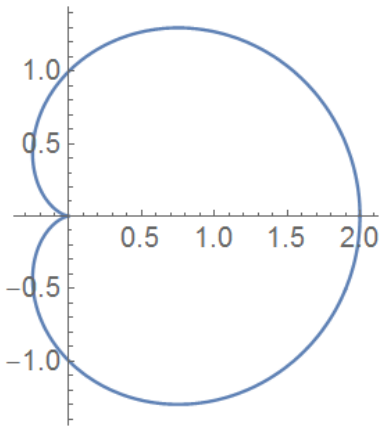


Figure 7.23. The cardioid $r=1+\cos\theta$.

Evaluate the line integral directly:

$$\oint_C 5\,y\,dx - 3\,x\,dy = \int_{t=0}^{2\pi} 5\,[(1+\cos\theta)\sin\theta]\,(-\sin\theta - 2\cos\theta\sin\theta)\,d\theta$$
$$- \int_{t=0}^{2\pi} 3\,[(1+\cos\theta)\cos\theta]\,(\cos\theta - \sin^2\theta + \cos^2\theta)\,d\theta$$
$$= -12\,\pi$$

Integrate[5((1+Cos[θ])Sin[θ])(-Sin[θ]-2Cos[θ]Sin[θ]),{θ,0,2π}]-
Integrate[3((1+Cos[θ])Cos[θ])(Cos[θ]-Sin[θ]^2+Cos[θ]^2),{θ,0,2π}]

We may employ the vectorLineInt2D module to compute the line integral:

vectorLineInt2D[c,F,{0,2π}]

plot[1]=PolarPlot[1+Cos[θ],{θ,0,2π}];

plot[2]=VectorPlot[F[x,y],{x,-2,2},{y,-2,2},VectorPoints->10,VectorScale->
{0.2, 0.2, Automatic},VectorStyle->Green];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->500]

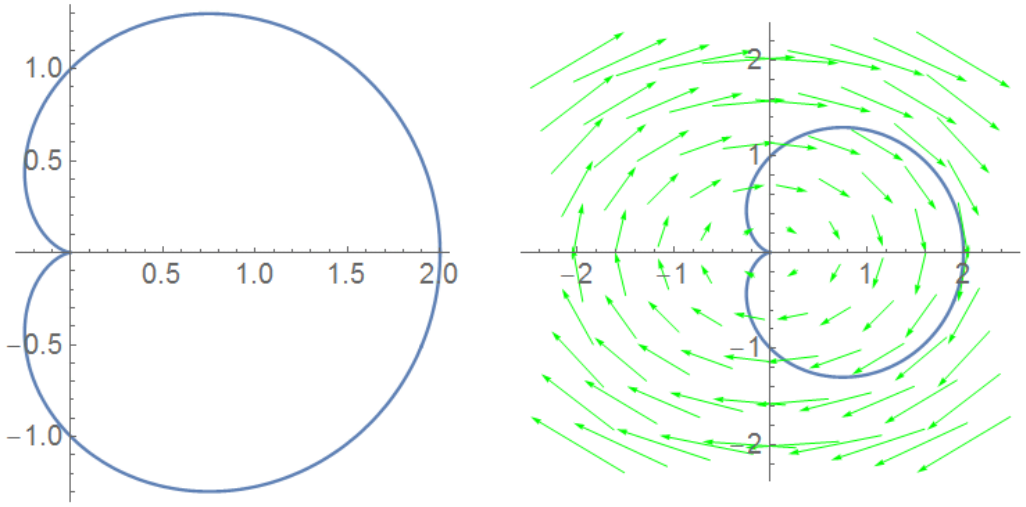


Figure 7.24. The cardioid r=1+cos θ in the vector field F=5y i-3x j.

Solution (b)

Using Green's Theorem, equation (7.13):

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

.

In this example M(x,y)=5y and N(x,y)=-3x. So,

$$\oint_C 5y\,dx - 3x\,dy = \iint_D (-3-5)\,dx\,dy = -8\iint_D dx\,dy$$
$$= -8 \times \text{area} = -8 \times \frac{3\pi}{2}$$
$$= -12\pi$$

Alternatively, we may evaluate this integral using polar coordinates:

$$\oint_C 5y\,dx - 3x\,dy = -8\int_{\theta=0}^{2\pi}\int_{r=0}^{1+\cos\theta} r\,dr\,d\theta$$
$$= -12\pi$$

-8*Integrate[r,{θ,0,2π},{r,0,1+Cos[θ]}]

Green's Theorem has other form that uses the curl of F. This form resembles Stokes's Theorem in $\mathbb{R}^3$ which we shall study in the next chapter.

Consider the vector field F=M(x,y)i+N(x,y)j. Let D be a region in $\mathbb{R}^2$ to which Green's Theorem applies and F has continuous partial derivatives. We can define F in $\mathbb{R}^3$ by setting its k-component to be zero. So that we may compute the curl of F:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y) & N(x,y) & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

.

Since k·k=1, we get

$$(\nabla \times \mathbf{F})\cdot\mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$.

Therefore, Green's theorem may be written as

(7.15)

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \oint_C M\,dx + N\,dy = \iint_D (\nabla \times \mathbf{F})\cdot\mathbf{k}\,dA$$

.

Note that k is a unit normal to the region D. The direction of k is determined by the right-hand rule so that D is on the left as one traverses C. Equation (7.15) tells us that we can compute the line integral of F over a simple closed curve C by integrating over region D the component of $\nabla \times \mathbf{F}$ in the direction of the unit normal k to the enclosed region D.

Green's Theorem has another form that uses the divergence of F. This form resembles Gauss's Theorem in $\mathbb{R}^3$ which we shall study in the next chapter.

Let F=M(x,y)i+N(x,y)j. Assume once again that D be a region in $\mathbb{R}^2$ to which Green's Theorem applies and F has continuous partial derivatives. Let n denote the unit vector that is perpendicular to C and points away from D.

If C is parametrized by c(t)=(x(t),y(t)) for a<=t<=b, then n is always perpendicular to the unit tangent vector c'(t)/||c'(t)||. So, we obtain

(7.16)

$$\mathbf{n} = \frac{y'(t)\,\mathbf{i} - x'(t)\,\mathbf{j}}{\|\mathbf{c}'(t)\|} = \frac{y'(t)\,\mathbf{i} - x'(t)\,\mathbf{j}}{\sqrt{x'(t)^2 + y'(t)^2}}$$

Notice that

$$\frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} \cdot \mathbf{n} = 0$$

.

Write $\tilde{\mathbf{F}} \equiv -N(x, y)\,\mathbf{i} + M(x, y)\,\mathbf{j}$.

Applying Green's Theorem to $\tilde{\mathbf{F}}$ we get

$$\oint_{\partial D}\tilde{\mathbf{F}}\cdot d\mathbf{s}=\oint_C -N\,dx+M\,dy=\iint_D\left(\frac{\partial M}{\partial x}-\frac{\partial(-N)}{\partial y}\right)dx\,dy$$
$$=\iint_D\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dx\,dy$$
$$=\iint_D \nabla\cdot\mathbf{F}\,dA$$

Now consider the line integral on the left side $\oint_{\partial D}\tilde{\mathbf{F}}\cdot d\mathbf{s}$:

$$\oint_{\partial D}\tilde{\mathbf{F}}\cdot d\mathbf{s}=\int_a^b[-N(x(t),y(t))\,\mathbf{i}+M(x(t),y(t))\,\mathbf{j}]\cdot\mathbf{c}'(t)\,dt$$
$$=\int_a^b[-N(x(t),y(t))\,\mathbf{i}+M(x(t),y(t))\,\mathbf{j}]\cdot(x'(t)\,\mathbf{i}+y'(t)\,\mathbf{j})\,dt$$
$$=\int_a^b[-N(x(t),y(t))\,x'(t)+M(x(t),y(t))\,y'(t)]\,dt$$
$$=\int_a^b\left[M(x(t),y(t))\,\frac{y'(t)}{\|\mathbf{c}'(t)\|}+N(x(t),y(t))\,\frac{-x'(t)}{\|\mathbf{c}'(t)\|}\right]\|\mathbf{c}'(t)\|\,dt$$

Since F=M(x,y)i+N(x,y)j, $\mathbf{n}=\frac{y'(t)\,\mathbf{i}-x'(t)\,\mathbf{j}}{\|\mathbf{c}'(t)\|}$ and $ds=\|\mathbf{c}'(t)\|\,dt$, we obtain

$$\oint_{\partial D}\tilde{\mathbf{F}}\cdot d\mathbf{s}=\oint_{\partial D}\mathbf{F}\cdot\mathbf{n}\,ds$$.

Thus, we have

(7.17)

$$\oint_{\partial D}\mathbf{F}\cdot\mathbf{n}\,ds=\iint_D\nabla\cdot\mathbf{F}\,dA$$

.

This form of Green's Theorem (equation 7.17) is sometimes called the "Divergence Theorem in the Plane". The line integral on the left side of equation (7.17) is called the flux of vector field F across C. If F represents the velocity of the flow of a fluid in D, the flux is the sum total of the scalar component of the fluid's velocity in the direction of the outward normal along the boundary curve C= ∂D **.**

Example 7.26. Let F=(2x+y^3) i+(3y-x^4) j represents the velocity of the flow of fluid in the disk D={(x,y)|x^2+y^2<1}. Use the divergence theorem (equation 7.17) to find the flux across the boundary C= ∂D .

Solution:

The divergence of F=M(x,y)i+N(x,y)j=(2x+y^3) i+(3y-x^4) j gives

$$\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2 + 3 = 5$$

.

In this example, the boundary is the unit circle, x^2+y^2=1.

So, the flux across this circle is

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA = 5 \iint_D dA$$
$$= 5 \cdot (\text{area of the unit circle}) = 5\pi$$

.

The flux is positive, therefore more fluid is flowing out of the region than into it (this means that there is a source in the region). This is illustrated in Figure 7.25. If the flux is negative, then there is more fluid flowing into the region than out of it (indicating that there is a sink in the region).

Clear[F]

```
F[x_,y_]:={2x+y^3,3y-x^4}


plot[1]=Show[Graphics[{EdgeForm[Blue],Yellow,Disk[{0,0},1]}],Axes->True,PlotRange->{{-2,2},{-2,2}}];

plot[2]=VectorPlot[F[x,y],{x,-2,2},{y,-2,2},VectorPoints->10,VectorScale->{0.3, 0.2, Automatic},VectorStyle->Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->500]
```

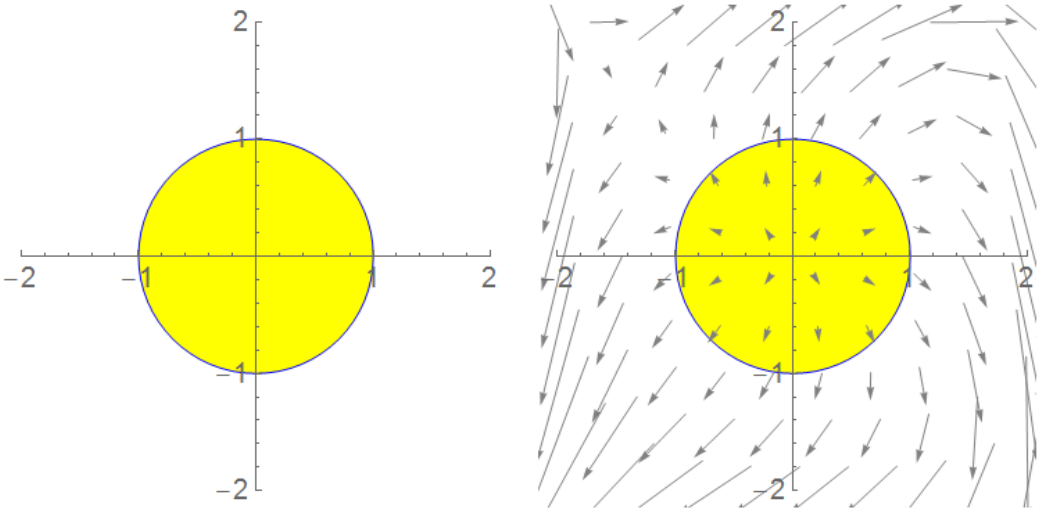


Figure 7.25. Total (net) flux flow is out of the disk x^2+y^2<1.

Example 7.27. Compute the flux of F=cos y i+sin y j across the region D that is shown in Figure 7.26.

Solution:

```
Clear[F]

F[x_,y_]:={Cos[y],Sin[y]}


plot[1]=Plot[1-x,{x,0,1},PlotRange->{{0,1.2},{0,1.2}},Epilog->Text["y=1-
x",{0.35,0.5}],AspectRatio->1];

plot[2]=Graphics[{Text["x^2+y^2=1",
{0.8,0.82}],Blue,Thickness[0.005],Circle[{0,0},1,{0,π/2}]}];

plot[3]=VectorPlot[F[x,y],{x,0,1.2},{y,0,1.2},VectorPoints->8,VectorScale->
{0.3, 0.2, Automatic},VectorStyle->Gray];

plot[4]=Show[Array[plot,2]];

plot[5]=Show[Array[plot,3]];

Show[GraphicsGrid[{{plot[4],plot[5]}}],ImageSize->500]
```

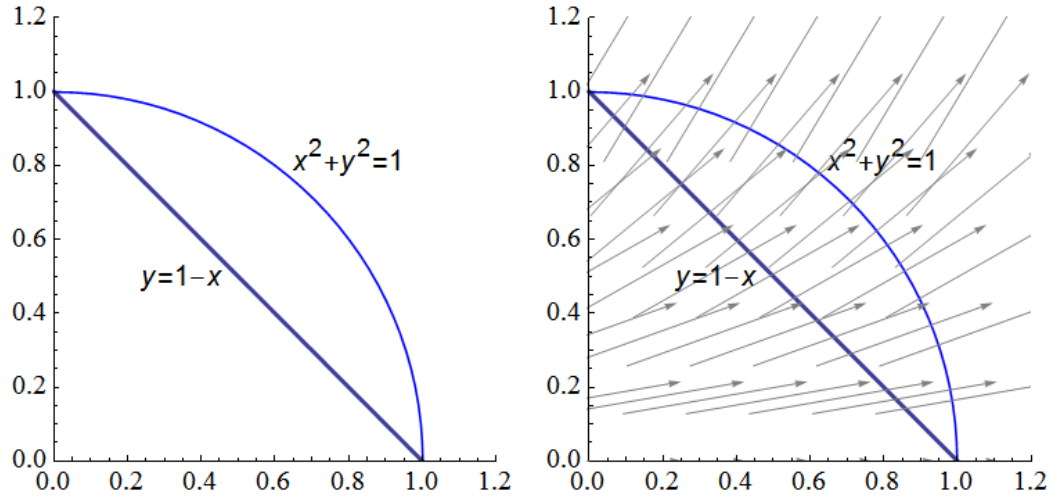


Figure 7.26. D is the region in the first quadrant bounded by the line y=1-x and the circular arc of unit radius, in the vector field F=cos y i+sin y j.

The flux of F across region D is defined by $\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds$ **.**

The boundary curve C consists of two paths:

1. the circular arc C1: c1(t)=cos t i+sin t j, 0<=t<=π/2; and

2. the line segment C2: c2(t)=t i+(1-t)j, 0<=t<=1.

Orienting the curve counterclockwise, we have

$$\text{flux} = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{C_1} \mathbf{F} \cdot \mathbf{n}_1\, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{n}_2\, ds$$

.

Consider the first line integral along the path C1:

$$\int_{C_1} \mathbf{F} \cdot \mathbf{n}_1\, ds = \int_0^{\pi/2} (\cos y\, \mathbf{i} + \sin y\, \mathbf{j}) \cdot \frac{y'(t)\, \mathbf{i} - x'(t)\, \mathbf{j}}{\|\mathbf{c}_1'(t)\|} \|\mathbf{c}_1'(t)\|\, dt$$

$$\int_{C_1} \mathbf{F} \cdot \mathbf{n}_1\, ds = \int_0^{\pi/2} (\cos(\sin t)\, \mathbf{i} + \sin(\sin t)\, \mathbf{j}) \cdot (\cos t\, \mathbf{i} + \sin t\, \mathbf{j})\, dt$$

$$= \int_0^{\pi/2} [(\cos(\sin t)) \cos t + (\sin(\sin t)) \sin t]\, dt$$

$$\simeq 1.5327$$

The analytical solution involves the Bessel function of the first kind.
However, for this example we are more interested in the numerical solution.

Integrate[Cos[Sin[t]]Cos[t]+Sin[Sin[t]]Sin[t],{t,0,π/2}]

NIntegrate[Cos[Sin[t]]Cos[t]+Sin[Sin[t]]Sin[t],{t,0,π/2}]

Next, we evaluate the second line integral along the path C2:

$$\int_{C_2} \mathbf{F}\cdot\mathbf{n}_2\, ds = \int_0^1 (\cos y\,\mathbf{i} + \sin y\,\mathbf{j})\cdot \frac{y'(t)\,\mathbf{i} - x'(t)\,\mathbf{j}}{\|\mathbf{c}_2'(t)\|}\,\|\mathbf{c}_2'(t)\|\,dt$$

$$\int_{C_2} \mathbf{F}\cdot\mathbf{n}_2\, ds = \int_0^1 (\cos(1-t)\,\mathbf{i} + \sin(1-t)\,\mathbf{j})\cdot(-\mathbf{i}-\mathbf{j})\,dt$$

$$= \int_0^1 [-\cos(1-t) - \sin(1-t)]\,dt$$

$$\simeq -1.30117$$

Integrate[-Cos[1-t]-Sin[1-t],{t,0,1}]

NIntegrate[-Cos[1-t]-Sin[1-t],{t,0,1}]

Thus,

$$\text{flux} = \oint_{\partial D} \mathbf{F}\cdot\mathbf{n}\, ds = \int_{C_1} \mathbf{F}\cdot\mathbf{n}_1\, ds + \int_{C_2} \mathbf{F}\cdot\mathbf{n}_2\, ds$$

$$\simeq 1.5327 - 1.30117$$

$$\simeq 0.23153$$

Since the flux is positive, there is more fluid flowing out of the region than into it.

Alternatively we may evaluate the flux by using equation (7.17):

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{n}\, ds = \iint_D \nabla\cdot\mathbf{F}\, dA$$

.

Let us evaluate the right-hand side of equation (7.17) and at the same time to verify our result obtained above.

The divergence of F is

$$\nabla \cdot \mathbf{F} = \frac{\partial(\cos y)}{\partial x} + \frac{\partial(\sin y)}{\partial y} = \cos y$$

.

Div[F[x,y],{x,y},"Cartesian"]

$$\iint_D \nabla \cdot \mathbf{F}\, dA = \int_{x=0}^{1} \int_{y=1-x}^{\sqrt{1-x^2}} \cos y \, dy \, dx$$

$$= \int_0^1 [\sin y]_{1-x}^{\sqrt{1-x^2}} \, dx$$

$$= \int_0^1 \left(\sin\left(\sqrt{1-x^2}\right) - \sin(1-x) \right) dx$$

$$\simeq 0.231532$$

NIntegrate[Sin[Sqrt[1-x^2]]-Sin[1-x],{x,0,1}]

Example 7.28. Verify Green's Theorem for F=(1+x^2+y^2)^(-1/2) i+ (1+x^2+y^2)^(1/2) j and the region D={ (x,y)|(x-1)^2

Solution:

Clear[F]

F[x_,y_]:={(1+x^2+y^2)^(-1/2),(1+x^2+y^2)^(1/2)}

plot[1]=ContourPlot[{y==(x-1)^2,y==1-x+x^2-x^5},{x,0,1}, {y,0,1},Epilog->{Text["y=(x-1)^2",{0.2,0.4}],Text["y=1-x+x^2-x^5", {0.8,0.7}]}];

plot[2]=VectorPlot[F[x,y],{x,0,1},{y,0,1},VectorPoints->8,VectorScale->{0.3, 0.2, Automatic},VectorStyle->Gray];

plot[3]=Show[Array[plot,2]];

Show[GraphicsGrid[{{plot[1],plot[3]}}],ImageSize->600]

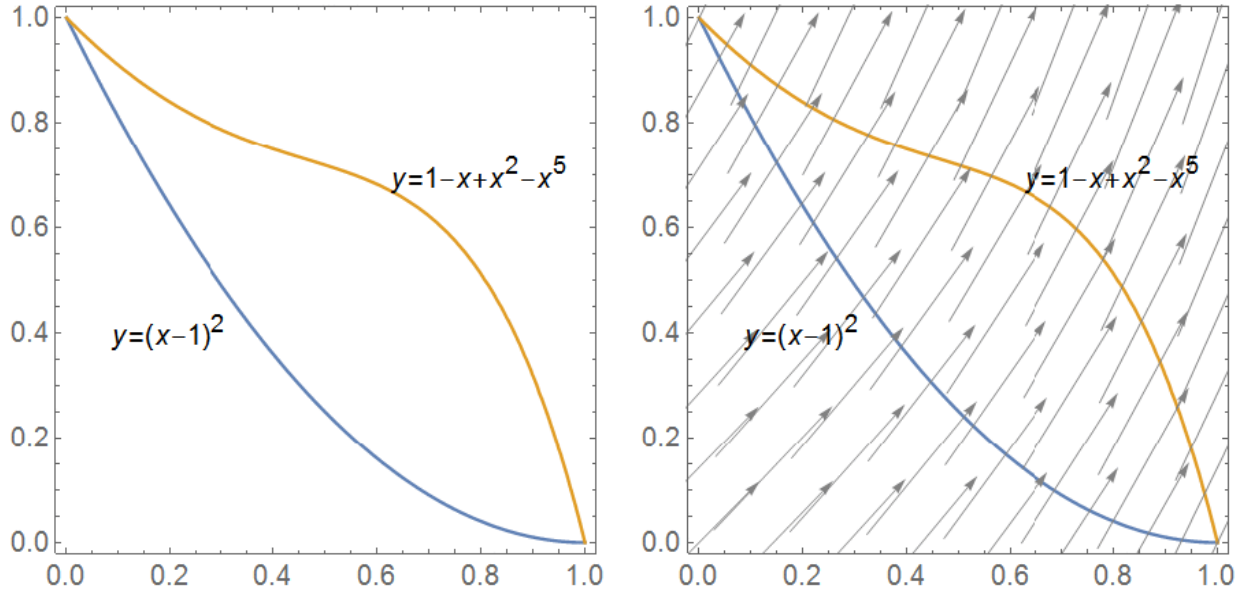


Figure 7.27. The region D={ (x,y)|(x-1)^2

According to Green's Theorem (Theorem 7.9):

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

In this example:

M(x,y)=(1+x^2+y^2)^(-1/2), N(x,y)=(1+x^2+y^2)^(1/2)

The boundary curve C= ∂D consists of two paths:

C1={ (x,y)|y=1-x+x^2-x^5,0<=x<=1 }

and

C2={ (x,y)|y=(x-1)^2,0<=x<=1 }

Orienting the curve counterclockwise:

$$\oint_{\partial D} M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$

Consider the first line integral on the right:

$$\begin{aligned}
&\int_{C_1} M\,dx + N\,dy \\
&\quad = \int_1^0 \left(1 + x^2 + \left(1 - x + x^2 - x^5\right)^2\right)^{-1/2} dx \\
&\qquad + \left(1 + x^2 + \left(1 - x + x^2 - x^5\right)^2\right)^{1/2} d\left(1 - x + x^2 - x^5\right) \\
&\quad = \int_1^0 \left[\left(1 + x^2 + \left(1 - x + x^2 - x^5\right)^2\right)^{-1/2}\right. \\
&\qquad \left. + \left(-1 + 2x - 5x^4\right)\left(1 + x^2 + \left(1 - x + x^2 - x^5\right)^2\right)^{1/2}\right] dx \\
&\quad \simeq 0.630702
\end{aligned}$$

NIntegrate[(1+x^2+(1-x+x^2-x^5)^2)^(-1/2)+(-1+2x-5x^4)(1+x^2+(1-x+x^2-x^5)^2)^(1/2),{x,1,0}]

Next, the second line integral on the right:

$$\begin{aligned}
&\int_{C_2} M\,dx + N\,dy \\
&\quad = \int_0^1 \left(1 + x^2 + (x-1)^4\right)^{-1/2} dx + \left(1 + x^2 + y^2\right)^{1/2} d\left((x-1)^2\right) \\
&\quad = \int_0^1 \left[\left(1 + x^2 + (x-1)^4\right)^{-1/2} + 2(x-1)\left(1 + x^2 + (x-1)^4\right)^{1/2}\right] dx \\
&\quad \simeq -0.408899
\end{aligned}$$

NIntegrate[(1+x^2+(x-1)^4)^(-1/2)+2(x-1)(1+x^2+(x-1)^4)^(1/2),{x,0,1}]

Thus, we have

$$\oint_{\partial D} M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$
$$\simeq 0.630702 - 0.408899$$
$$\simeq 0.221803$$

Now, we evaluate the double integrals in Green's Theorem:

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \int_{y=(x-1)^2}^{1-x+x^2-x^5} \left(\frac{x}{\sqrt{1+x^2+y^2}} + \frac{y}{\left(1+x^2+y^2\right)^{3/2}}\right) dy\,dx$$
$$\simeq 0.221803$$

Thus, we have verified Green's Theorem:

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \simeq 0.221803$$

NIntegrate[x/(1+x^2+y^2)^(1/2)+y/(1+x^2+y^2)^(3/2),{x,0,1},{y,(x-1)^2,1-x+x^2-x^5}]

Chapter 8. Stokes's and Gauss's Theorems

It was mentioned in Chapter 7 when we studied Green's Theorem, that Stokes's Theorem is a generalization of Green's Theorem in $\mathbb{R}^2$ to orientable surfaces in $\mathbb{R}^3$. In this chapter we shall study Stokes's and Gauss's Theorem. Together with Green's Theorem, Stokes's and Gauss's Theorem constitute the core of integral vector analysis.

Initialization

```
Clear["Global"]

(* The standard normal vector to a parametrized surface S *)
normal[S_][s_,t_]:=
Module[{u,v},
Cross[D[S[u,v],u],D[S[u,v],v]]/.{u->s,v->t}
]

(* Vector line integral as defined in Definition 7.4 *)
(* for three-dimensional case
c is given parametrically by (x(t),y(t),z(t))
*)
vectorLineInt3D[c_,F_,{a_,b_}]:=
Module[{x,y,z,v},
x=c[t][[1]];
y=c[t][[2]];
z=c[t][[3]];
```

```
v=c'[t];

Integrate[F[x,y,z].v,{t,a,b}]

]


(* Vector surface integral by equation 4.6 of Section 4.2.3

  sign_: the default sign of normal vector is positive

*)

vectorSurfIntegral[X_,F_,{s0_,s1_},{t0_,t1_},sign_:1]:=

Module[{u,v,F1,F2,F3,normalVector},

normalVector=sign*Cross[D[X[u,v],u],D[X[u,v],v]]/.{u->s,v->t};

{F1,F2,F3}={F[X[u,v][[1]],X[u,v][[2]],X[u,v][[3]]][[1]],F[X[u,v]
[[1]],X[u,v][[2]],X[u,v][[3]]][[2]],F[X[u,v][[1]],X[u,v][[2]],X[u,v][[3]]]
[[3]]}/.{u->s,v->t};

Integrate[Dot[{F1,F2,F3},normalVector],{s,s0,s1},{t,t0,t1}]

]
```

8.1 Stokes's Theorem

A surface S in $\mathbb{R}^3$ is orientable if there is a continuous vector field N such that N is nonzero and normal to S at each point of S. We say that N is a normal vector field to S. Let C be any simple, closed curve that has continuous derivative lying in S. We say that C is oriented consistently with S if N orients C by a right-hand rule, that is, if the thumb of your right hand points along N, then the fingers curl in the direction of the orientation of C.

As an example, let S be the sphere of radius a: x^2+y^2+z^2=a^2. The normal vector field to S may be computed using the gradient:

$$\mathbf{N}(x, y, z) = \nabla\left(x^2 + y^2 + z^2\right) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

.

The sphere is orientable, since N(x,y,z) is continuous, nonzero and normal to the sphere at each point. In this example N(x,y,z) is outward normal and -N(x,y,z) is inward normal vector to S.

Let ∂S denote the boundary of S. suppose ∂S consists of finitely many simple, closed curves that have continuous derivatives. We say that ∂S is oriented consistently with S if each of its closed curves is oriented consistently with S.

We now state Stokes's Theorem:

Theorem 8.1 (Stokes's Theorem)

Let S be an orientable surface in $\mathbb{R}^3$ whose boundary ∂S consists of finitely many simple, closed curves that have continuous derivatives,

each of which is oriented consistently with S. Let F be a vector field that has continuous derivatives whose domain includes S. Then

(8.1)

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

Here are some important points to note:

1. Stokes's Theorem equates the surface integral of the "curling of vector field F" over an oriented surface S to a line integral of F along just the boundary of S. Thus, here lies the power of Stokes's Theorem: if S_1 is any orientable surface (that satisfies Stokes's Theorem) whose boundary ∂S_1 is the same as ∂S then, we have

(8.2)

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S_1} \mathbf{F} \cdot d\mathbf{s} = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

.

This implies that we may evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ by using a different surface!

2. Suppose there is a vector field A such that $\mathbf{F} = \nabla \times \mathbf{A}$. The vector field A is called a vector potential for F. According to Stokes's Theorem:

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{s}$$

,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{s}$$.

Stokes's Theorem also tells us that for another surface S' with the same oriented boundary ∂S (see point 1 above), we have

(8.3)

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{s}$$

.

This implies that the surface integral of a vector field with vector potential A is surface independent, just as a vector field with a potential function V is path independent. The surface integral of F on the left-hand side of equation (8.3) is called the flux of F through the surface S.

Since $d\mathbf{S} = \mathbf{N}\, dS$, we may write equation (8.3) as

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS - \iint_{S'} \mathbf{F} \cdot \mathbf{N}\, dS' = 0.$$

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS + \iint_{S'} \mathbf{F} \cdot (-\mathbf{N})\, dS' = 0.$$

The last equation tells us that if the normal vectors of surfaces S' and S have opposite signs, then the surface integral vanishes. Since S and S' share the same oriented boundary ∂S , the surface S is closed. Conversely, if S is closed, it can be shown by Stokes's Theorem then, the flux of F through the surface S vanishes.

3. Let F be a vector field that is independent of z and that its k-component is identically zero. If S is a surface in the xy-plane. Then

equation (8.1) reduces to equation (7.13) of Green's Theorem. Hence, Stokes's Theorem implies Green's Theorem (see Example 8.6).

Example 8.1. Let the vector field be given by

F(x,y,z)=(cos y+y cos x)i+(sin x-x sin y)j+x y z k.

Verify Stokes's Theorem if the surface S is defined by x^2+y^2+5z=1, z>=0, oriented by upward normal.

Solution:

Write x^2+y^2+5z=1 as

z=1/5 (1-x^2-y^2).

Since z>=0, the boundary ∂S resides in the xy-plane at z=0. Thus, ∂S is the unit circle

C={ (x,y,z)|x^2+y^2=1,z=0 };

parametrized by

c(t)=(cos t,sin t,0), for 0<=t<=2π,

which is oriented consistently with S.

The surface S is the graph of f(x,y)=1/5 (1-x^2-y^2) defined over region D:

D={ (x,y)|x^2+y^2<=1 }.

We calculate the curl of F:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\cos y + y \cos x) & (\sin x - x \sin y) & x\,y\,z \end{vmatrix}$$

$$= (x\,z,\ -y\,z,\ 0)$$

```
Clear[c,F]

c[t_]:={Cos[t],Sin[t],0}

F[x_,y_,z_]:={Cos[y]+y Cos[x],Sin[x]-x Sin[y],x y z}


Curl[F[x,y,z],{x,y,z},"Cartesian"]


plot[1]=ContourPlot3D[x^2+y^2+5z==1,{x,-2,2},{y,-2,2},{z,0,0.2},Mesh-
>None,ContourStyle->{Green,Opacity[0.3]},PlotPoints->40,AxesLabel->
{"x","y","z"},PlotTheme->"Classic"];

plot[2]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[3]=VectorPlot3D[F[x,y,z],{x,-2,2},{y,-2,2},{z,0,0.2},VectorPoints-
>7,VectorScale->{0.06, 0.1, Automatic},VectorStyle->Gray];

plot[4]=Show[Array[plot,2]];

plot[5]=Show[Array[plot,3]];

Show[GraphicsGrid[{{plot[4],plot[5]}}],ImageSize->600]
```

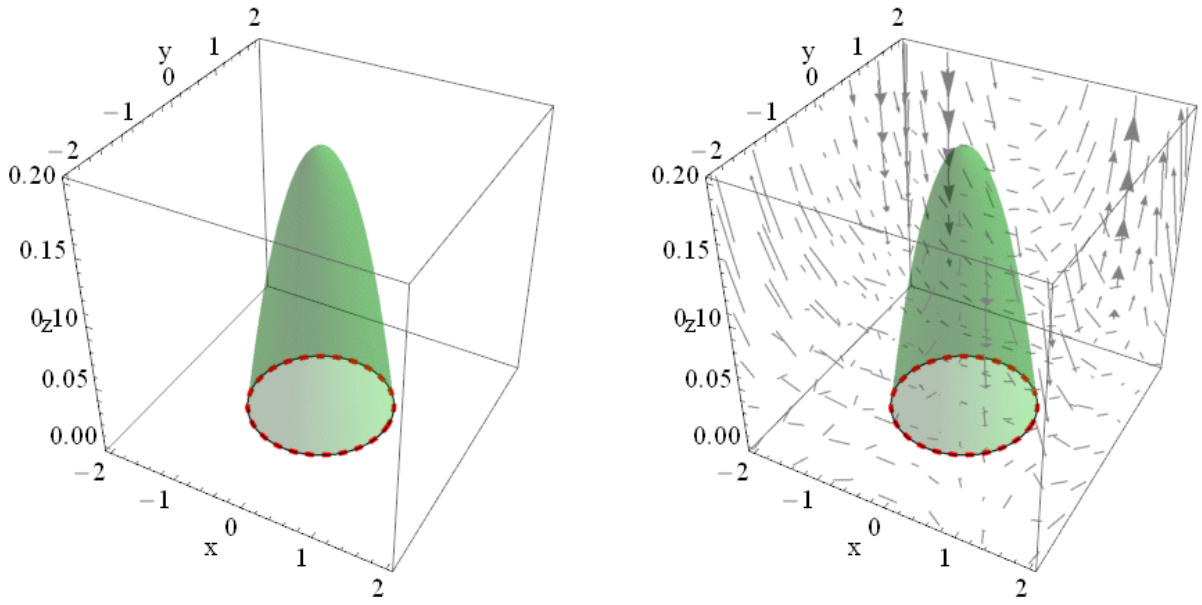


Figure 8.1. The surface S defined by x^2+y^2+5z=1, z>=0, oriented by upward normal; in the vector field F(x,y,z)=(cos y+y cos x)i+(sin x-x sin y)j+x y z k.

Using the level set F(x,y,z)=x^2+y^2+5z, an upward-pointing normal vector N is given by

$$\nabla F \equiv \mathbf{N} = (2x, 2y, 5)$$.

According to Stokes's Theorem, equation (8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

First we evaluate the left-hand side of the Stokes's Theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \iint_S (x z, -y z, 0) \cdot (2x, 2y, 5)\, dx\, dy$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x^2 z - 2y^2 z)\, dx\, dy$$

$$= \frac{1}{5} \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)(2x^2 - 2y^2)\, dx\, dy$$

$$= 0$$

1/5*Integrate[(1-x^2-y^2)(2x^2-2y^2),{x,-1,1},{y,-Sqrt[1-x^2],Sqrt[1-x^2]}]

Next we evaluate the right-hand side of the Stokes's Theorem:

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

$$= \int_{t=0}^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

$$= \int_{t=0}^{2\pi} (\cos(\sin t) + \sin t \cos(\cos t),\ \sin(\cos t) - \cos t \sin(\sin t),\ 0) \cdot$$
$$(-\sin t,\ \cos t,\ 0)\, dt$$

$$= \int_{t=0}^{2\pi} [-\sin t (\cos(\sin t) + \sin t \cos(\cos t))$$
$$+ \cos t (\sin(\cos t) - \cos t \sin(\sin t))]\, dt$$

$$= 0$$

which checks.

Integrate[-Sin[t](Cos[Sin[t]]+Sin[t]Cos[Cos[t]])+Cos[t](Sin[Cos[t]]-Cos[t]Sin[Sin[t]]),{t,0,2π}]

Since the right-hand side of equation (8.1) is a vector line integral, we may compute it using vectorLineInt3D module:

vectorLineInt3D[c,F,{0,2π}]

Example 8.2. Verify Stokes's Theorem for the vector field F=2x y i+x j+(y+z)k where the surface S is a helicoid parametrized by X(s,t)=(s cos t,s sin t,t), 0<=s<=1, 0<=t<=π/2.

Solution:

The boundary ∂S consists of four curves of each is simple and oriented consistently with S according to the right-hand rule. This is illustrated in Figure 8.2. The parametrizations are:

c1(s)=(s,0,0), 0<=s<=1,

c2(t)=(cos t,sin t,t), 0<=t<=π/2,

c3(s)=(0,1-s,π/2), 0<=s<=1,

c4(t)=(0,0,π/2-t), 0<=t<=π/2.

Clear[X,c1,c2,c3,c4,F]

X[s_,t_]:={s Cos[t],s Sin[t],t}

c1[s_]:={s,0,0}

c2[t_]:={Cos[t],Sin[t],t}

c3[s_]:={0,1-s,π/2}

c4[t_]:={0,0,π/2-t}

```
F[x_,y_,z_]:={2x y,x,y+z}


plot[1]=ParametricPlot3D[X[s,t],{s,0,1},{t,0,π/2},PlotStyle-
>Opacity[0.8],AxesLabel->{"x","y","z"},PlotRange->{{-1,1},{-1,1},
{-0.5,2}},PlotTheme->"Classic"];

plot[2]=ParametricPlot3D[c1[s],{s,0,1},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[3]=ParametricPlot3D[c2[t],{t,0,π/2},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[4]=ParametricPlot3D[c3[s],{s,0,1},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[5]=ParametricPlot3D[c4[t],{t,0,π/2},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

Show[Array[plot,5]]
```

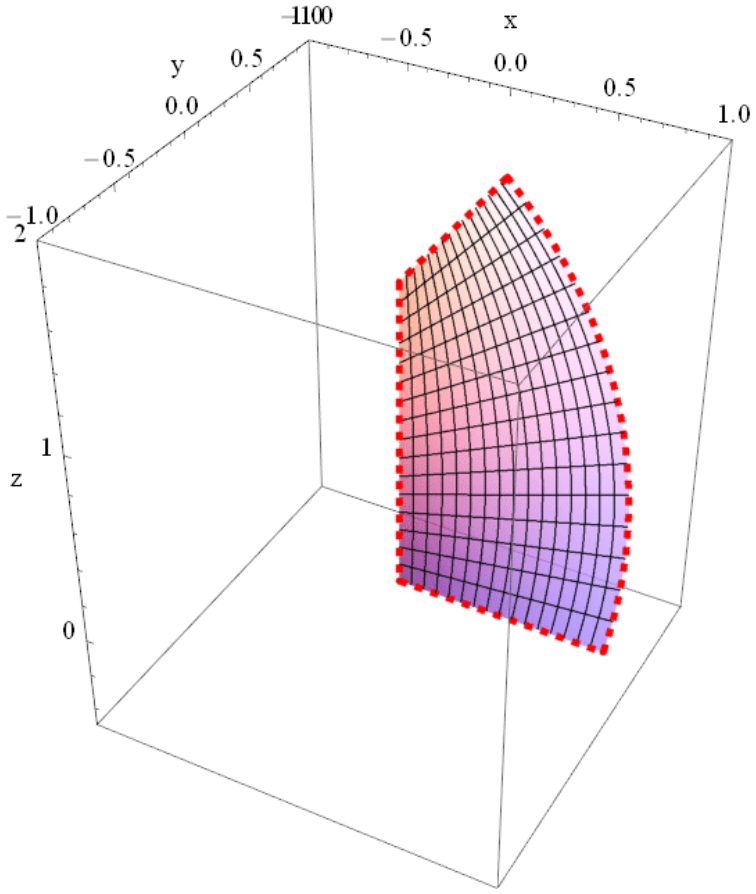


Figure 8.2. The surface S parametrized by X(s,t)=(s cos t,s sin t,t), 0<=s<=1, 0<=t<=π/2; and its boundary curves.

The surface S is defined over region D:

D={ (x,y)|x^2+y^2<=1 }.

The normal vector N is given by

$$\mathbf{N} = \partial_s \mathbf{X} \times \partial_t \mathbf{X} = \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix}$$

$$= (\sin t,\ -\cos t,\ s)$$

and

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x & y+z \end{vmatrix} = (1,\ 0,\ 1-2x)$$

.

(* The normal vector: N *)

normal[X][s,t]//Simplify

Curl[F[x,y,z],{x,y,z},"Cartesian"]

According to Stokes's Theorem, equation (8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

First we evaluate the left-hand side of the Stokes's Theorem:

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, ds\, dt \\ &= \iint_S (1,\ 0,\ 1-2x) \cdot (\sin t,\ -\cos t,\ s)\, ds\, dt \\ &= \int_{s=0}^{1} \int_{t=0}^{t=\pi/2} (1,\ 0,\ 1-2s\cos t) \cdot (\sin t,\ -\cos t,\ s)\, dt\, ds \\ &= \int_{s=0}^{1} \int_{t=0}^{t=\pi/2} [\sin t + s(1-2s\cos t)]\, dt\, ds \\ &= \frac{1}{3} + \frac{\pi}{4} \end{aligned}$$

Integrate[Sin[t]+s(1-2s Cos[t]),{s,0,1},{t,0,π/2}]

Next, we calculate the right-hand side of equation (8.1) which consists of four parts:

c1(s)=(s,0,0), 0<=s<=1,

c2(t)=(cos t,sin t,t), 0<=t<=π/2,

c3(s)=(0,1-s,π/2), 0<=s<=1,

c4(t)=(0,0,π/2-t), 0<=t<=π/2.

(8.4)

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{s=0}^{1} \mathbf{F}(\mathbf{c}_1(s)) \cdot \mathbf{c}_1'(s)\, ds + \int_{t=0}^{\pi/2} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt + \int_{s=0}^{1} \mathbf{F}(\mathbf{c}_3(s)) \cdot \mathbf{c}_3'(s)\, ds + \int_{t=0}^{\pi/2} \mathbf{F}(\mathbf{c}_4(t)) \cdot \mathbf{c}_4'(t)\, dt$$

Consider the first integral on the right-hand side of equation (8.4):

$$\begin{aligned} \int_{s=0}^{1} \mathbf{F}(\mathbf{c}_1(s)) \cdot \mathbf{c}_1'(s)\, ds &= \int_{s=0}^{1} \mathbf{F}(s, 0, 0) \cdot (1, 0, 0)\, ds \\ &= \int_{s=0}^{1} (0, s, 0) \cdot (1, 0, 0)\, ds \\ &= \int_{s=0}^{1} 0\, ds \\ &= 0 \end{aligned}$$

The second integral on the right-hand side of equation (8.4):

$$\int_{t=0}^{\pi/2} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt = \int_{t=0}^{\pi/2} \mathbf{F}(\cos t,\ \sin t,\ t) \cdot (-\sin t,\ \cos t,\ 1)\, dt$$
$$= \int_{t=0}^{\pi/2} (2 \cos t \sin t,\ \cos t,\ \sin t + t) \cdot (-\sin t,\ \cos t,\ 1)\, dt$$
$$= \int_{t=0}^{\pi/2} \left(-2 \cos t \sin^2 t + \cos^2 t + \sin t + t\right) dt$$
$$= \frac{1}{24} \left(8 + 6\,\pi + 3\,\pi^2\right)$$

Integrate[-2Cos[t]Sin[t]^2+Cos[t]^2+Sin[t]+t,{t,0,π/2}]

The third integral on the right-hand side of equation (8.4):

$$\int_{s=0}^{1} \mathbf{F}(\mathbf{c}_3(s)) \cdot \mathbf{c}_3'(s)\, ds = \int_{s=0}^{1} \mathbf{F}(0,\ 1 - s,\ \pi/2) \cdot (0,\ -1,\ 0)\, ds$$
$$= \int_{s=0}^{1} (0,\ 0,\ 1 - s + \pi/2) \cdot (0,\ -1,\ 0)\, ds$$
$$= \int_{s=0}^{1} 0\, ds$$
$$= 0$$

The fourth integral on the right-hand side of equation (8.4):

$$\int_{t=0}^{\pi/2} \mathbf{F}(\mathbf{c}_4(t)) \cdot \mathbf{c}_4'(t)\, dt = \int_{t=0}^{\pi/2} \mathbf{F}(0,\ 0,\ \pi/2 - t) \cdot (0,\ 0,\ -1)\, dt$$
$$= \int_{t=0}^{\pi/2} \mathbf{F}(0,\ 0,\ \pi/2 - t) \cdot (0,\ 0,\ -1)\, dt$$
$$= \int_{t=0}^{\pi/2} (0,\ 0,\ \pi/2 - t) \cdot (0,\ 0,\ -1)\, dt$$
$$= \int_{t=0}^{\pi/2} (-\pi/2 + t)\, dt$$
$$= -\frac{\pi^2}{8}$$

Integrate[-(π/2)+t,{t,0,π/2}]

So,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{s=0}^{1} \mathbf{F}(\mathbf{c}_1(s)) \cdot \mathbf{c}_1'(s)\, ds + \int_{t=0}^{\pi/2} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt$$
$$+ \int_{s=0}^{1} \mathbf{F}(\mathbf{c}_3(s)) \cdot \mathbf{c}_3'(s)\, ds + \int_{t=0}^{\pi/2} \mathbf{F}(\mathbf{c}_4(t)) \cdot \mathbf{c}_4'(t)\, dt$$
$$= 0 + \frac{1}{24}\left(8 + 6\pi + 3\pi^2\right) + 0 - \frac{\pi^2}{8}$$
$$= \frac{1}{3} + \frac{\pi}{4}$$

The two integrals agree:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{3} + \frac{\pi}{4}$$

.

Alternatively, we may compute the line integral using vectorLineInt3D module:

```
vectorLineInt3D[c1,F,{0,1}]+vectorLineInt3D[c2,F,
{0,π/2}]+vectorLineInt3D[c3,F,{0,1}]+vectorLineInt3D[c4,F,
{0,π/2}]//Simplify
```

```
plot[1]=ParametricPlot3D[X[s,t],{s,0,1},{t,0,π/2},PlotStyle-
>Opacity[0.8],AxesLabel->{"x","y","z"},PlotRange->All,PlotTheme-
>"Classic"];

plot[2]=ParametricPlot3D[c1[s],{s,0,1},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[3]=ParametricPlot3D[c2[t],{t,0,π/2},PlotStyle->
{Red,Dashed,Thickness[0.01]}];
```

```
plot[4]=ParametricPlot3D[c3[s],{s,0,1},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[5]=ParametricPlot3D[c4[t],{t,0,π/2},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[6]=VectorPlot3D[F[x,y,z],{x,-1,1},{y,-1,1},{z,-0.5,1.5},VectorPoints-
>6,VectorScale->{0.2, 0.2, Automatic},VectorStyle->Gray];

Show[Array[plot,6]]
```

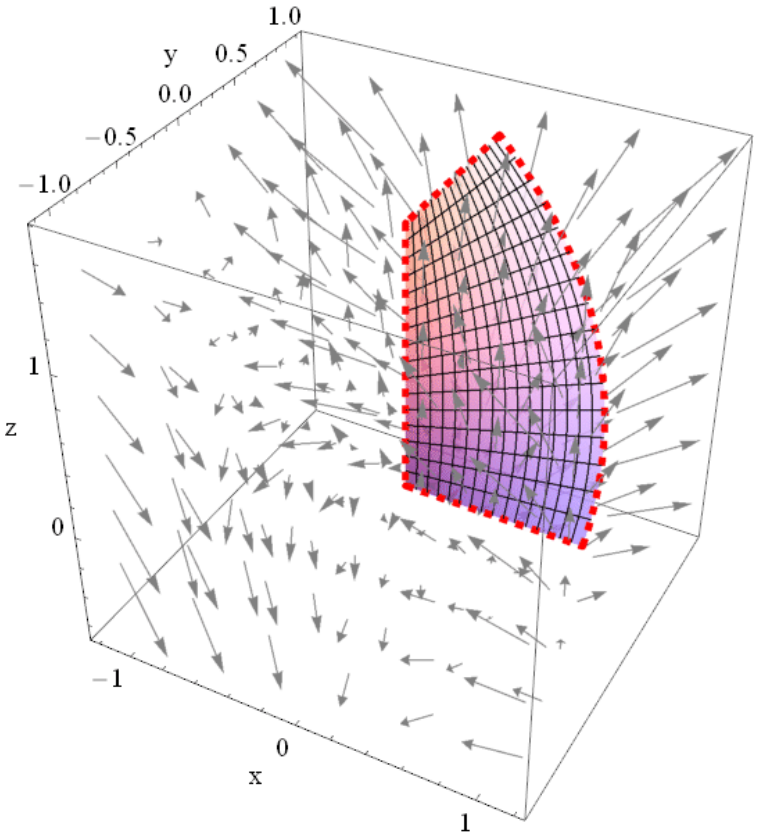


Figure 8.3. The surface S parametrized by X(s,t)=(s cos t,s sin t,t), 0<=s<=1, 0<=t<=π/2, in the vector field F=2x y i+x j+(y+z)k.

Example 8.3. Verify Stokes's Theorem for the given surface and vector field:

S is defined by y=Sqrt[16-x^2-z^2] and F=z i+x j+y k.

Solution:

In this example, since y>=0, the boundary ∂S resides in the xz-plane at y=0. Thus, ∂S consists of the circle

C={ (x,y,z)|x^2+z^2=16,y=0 }.

The parametrization of C is

c(t)=(4 cos t,0,4 sin t), 0<=t<=2π.

The surface S is the graph of y=f(x,z)=Sqrt[16-x^2-z^2] defined over region D:

D={ (x,z)|x^2+z^2<=16 }.

S may be parametrized by

X(s,t)=(s cos t,Sqrt[16-s^2],s sin t), for 0<=s<=4, 0<=t<=2π.

The normal vector N to S:

$$\mathbf{N} = \partial_s \mathbf{X} \times \partial_t \mathbf{X} = \left(\cos t, -\frac{s}{\sqrt{16-s}}, \sin t\right) \times (-s \sin t, 0, s \cos t)$$

$$= \left(-\frac{s^2 \cos t}{\sqrt{16-s^2}}, -s, -\frac{s^2 \sin t}{\sqrt{16-s^2}}\right)$$

$$= -\left(\frac{s^2 \cos t}{\sqrt{16-s^2}}, s, \frac{s^2 \sin t}{\sqrt{16-s^2}}\right)$$

Notice that the normal vector is pointing inwardly into the surface (Figure 8.4). However, we choose an opposite normal vector such that it always point away from the surface. Then, as stipulated by Stokes's Theorem the boundary curve C= ∂S has to be oriented consistently using the right-hand rule. As a result, the direction is clockwise in the xz-plane.

Evaluate the curl of F:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = (1, 1, 1)$$

.

```
Clear[c,X,F]

(* The boundary of the surface *)

c[t_]:={4Cos[t],0,4Sin[t]}

(* The parametrized surface S *)

X[s_,t_]:={s Cos[t],Sqrt[16-s^2],s Sin[t]}

F[x_,y_,z_]:={z,x,y}


Curl[F[x,y,z],{x,y,z},"Cartesian"]


normal[X][s,t]//Simplify


{s0,t0}={2,3π/2};

plot[1]=ParametricPlot3D[X[s,t],{s,0,4},{t,0,2π},Mesh->None,PlotStyle-
>Opacity[0.6],BoxRatios->{1, 1, 1},PlotTheme->"Classic",AxesLabel->
{"x","y","z"}];
```

```
plot[2]=Graphics3D[{PointSize[0.02],Point[X[s0,t0]],Blue,Arrow[{X[s0,t0],
X[s0,t0]+normal[X][s0,t0]}]}];

plot[3]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Red,Dashed,Thickness[0.01]}];

plot[4]=VectorPlot3D[F[x,y,z],{x,-4,4},{y,0,4},{z,-4,4},VectorPoints-
>5,VectorScale->{0.2, 0.2, Automatic},VectorStyle->Gray];

plot[5]=Show[Array[plot,3]];

plot[6]=Show[Array[plot,4]];

Show[GraphicsGrid[{{plot[5],plot[6]}}],ImageSize->600]
```

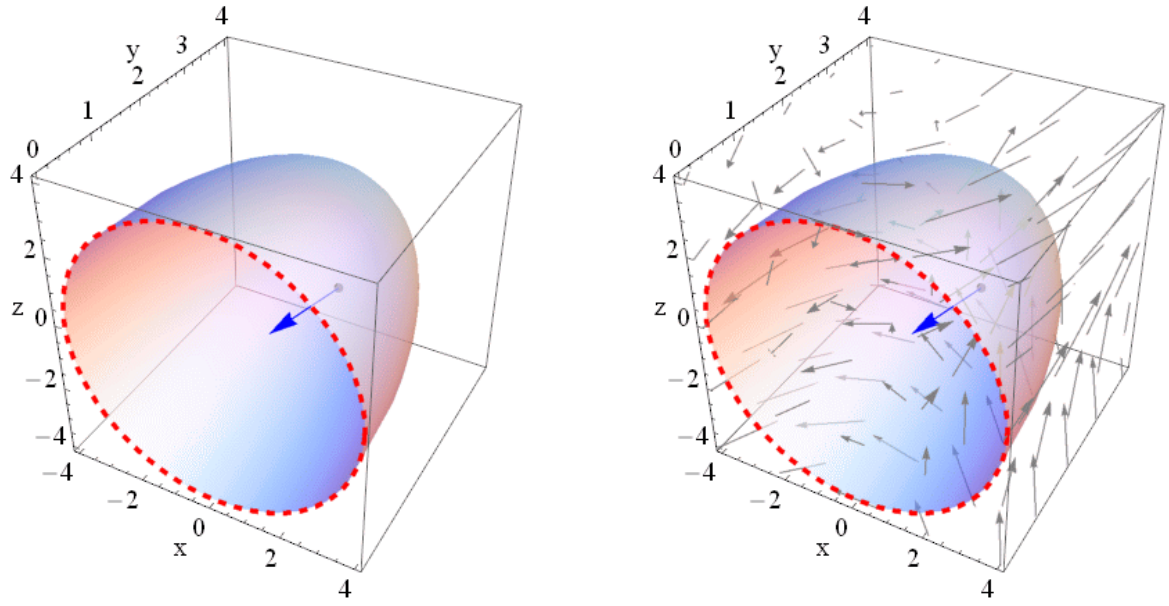


Figure 8.4. The surface S defined by y=Sqrt[16-x^2-z^2] in the vector field F=z i+x j+y k. The original normal vector points inwardly (blue arrow).

According to Stokes's Theorem, equation (8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

Evaluate the left-hand side integral using an outward-pointing normal vector:

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_S (\nabla \times \mathbf{F}) \cdot (-\mathbf{N})\, ds\, dt \\ &= \int_{s=0}^{4} \int_{t=0}^{2\pi} (1,\ 1,\ 1) \cdot \left(\frac{s^2 \cos t}{\sqrt{16 - s^2}},\ s,\ \frac{s^2 \sin t}{\sqrt{16 - s^2}} \right) dt\, ds \\ &= \int_{s=0}^{4} \int_{t=0}^{2\pi} \left(\frac{s^2 \cos t}{\sqrt{16 - s^2}} + s + \frac{s^2 \sin t}{\sqrt{16 - s^2}} \right) dt\, ds \\ &= 16\pi\end{aligned}$$

Integrate[(s^2 Cos[t])/Sqrt[16-s^2]+s+(s^2 Sin[t])/Sqrt[16-s^2],{s,0,4},{t,0,2π}]

To calculate the right-hand side of equation (8.1), we orient the closed curve consistently with S, so the direction is clockwise:

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_{t=2\pi}^{0} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\ &= \int_{t=2\pi}^{0} \mathbf{F}(4\cos t,\ 0,\ 4\sin t) \cdot (-4\sin t,\ 0,\ 4\cos t)\, dt \\ &= \int_{t=2\pi}^{0} (4\sin t,\ 4\cos t,\ 0) \cdot (-4\sin t,\ 0,\ 4\cos t)\, dt \\ &= \int_{t=2\pi}^{0} (-16\sin^2 t)\, dt \\ &= 16\pi\end{aligned}$$

which checks with the previous result:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 16\,\pi$$

.

Integrate[-16 Sin[t]^2,{t,2π,0}]

We may also use vectorLineInt3D module to calculate the line integral:

vectorLineInt3D[c,F,{2π,0}]

Example 8.4. Let S be the surface defined by z=4-4x^2-y^2 with z>=0. Use Stokes's Theorem to compute the flux of $\nabla \times \mathbf{F}$, where F=(E^z^2-y)i+ (E^z^3+x)j+cos (x z)k.

Solution:

$$\nabla \times \mathbf{F} = \left(-3\,e^{z^3}\,z^2\right)\mathbf{i} + \left(2\,e^{z^2}\,z + z\sin(x\,z)\right)\mathbf{j} + 2\,\mathbf{k}$$

.

ContourPlot3D[z==4-4x^2-y^2,{x,-2,2},{y,-2,2},{z,0,4},PlotPoints->40,AxesLabel->{"x","y","z"},PlotTheme->"Classic"]

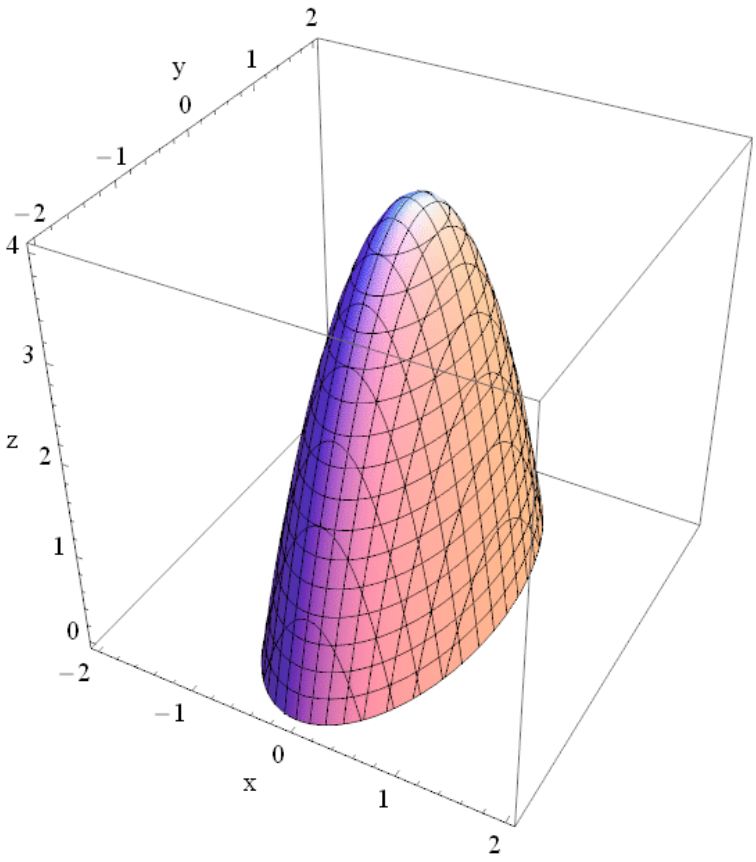


Figure 8.5. The surface S is defined by z=4-4x^2-y^2 with z>=0,

Write

z=4-4x^2-y^2=4-(4x^2+y^2)

The bottom of the surface S is at z=0. So, the boundary C= ∂S is an ellipse:

4x^2+y^2=4,

x^2+(y/2)^2=1.

∂S ={ (x,y,z)|x^2+(y/2)^2=1,z=0 }.

The semi-minor and semi-major of the ellipse is 1 and 2, respectively.

We may parametrize the surface S by X(s,t):

X(s,t)=(s/2 cos t,s sin t,4-s^2), 0<=s<=2, 0<=t<=2π;

and the boundary ∂S at z=0 by c(t):

c(t)=(cos t,2 sin t,0), 0<=t<=2π.

The normal vector to S:

N=(2 s^2 cos t,s^2 sin t,1/2 s).

The flux of $\nabla \times \mathbf{F}$ is just the double integral on the left-hand side of equation (8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

Let us first evaluate it directly:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

$$= \int_{s=0}^{2} \int_{t=0}^{2\pi} \left(-3 e^{z^3} z^2,\ 2 e^{z^2} z + z \sin(x z),\ 2\right) \cdot \left(2 s^2 \cos t,\ s^2 \sin t,\ \frac{1}{2} s\right) dt\, ds$$

$$= \int_{s=0}^{2} \int_{t=0}^{2\pi} \left(-3 e^{\left(4-s^2\right)^3} \left(4-s^2\right)^2,\ 2 e^{\left(4-s^2\right)^2} \left(4-s^2\right) + \left(4-s^2\right) \sin\left(\frac{s}{2} \cos t \left(4-s^2\right)\right),\ 2\right) \cdot \left(2 s^2 \cos t,\ s^2 \sin t,\ \frac{1}{2} s\right) dt\, ds$$

$$= 4\pi$$

It is not easy to compute this double integral by hand, however Mathematica is able to get the result analytically.

```
Integrate[{-3 E^(4-s^2)^3 (4-s^2)^2,2 E^(4-s^2)^2 (4-s^2)+(4-s^2) Sin[s/2 Cos[t](4-s^2)],2}.{2 s^2 Cos[t],s^2 Sin[t],1/2 s},{s,0,2},{t,0,2π}]
```

```
Clear[X,c,F]

X[s_,t_]:={s/2 Cos[t],s Sin[t],4-s^2}

c[t_]:={Cos[t],2Sin[t],0}

F[x_,y_,z_]:={Exp[z^2]-y,Exp[z^3]+x,Cos[x z]}
```

```
Curl[F[x,y,z],{x,y,z},"Cartesian"]
```

```
normal[X][s,t]//Simplify
```

```
{s0,t0}={1,π/4};

plot[1]=ParametricPlot3D[X[s,t],{s,0,2},{t,0,2π},Mesh->None,PlotStyle->Opacity[0.7],AxesLabel->{"x","y","z"},PlotTheme->"Classic",PlotRange->{{-3,3},{-3,3},{0,4}}];

plot[2]=Graphics3D[{PointSize[0.02],Point[X[s0,t0]],Blue,Arrow[{X[s0,t0],X[s0,t0]+normal[X][s0,t0]}]}];

plot[3]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->{Red,Dashed,Thickness[0.01]}];
```

```
plot[4]=VectorPlot3D[F[x,y,z],{x,-3,3},{y,-3,3},{z,0,4},VectorPoints->10,VectorScale->{0.2, 0.2, Automatic},VectorStyle->Gray];

plot[5]=Show[Array[plot,3]];

plot[6]=Show[Array[plot,4]];

Show[GraphicsGrid[{{plot[5],plot[6]}}],ImageSize->600]
```

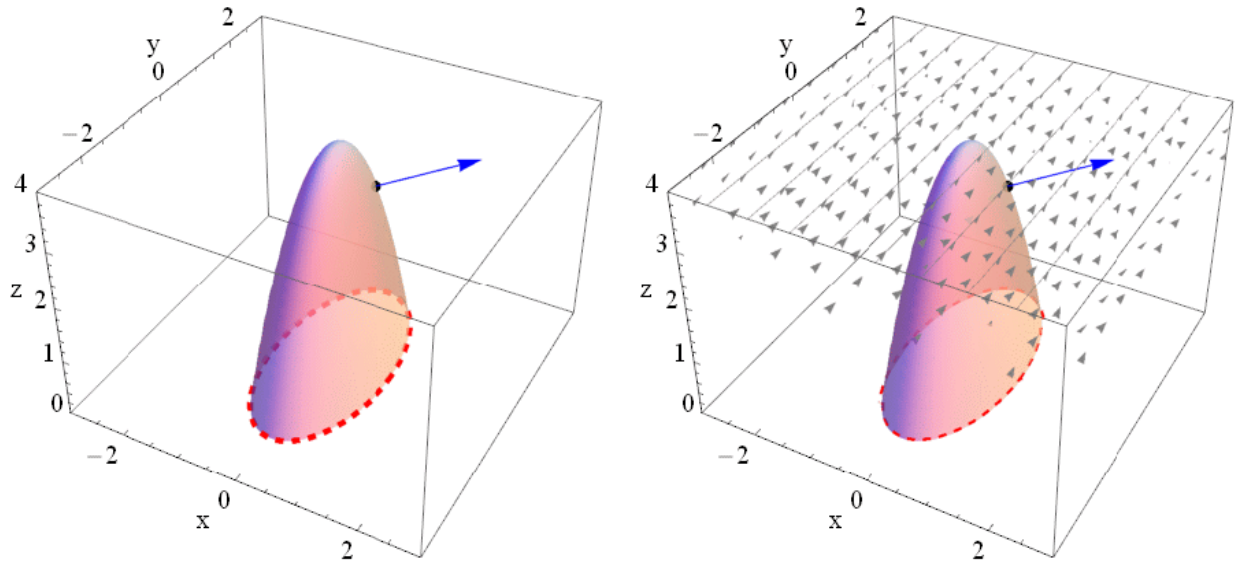


Figure 8.6. The surface S defined by z=4-4x^2-y^2 with z>=0, in the vector field F=(E^z^2-y)i+(E^z^3+x)j+cos (x z)k. The blue arrow is the outward-pointing normal.

Now, we evaluate the flux of $\nabla \times \mathbf{F}$ using the line integral, namely the right-hand side of equation (8.1):

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$
$$= \int_{t=0}^{2\pi} \mathbf{F}(\cos t,\, 2\sin t,\, 0) \cdot (-\sin t,\, 2\cos t,\, 0)\, dt$$
$$= \int_{t=0}^{2\pi} (1 - 2\sin t,\, 1 + \cos t,\, 1) \cdot (-\sin t,\, 2\cos t,\, 0)\, dt$$
$$= \int_{t=0}^{2\pi} [-(1 - 2\sin t)\sin t + 2(1 + \cos t)\cos t)]\, dt$$
$$= 4\pi$$

The line integral is much easier to compute by hand.

Integrate[-(1-2 Sin[t])Sin[t]+2(1+Cos[t])Cos[t],{t,0,2π}]

Using vectorLineInt3D module we verify our result:

vectorLineInt3D[c,F,{0,2π}]

Example 8.5. Let S be the surface defined by x=10-y^2-z^2 with x>=1, oriented with outward-pointing normal. Let F=(2x y z+5z)i+E^y cos (x z) j+x y^2 k.

Find the flux of $\nabla \times \mathbf{F}$:

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$.

Solution:

Evaluate the curl of F:

$$\nabla \times \mathbf{F} = \left(2\,x\,y + e^{y}\,x \sin(x\,z),\ 5 + 2\,x\,y - y^{2},\ -2\,x\,z - e^{y}\,z \sin(x\,z)\right)$$

.

ContourPlot3D[x==10-y^2-z^2,{x,1,10},{y,-Sqrt[10],Sqrt[10]},{z,-Sqrt[10],Sqrt[10]},PlotRange->{{0,10},{-Sqrt[10],Sqrt[10]},{-Sqrt[10],Sqrt[10]}},AxesLabel->{"x","y","z"},PlotTheme->"Classic",ViewPoint->{-1,-2,1}]

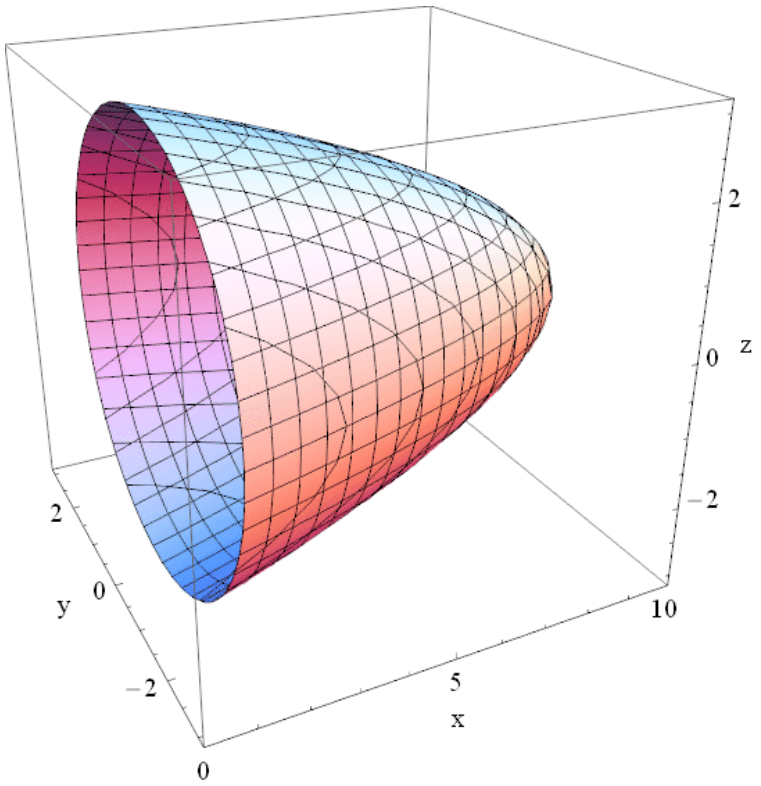


Figure 8.7. The surface S is defined by x=10-y^2-z^2 with x>=1.

The surface S is defined by

S={ (x,y,z)|x=10-y^2-z^2,x>=1 }.

Since x>=1, the parametrization of S is given by

X(s,t)=(10-s^2,s cos t,s sin t), 0<=s<=3, 0<=t<=2π.

The normal vector to S:

$$\mathbf{N} = \partial_s \mathbf{X} \times \partial_t \mathbf{X} = \left(s,\, 2\,s^2 \cos t,\, 2\,s^2 \sin t\right)$$

,

is outward-pointing (Figure 8.8).

The boundary C= ∂S is the circle of radius 3 in the yz-plane at x=1 given by

C={ (x,y,z)|y^2+z^2=9,x=1 }

To be consistent with the outward-pointing normal vector of S, the parametrization of C is

c(t)=(1,3 cos t,3 sin t), 0<=t<=2π.

```
Clear[X,c,F]

X[s_,t_]:={10-s^2,s Cos[t],s Sin[t]}

c[t_]:={1,3Cos[t],3Sin[t]}

F[x_,y_,z_]:={2x y z+5z,E^y Cos[x z],x y^2}

normal[X][s,t]//Simplify

Curl[F[x,y,z],{x,y,z},"Cartesian"]
```

```
{s0,t0}={1,5π/6};

plot[1]=ParametricPlot3D[X[s,t],{s,0,3},{t,0,2π},Mesh->None,PlotStyle-
>Opacity[0.6],PlotTheme->"Classic",AxesLabel->{"x","y","z"},ViewPoint-
>{-1,-2,1}];

plot[2]=Graphics3D[{PointSize[0.02],Point[X[s0,t0]],Blue,Arrow[{X[s0,t0],
X[s0,t0]+normal[X][s0,t0]}]}];

plot[3]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Dashed,Red,Thickness[0.01]}];

plot[4]=VectorPlot3D[F[x,y,z],{x,0,10},{y,-Sqrt[10],Sqrt[10]},{z,-
Sqrt[10],Sqrt[10]},VectorPoints->8,VectorScale->{0.2, 0.2,
Automatic},VectorStyle->Gray];

plot[5]=Show[Array[plot,3]];

plot[6]=Show[Array[plot,4]];

Show[GraphicsGrid[{{plot[5],plot[6]}}],ImageSize->600]
```

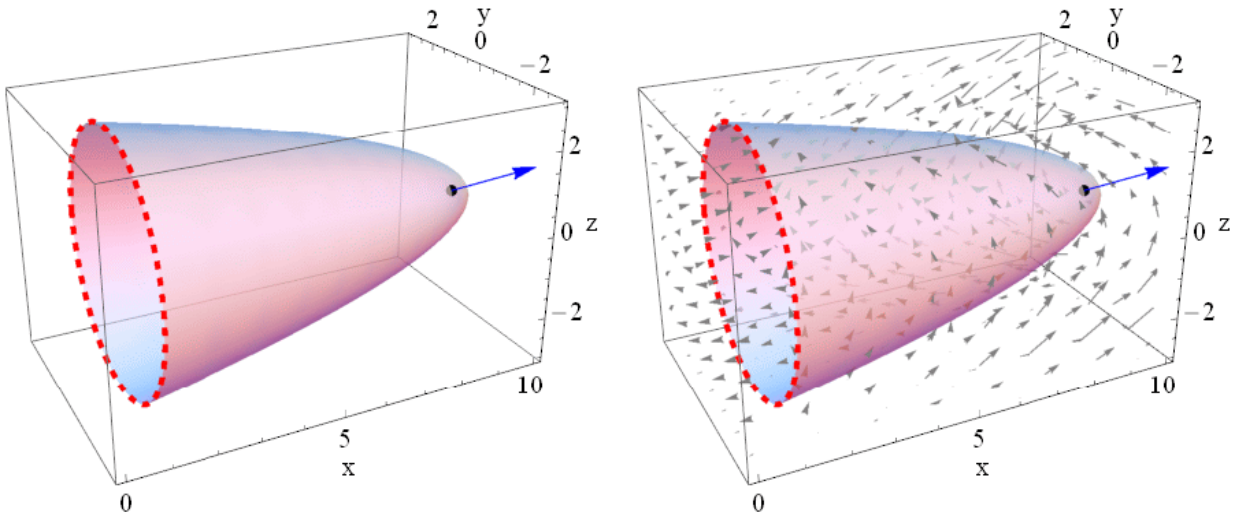


Figure 8.8. The surface S defined by x=10-y^2-z^2 with x>=1 with its outward-pointing normal vector (blue arrow); in the vector field F=(2x y

z+5z)i+E^y cos (x z) j+x y^2 k.

Now, we compute the flux of $\nabla \times \mathbf{F}$ directly:

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$
$$= \int_{s=0}^{3} \int_{t=0}^{2\pi} [2s(10 - s^2)\cos t + e^{s\cos t}(10 - s^2)\sin(s(10 - s^2)\sin t),$$
$$5 + 2s(10 - s^2)\cos t - s^2\cos^2 t,$$
$$-2s(10 - s^2)\sin t - e^{s\cos t}\, s \sin t \sin(s(10 - s^2)\sin t)] \cdot$$
$$(s, 2s^2\cos t, 2s^2 \sin t)\, dt\, ds$$

We employed Mathematica to do the integration using Integrate command. However, it did not respond for long time, so we aborted the evaluation. The flux of $\nabla \times \mathbf{F}$ looks impossible to evaluate analytically. We may try to evaluate this integral numerically using NIntegrate command. However, we will not attempt to proceed any further with this calculation.

Recall from our notes above, the power of Stokes's Theorem is that if S_1 is any orientable surface (that satisfies Stokes's Theorem) whose boundary ∂S_1 is the same as ∂S then, by equation (8.2):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S_1} \mathbf{F} \cdot d\mathbf{s} = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

.

Our alternative surface that satisfies Stokes's Theorem and share the same boundary ∂S is S_1 , namely the disk y^2+z^2<=9 at x=1,

S_1 ={ (x,y,z)|y^2+z^2<=9,x=1 }.

Since x>=1, the parametrization of S_1 is

$$\mathbf{X}_1(s, t) = (1, s\cos t, s\sin t)$$

, 0<=s<=3, 0<=t<=2π.

It is so parametrized such that its normal vector $\mathbf{N}_1 = (s, 0, 0)$ is oriented consistently with the direction of ∂S as stipulated by Stokes's Theorem. Note that this is also consistent with the right-hand rule.

```
(* Alternative surface S1: a disk surface x^2+z^2=9 *)

Clear[X1]

X1[s_,t_]:={1,s Cos[t],s Sin[t]}


normal[X1][s,t]//Simplify


plot[1]=ParametricPlot3D[X[s,t],{s,0,3},{t,0,2π},Mesh->None,PlotStyle-
>Opacity[0.5],PlotTheme->"Classic",AxesLabel->{"x","y","z"}];

plot[2]=ParametricPlot3D[X1[s,t],{s,0,3},{t,0,2π},PlotStyle->Opacity[0.2]];

plot[3]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Dashed,Red,Thickness[0.01]}];

Show[Array[plot,3],ViewPoint->{-1,-2,1}]
```

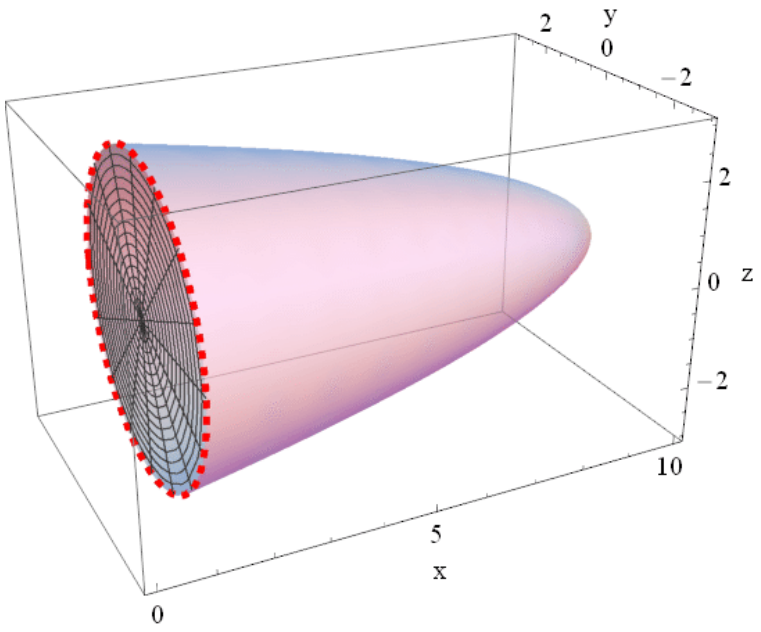


Figure 8.9. The surface S (x=10-y^2-z^2 with x>=1) and S_1 (the disk y^2+z^2<=9 at x=1) share the same boundary ∂S that satisfy Stokes's Theorem.

Thus, the flux of $\nabla \times \mathbf{F}$ over S may now be computed using different, yet simple surface S_1 :

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{N}_1 \, dS$$

$$= \int_{s=0}^{3} \int_{t=0}^{2\pi} \left[2 s \cos t + e^{s \cos t} \sin (s \sin t), \right.$$

$$5 + 2 s \cos t - s^2 \cos^2 t,$$

$$\left. -2 s \sin t - e^{s \cos t} s \sin t \sin (s \sin t) \right] \cdot$$

$$(s, 0, 0) \, dt \, ds$$

$$= 0$$

Integrate[{2 s Cos[t]+E^(s Cos[t]) Sin[s Sin[t]],5+2 s Cos[t]-s^2 Cos[t]^2,-2 s Sin[t]-E^(s Cos[t]) s Sin[t] Sin[s Sin[t]]}.{s,0,0},{s,0,3},{t,0,2π}]

To verify our result, we evaluate the flux of $\nabla \times \mathbf{F}$ indirectly using the line integral in Stokes's Theorem (equation 8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

$$\text{flux of } \nabla \times \mathbf{F} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

$$= \int_{t=0}^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

$$= \int_{t=0}^{2\pi} \mathbf{F}(1, 3\cos t, 3\sin t) \cdot (0, -3\sin t, 3\cos t)\, dt$$

$$= \int_{t=0}^{2\pi} \left(18\cos t \sin t + 15\sin t,\ e^{3\cos t}\cos(3\sin(t)),\ 9\cos^2 t\right) \cdot (0, -3\sin t, 3\cos t)\, dt$$

$$= 0$$

Integrate[{18 Cos[t] Sin[t]+15 Sin[t],E^(3 Cos[t]) Cos[3 Sin[t]],9 Cos[t]^2}. {0,-3 Sin[t],3Cos[t]},{t,0,2π}]

Using vectorLineInt3D module we verify the line integral:

vectorLineInt3D[c,F,{0,2π}]

Example 8.6. Let F(x,y)=M(x,y)i+N(x,y)j. Verify that Stokes's Theorem implies Green's Theorem.

Solution:

Let F(x,y)=M(x,y) i+N(x,y) j. Write

F(x,y)=M(x,y) i+N(x,y) j+0 k.

So,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

.

Let S be a surface in the xy-plane, then the unit normal vector to S is k. According to Stokes's Theorem, equation (8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

;

the left-hand side gives

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \cdot d\mathbf{S}$$

.

Since $\mathbf{k} \cdot d\mathbf{S} = dx\,dy$, therefore

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

.

Whereas the right-hand side gives

$$\begin{aligned} \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \oint_{\partial S} (M\mathbf{i} + N\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) \\ &= \oint_{\partial S} (M\,dx + N\,dy) \end{aligned}$$

Hence, for F(x,y)=M(x,y)i+N(x,y)j, where S is a surface in the xy-plane, Stokes's Theorem (Theorem 8.1) implies Green's Theorem (Theorem 7.9):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

$$\Rightarrow \quad \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \oint_{\partial S} (M\, dx + N\, dy)$$

Example 8.7. An oriented surface is given by S={ (x,y,z)|z=E^(x^2+y^2),x^2+y^2<1 }. Let F=x E^-y j. Verify Stokes's Theorem.

Solution:

Evaluate the curl of F:

$$\nabla \times \mathbf{F} = (0,\ 0,\ e^{-y})$$.

ContourPlot3D[z==Exp[x^2+y^2],{x,-1.5,1.5},{y,-1.5,1.5}, {z,0,E+1},AxesLabel->{"x","y","z"},PlotTheme->"Classic"]

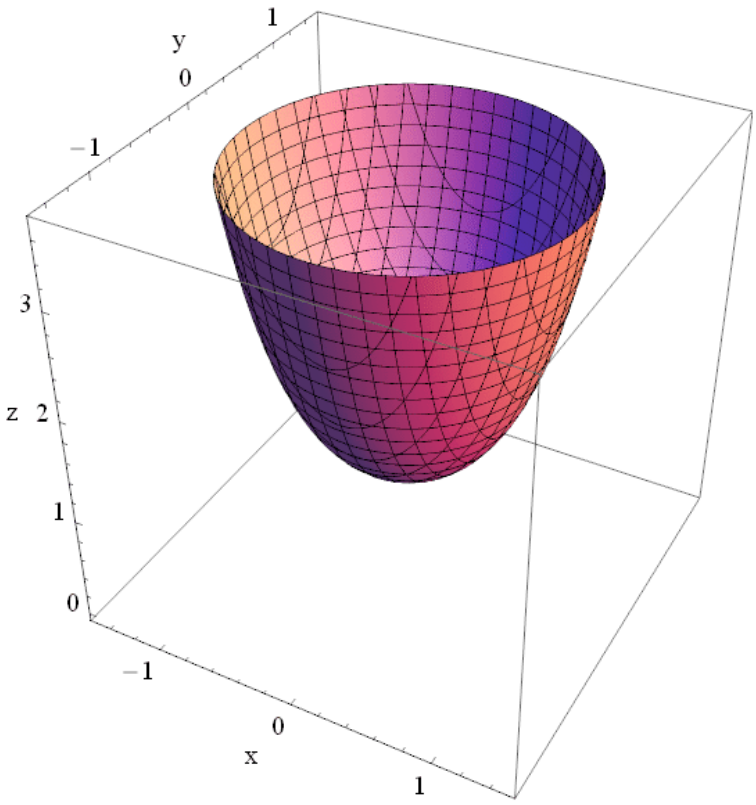


Figure 8.10. The surface S={ (x,y,z)|z=E^(x^2+y^2),x^2+y^2<1 }.

The surface S is given by

S={ (x,y,z)|z=E^(x^2+y^2),x^2+y^2<1 }.

We may parametrize of S by

X(s,t)=(s cos t,s sin t,E^s^2), 0<=s<=1, 0<=t<=2π.

The normal vector to S:

$$\mathbf{N} = \partial_s \mathbf{X} \times \partial_t \mathbf{X} = \left(-2\, e^{s^2}\, s^2 \cos t,\ -2\, e^{s^2}\, s^2 \sin t,\ s\right)$$

.

In this example N is an inward-pointing normal vector (see Figure 8.11).

The boundary C= ∂S is the unit circle in the xy-plane at z=E given by

C={ (x,y,z)|x^2+y^2=1,z=E }

To be consistent with the inward-pointing normal vector of S, the parametrization of C is

c(t)=(cos t,sin t,E), 0<=t<=2π.

```
Clear[X,X1,c,F]

X[s_,t_]:={s Cos[t],s Sin[t],Exp[s^2]}

c[t_]:={Cos[t],Sin[t],E}

F[x_,y_,z_]:={0,x E^-y,0}


normal[X][s,t] // Simplify


Curl[F[x,y,z],{x,y,z},"Cartesian"]


{s0,t0}={0.5,π/3};

plot[1]=ParametricPlot3D[{X[s,t]},{s,0,1},{t,0,2π},Mesh->None,PlotStyle-
>Opacity[0.3],AxesLabel->{"x","y","z"},PlotRange->{{-1.5,1.5},{-1.5,1.5},
{0,E+1}}];
```

```
plot[2]=Graphics3D[{PointSize[0.02],Point[X[s0,t0]],Blue,Arrow[{X[s0,t0],
X[s0,t0]+normal[X][s0,t0]}]}];

plot[3]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Dashed,Red,Thickness[0.01]}];

plot[4]=VectorPlot3D[F[x,y,z],{x,-1.5,1.5},{y,-1.5,1.5},
{z,0,E+1},VectorPoints->6,VectorScale->{0.2, 0.2, Automatic},VectorStyle-
>Gray];

plot[5]=Show[Array[plot,3]];

plot[6]=Show[Array[plot,4]];

Show[GraphicsGrid[{{plot[5],plot[6]}}],ImageSize->600]
```

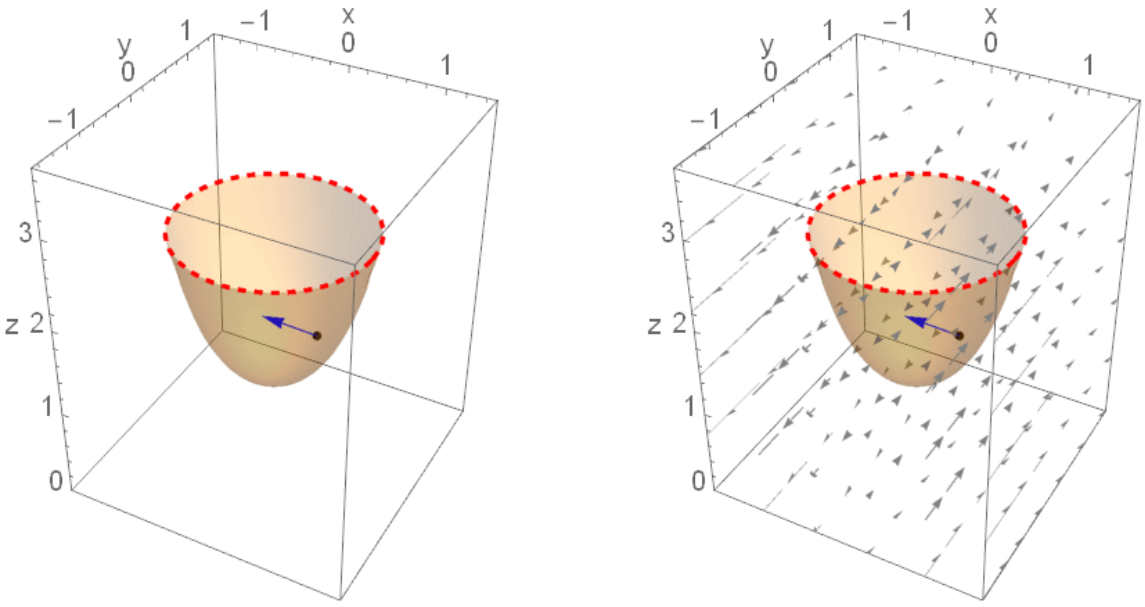


Figure 8.11. The surface S={(x,y,z)|z=E^(x^2+y^2),x^2+y^2<1} with its inward-pointing normal vector (blue arrow); in the vector field F=x E^-y j.

According to Stokes's Theorem, equation (8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

Evaluate the left-hand side integral:

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, ds\, dt \\ &= \int_{s=0}^{1} \int_{t=0}^{2\pi} (0, 0, e^{-s \sin t}) \cdot \left(-2 e^{s^2} s^2 \cos t, -2 e^{s^2} s^2 \sin t, s\right) dt\, ds \\ &= \int_{s=0}^{1} \int_{t=0}^{2\pi} (s e^{-s \sin t})\, dt\, ds \\ &= 2\pi \,\text{BesselI}[1, 1]\end{aligned}$$

Note that the solution is a Bessel function of the first kind.

Integrate[s Exp[-s Sin[t]],{s,0,1},{t,0,2π}]

Next, we calculate the right-hand side of equation (8.1):

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_{t=0}^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\ &= \int_{t=0}^{2\pi} \mathbf{F}(\cos t, \sin t, e) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_{t=0}^{2\pi} (0, \cos t\, e^{-\sin t}, 0) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_{t=0}^{2\pi} \cos^2 t\, e^{-\sin t}\, dt \\ &= 2\pi \,\text{BesselI}[1, 1]\end{aligned}$$

The two answers agree:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 2\pi \,\text{BesselI}[1, 1]$$

.

Integrate[Cos[t]^2 Exp[-Sin[t]],{t,0,2π}]

Let us verify our result using the vectorLineInt3D module:

vectorLineInt3D[c,F,{0,2π}]

Example 8.8. (a) Evaluate

$$\oint_C \left(y^3 + \cos x\right) dx + \left(\sin y + z^2\right) dy + x\, dz$$

,

where C is the closed curve parametrized and oriented by the path c(t)=(cos t,sin t,sin 2t).

(b) Use Stokes's Theorem to evaluate the line integral.

Solution (a):

Since sin 2t=2 cos t sin t, the curve c(t)=(cos t,sin t,sin 2t) satisfies the equation z=2x y. This means that c(t) lies on the surface z=2x y. This is shown in Figure 8.12.

Clear[c,F]

c[t_]:={Cos[t],Sin[t],Sin[2t]}

```
F[x_,y_,z_]:={y^3+Cos[x],Sin[y]+z^2,x}


plot[1]=ContourPlot3D[z==2x y,{x,-2,2},{y,-2,2},{z,-2,2},Mesh-
>None,ContourStyle->Opacity[0.3],AxesLabel->{"x","y","z"},PlotTheme-
>"Detailed"];

plot[2]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Dashed,Red,Thickness[0.007]}];

Show[Array[plot,2]]
```

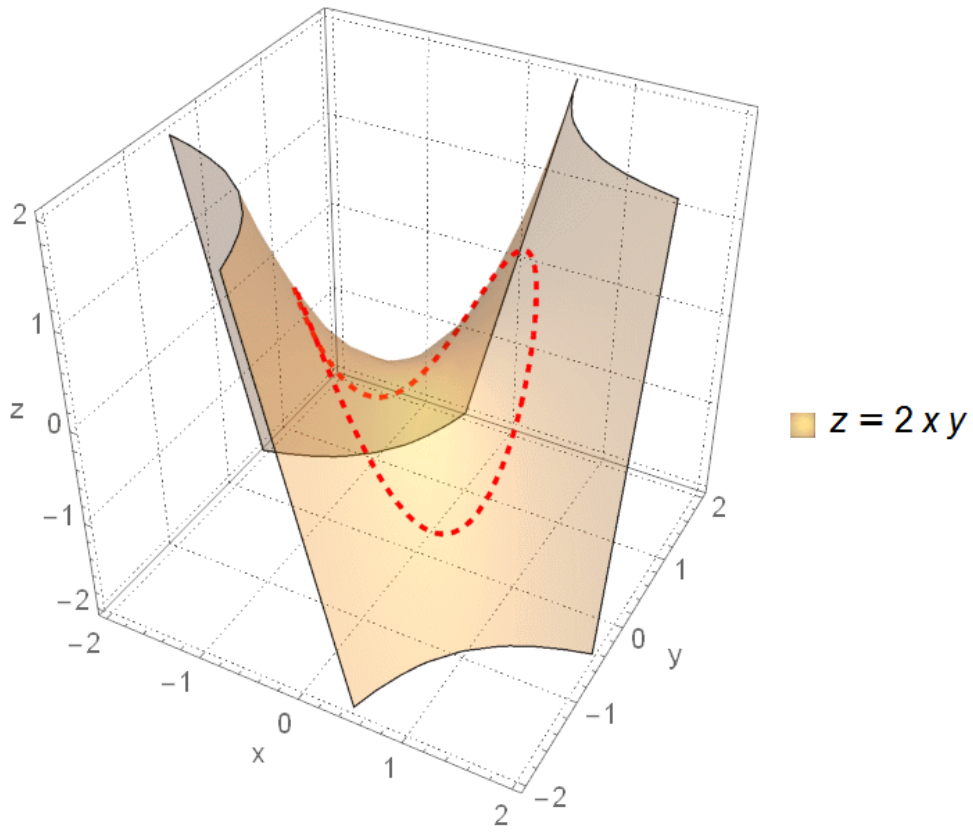


Figure 8.12. The curve c(t)=(cos t,sin t,sin 2t) lies on the surface z=2x y.

Let F=(y^3+cos x)i+(sin y+z^2)j+x k, then the closed integral:

$$\oint_C \left(y^3 + \cos x\right) dx + \left(\sin y + z^2\right) dy + x\, dz$$

,

may be written as the vector line integral:

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_C \left(y^3 + \cos x\right) dx + \left(\sin y + z^2\right) dy + x\, dz$$

,

where $d\mathbf{s} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

So,

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{s} &= \oint_C \left(y^3 + \cos x\right) dx + \left(\sin y + z^2\right) dy + x\, dz \\ &= \int_0^{2\pi} \left(\sin^3 t + \cos\left(\cos t\right)\right)\left(-\sin t\right) dt \\ &\quad + \int_0^{2\pi} \left(\sin\left(\sin t\right) + \sin^2(2t)\right)\left(\cos t\right) dt \\ &\quad + \int_0^{2\pi} \cos t\,\left(2\cos\left(2t\right)\right) dt \\ &= -\frac{3\pi}{4}\end{aligned}$$

Integrate[(Sin[t]^3+Cos[Cos[t]])(-Sin[t]),
{t,0,2π}]+Integrate[(Sin[Sin[t]]+Sin[2t]^2)Cos[t],
{t,0,2π}]+Integrate[2Cos[t]Cos[2t],{t,0,2π}]

We may also use vectorLineInt3D module to compute this line integral:

vectorLineInt3D[c,F,{0,2π}]

Solution (b):

According to Stokes's Theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

Evaluate the curl of F:

$$\nabla \times \mathbf{F} = \left(-2z, -1, -3y^2\right)$$.

Curl[F[x,y,z],{x,y,z},"Cartesian"]

The surface z=2 x y may be parametrized by

X(s,t)=(s cos t,s sin t,s^2 sin 2 t).

Now, the patch of S corresponding to the boundary C= ∂S is bounded by 0<=s<=1 and 0<=t<=2π.

The normal vector is:

N=(-2 s^2 sin t,-2 s^2 cos t,s).

Note that the normal vector always point upward in z-direction which is consistent with the orientation of the curve c(t).

Clear[X]

X[s_,t_]:={s Cos[t],s Sin[t],s^2 Sin[2t]}

```
normal[X][s,t]//Simplify


plot[1]=ParametricPlot3D[X[s,t],{s,0,1},{t,0,2π},PlotStyle-
>Yellow,AxesLabel->{"x","y","z"},BoxRatios->{1, 1, 1}];

plot[2]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Dashed,Red,Thickness[0.007]}];

Show[Array[plot,2],PlotRange->All]
```

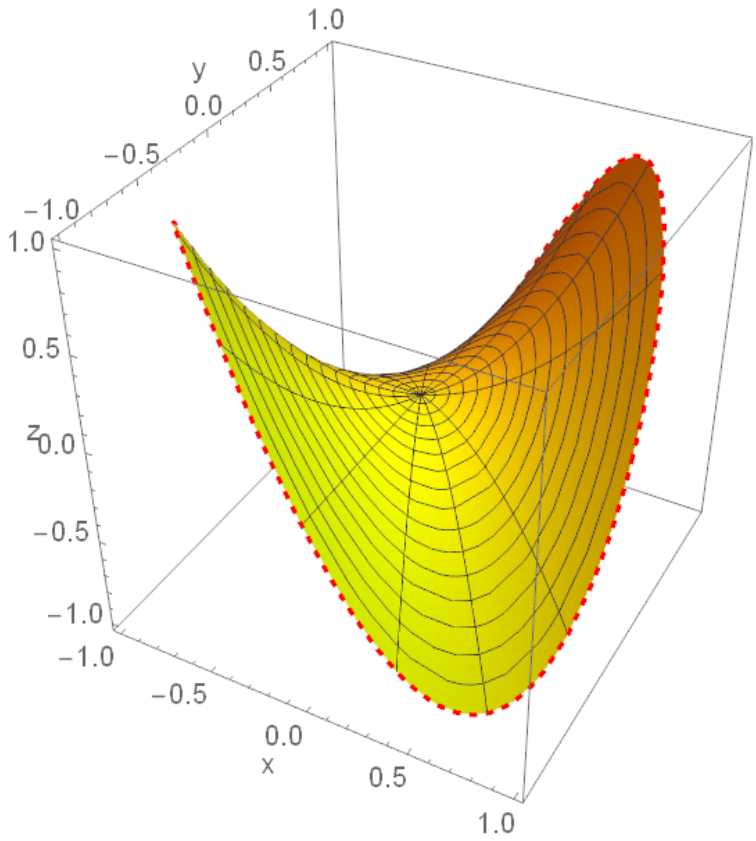


Figure 8.13. A patch of the surface z=2x y whose boundary curve is c(t)= (cos t,sin t,sin 2t).

So,

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$
$$= \int_{s=0}^{1} \int_{t=0}^{2\pi} \left(-2s^2 \sin 2t,\ -1,\ -3s^2 \sin^2 t\right) \cdot \left(-2s^2 \sin t,\ -2s^2 \cos t,\ s\right) dt\, ds$$
$$= \int_{s=0}^{1} \int_{t=0}^{2\pi} \left(4s^4 \sin t \sin 2t + 2s^2 \cos t - 3s^3 \sin^2 t\right) dt\, ds$$
$$= -\frac{3\pi}{4}$$

The two integrals agree:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = -\frac{3\pi}{4}$$

.

Integrate[4s^4 Sin[t]Sin[2t]+2s^2 Cos[t]-3s^3 Sin[t]^2,{s,0,1},{t,0,2π}]

Example 8.9. The surface S consists of four sides and a bottom face of a cube with vertices (±a,±a,±a). Orient S with outward-pointing normals. Let the vector field be F=x^2 y z^2 i+y z^2 j+z E^z sin (x y) k. Evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

Solution:

Note that the top face is open. Thus, we may use Stokes's Theorem.

a=1;

```
Show[Graphics3D[{Opacity[0.3],Cuboid[{-a,-a,-a},{a,a,a}]}],Axes->True,AxesLabel->{"x","y","z"},PlotRange->{{-2,2},{-2,2},{-2,2}}]

Clear[a]
```

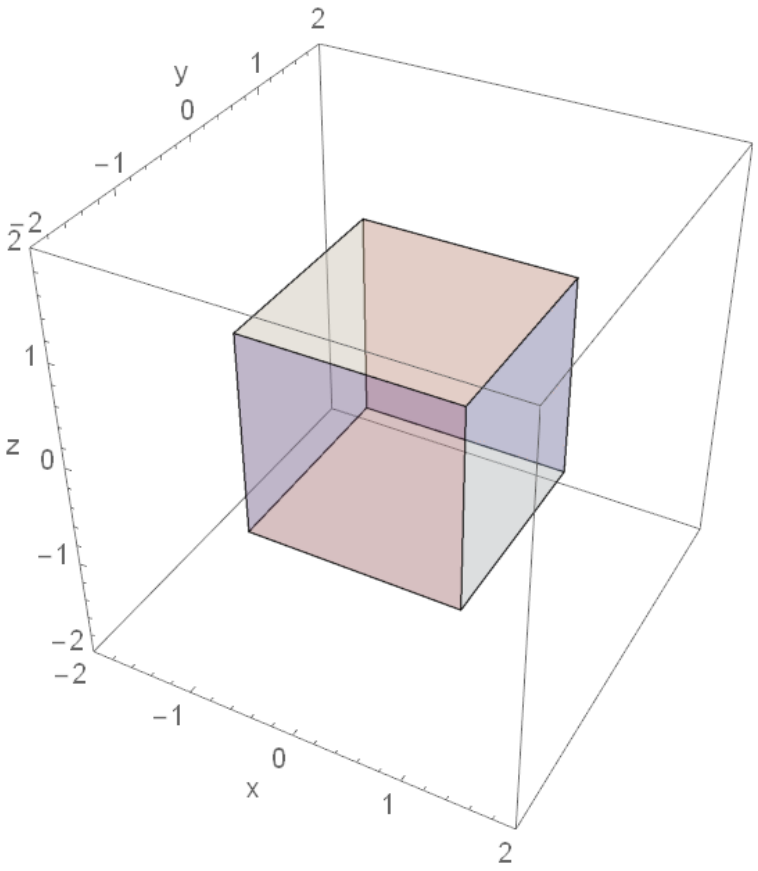


Figure 8.14. The cube with vertices (±1,±1,±1).

The surface S consists of five faces (parametrized respectively by):

1. S1 in the xz-plane, front: X1(s,t)=(s,-a,t), -a<=s<=a, -a<=t<=a,

Normal vector: N1=(0,-1,0).

2. S2 in the yz-plane, back: X2(s,t)=(-a,s,t), -a<=s<=a, -a<=t<=a,

Normal vector: N2=(1,0,0).

3. S3 in the xz-plane, back: X3(s,t)=(s,a,t), -a<=s<=a, -a<=t<=a,

Normal vector: N3=(0,-1,0).

4. S4 in the yz-plane, front: X4(s,t)=(a,s,t), -a<=s<=a, -a<=t<=a,

Normal vector: N4=(1,0,0).

5. S5 in the xy-plane, bottom: X5(s,t)=(s,t,-a), -a<=s<=a, -a<=t<=a.

Normal vector: N5=(0,0,1).

The boundary of S is the square of length 2a. It is illustrated in Figure 8.15 for a=1.

Evaluate the curl of F:

$$\nabla \times \mathbf{F} = \left(-2\,y\,z + e^{z}\,x\,z\cos(x\,y),\ 2\,x^{2}\,y\,z - e^{z}\,y\,z\cos(x\,y),\ -x^{2}\,z^{2}\right)$$

.

As required by this example, we orient S such that the normal vectors point outwardly. So, we need to multiply N2, N3 and N5 by -1 for this purpose.

Note that as one traverses the boundary path, the surface is always on the left.

```
Clear[X1,X2,X3,X4,X5,a]

X1[s_,t_]:={s,-a,t}

X2[s_,t_]:={-a,s,t}

X3[s_,t_]:={s,a,t}

X4[s_,t_]:={a,s,t}

X5[s_,t_]:={s,t,-a}

F[x_,y_,z_]:={x^2 y z^2,y z^2,z E^z Sin[x y]}


a=1;

plot[1]=ParametricPlot3D[{X1[s,t],X2[s,t],X3[s,t],X4[s,t],X5[s,t]},{s,-1,1},
{t,-1,1},Mesh->False,PlotStyle->{Opacity[0.2]},PlotRange->{{-2,2},{-2,2},
{-2,2}},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{Red,Dashed,Thickness[0.005],Arrow[{{1,-1,1},
{-1,-1,1}}],Arrow[{{1,1,1},{1,-1,1}}],Arrow[{{-1,1,1},
{1,1,1}}],Arrow[{{-1,-1,1},{-1,1,1}}],Black,Text["C1",
{0,-1,1.2}],Text["C2",{-1,0,1.3}],Text["C3",{0,1,1.2}],Text["C4",
{0.95,-0.2,1.3}],Blue,Text["S1",{0,-1,0.3}],Text["S2",
{-1,0,-0.1}],Text["S3",{0,1,0}],Text["S4",{0.95,0,0}],Text["S5",{0,0,-1}]}];

Show[Array[plot,2]]

Clear[a]
```

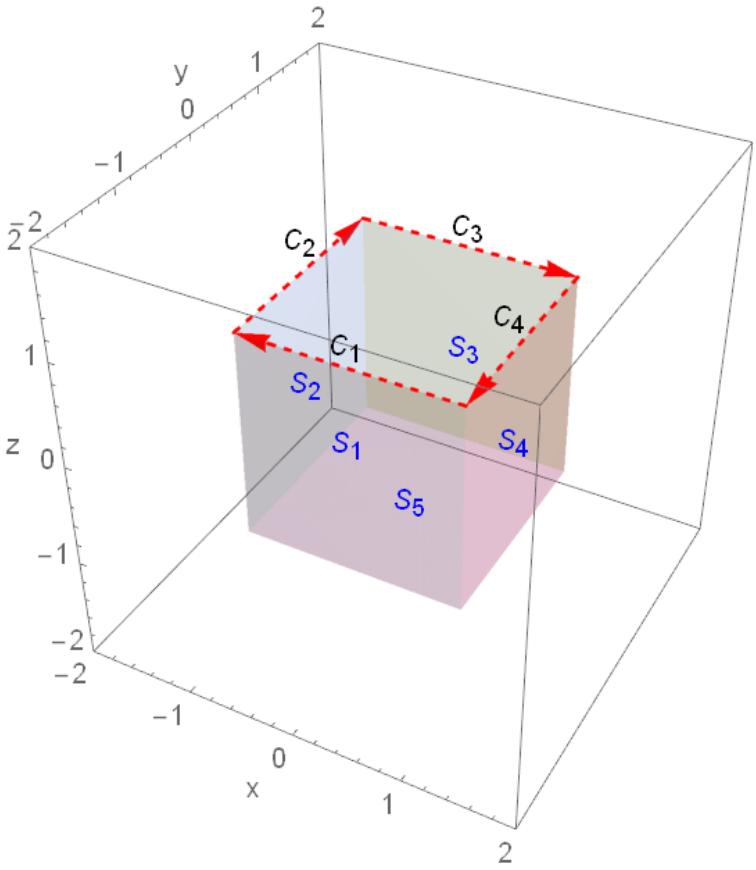


Figure 8.15. The surface S consists of four sides and a bottom face of a cube with vertices (±1,±1,±1); with the boundary ∂S as a square oriented in clockwise direction.

curlF[x_,y_,z_]=Curl[F[x,y,z],{x,y,z},"Cartesian"]

normal[X1][s,t]

normal[X2][s,t]

normal[X3][s,t]

normal[X4][s,t]

normal[X5][s,t]

It is rather tedious to compute these surface integrals by hand, so we employ vectorSurfIntegral module for the tasks (with normal vectors to X2, X3 and X5 redirected to point outward as reflected in the "sign" argument passing to the module):

vectorSurfIntegral[X1,curlF,{-a,a},{-a,a}]+vectorSurfIntegral[X2,curlF,{-a,a},{-a,a},-1]+vectorSurfIntegral[X3,curlF,{-a,a},{-a,a},-1]+vectorSurfIntegral[X4,curlF,{-a,a},{-a,a}]+vectorSurfIntegral[X5,curlF,{-a,a},{-a,a},-1]//Simplify

We may recall that the power of Stokes's Theorem lies in the fact that if S' is any orientable surface (that satisfies Stokes's Theorem) whose boundary $\partial S'$ is the same as ∂S , then we have equation (8.2):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial S'} \mathbf{F} \cdot d\mathbf{s} = \iint_{S'} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Hence, we may evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ by using a different surface S'. The surface that shares the same boundary with S is the top face of the cube, thus,

S'={ (x,y,z)|-a<=x<=a,-a<=y<=a,z=a }.

To be consistent with the orientation of the boundary ∂S (which is a square oriented clockwise), the normal vector N to S' has to point downward. So, N=(0,0,-1).

Thus, the surface integral becomes

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S'} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

$$= \iint_{S'} (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

$$= \int_{y=-a}^{a} \int_{x=-a}^{a} \left(-2ay + e^a a x \cos(xy),\ 2ax^2 y - e^a a y \cos(xy),\ -a^2 x^2\right) \cdot (0, 0, -1)\, dx\, dy$$

$$= \int_{y=-a}^{a} \int_{x=-a}^{a} \left(a^2 x^2\right) dx\, dy$$

$$= a^2 (2a) \left[\frac{1}{3} x^3\right]_{-a}^{a}$$

$$= 2a^3 \left[\frac{2}{3} a^3\right]$$

$$= \frac{4}{3} a^6$$

Integrate[a^2 x^2,{x,-a,a},{y,-a,a}]

There is still another way to verify this result is to use the line integral in Stokes's Theorem, equation (8.1):

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

.

The line integral $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$ consists of four paths:

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_3} \mathbf{F} \cdot d\mathbf{s} + \int_{C_4} \mathbf{F} \cdot d\mathbf{s}$$

.

1. C1: along the x-axis, where y=-a, z=a:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{x=-a}^{a} \left(-a^3 x^2, -a^3, a e^a \sin(-a x)\right) \cdot (-1, 0, 0)\, dx$$
$$= \int_{x=-a}^{a} \left(a^3 x^2\right) dx$$
$$= a^3 \left[\frac{1}{3} x^3\right]_{-a}^{a}$$
$$= \frac{2}{3} a^6$$

2. C2: along the y-axis, where x=-a, z=a:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{y=-a}^{a} \left(a^4 y, a^2 y, a e^a \sin(-a y)\right) \cdot (0, 1, 0)\, dy$$
$$= \int_{y=-a}^{a} \left(a^2 y\right) dy$$
$$= a^2 \left[\frac{1}{2} y^2\right]_{-a}^{a} = 0$$

3. C3: along the x-axis, where y=a, z=a:

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{s} = \int_{x=-a}^{a} \left(a^3 x^2, a^3, a e^a \sin(a x)\right) \cdot (1, 0, 0)\, dx$$
$$= \int_{x=-a}^{a} \left(a^3 x^2\right) dx$$
$$= a^3 \left[\frac{1}{3} x^3\right]_{-a}^{a}$$
$$= \frac{2}{3} a^6$$

4. C4: along the y-axis, where x=a, z=a:

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{s} = \int_{y=-a}^{a} \left(a^4 y, a^2 y, a e^a \sin(a y)\right) \cdot (0, -1, 0)\, dy$$
$$= \int_{y=-a}^{a} \left(-a^2 y\right) dy$$
$$= -a^2 \left[\frac{1}{2} y^2\right]_{-a}^{a}$$
$$= 0$$

So,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_3} \mathbf{F} \cdot d\mathbf{s} + \int_{C_4} \mathbf{F} \cdot d\mathbf{s}$$

$$= \frac{2}{3} a^6 + 0 + \frac{2}{3} a^6 + 0$$

$$= \frac{4}{3} a^6$$

8.2 Gauss's Theorems

Gauss's Theorem is also known as the divergence theorem. It is so called because the integrand is a divergence of a vector field $\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$. The divergence is a scalar function:

$$\nabla \cdot \mathbf{F} = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) \cdot \left(F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}\right)$$
$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Theorem 8.2 (Gauss's Theorem)

Let S be a closed surface that encloses a solid region D in $\mathbb{R}^3$. Assume that S consists of finitely many piecewise smooth, closed orientable surface, each of which is oriented by unit normals that point away from D. Let F be a vector field that has continuous partial derivatives whose domain includes D. We call S the boundary of D and denote S= ∂D. Then

(8.5)

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

.

The symbol $\oiint$ denotes a surface integral taken over a closed surface or finitely many piecewise smooth, closed orientable surface.

There are several important points to note:

1. Gauss's Theorem says that the flux of the vector field F across the boundary surface S is equal to the total divergence of F in a bounded, solid region D in space.

2. Let D be a region to which Gauss's Theorem applies. If F=x i+y j+ z k, then $\nabla \cdot \mathbf{F} = 3$. Equation (8.5) reduces to

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = 3 \iiint_D dV$$.

(8.6)

$$\text{volume of } D = \frac{1}{3} \oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$$

.

Hence, Gauss's Theorem may be used to calculate the volume of region D using closed surface integral. This is analogous to the way we used Green's Theorem to calculate areas of plane regions by means of suitable line integrals (Chapter 7). Geometrically, the volume of D is one-third of the flux of the vector field F across the boundary surface.

3. Let D be a region to which Gauss's Theorem applies. Suppose that the boundary of D consists of two surfaces S and S'. If $\nabla \cdot \mathbf{F} = 0$, we say that F is divergenceless. In such case equation (8.5) reduces to

(8.7) $$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = 0$$.

So,

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_{S'} \mathbf{F} \cdot d\mathbf{S} = 0$$

,

or

(8.8) $$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_{S'} \mathbf{F}\cdot d\mathbf{S}$$.

Equation (8.8) implies that we may evaluate the surface integral of either S or S' indirectly using the other one subject to an opposite sign. This allows us to choose from the two integrals one that is considerably easier to evaluate.

Example 8.10. Verify Gauss's Theorem (Theorem 8.2) for F=x i-y^2 j+z k for the solid region D={ (x,y,z)|x^2+y^2+1<=z<=5 }.

Solution:

The solid region D is defined by

D={ (x,y,z)|x^2+y^2+1<=z<=5 }

Since x^2 and y^2 are nonnegative, the minimum of z is 1.

As shown in Figure 8.16 below the boundary S= ∂D consists of two piecewise smooth, closed orientable surfaces:

1. a paraboloid surface: ∂D_1 ={ (x,y,z)|x^2+y^2+1=z },

∂D_1 is the graph of the function z=f(x,y)=x^2+y^2+1.

2. a disk with radius 2: ∂D_2 ={ (x,y,z)|x^2+y^2+1=5,z=5 }

∂D_1 and ∂D_2 may be parametrized (respectively) by

X1(s,t)=(s cos t,s sin t,s^2+1), 0<=s<=2, 0<=t<=2π

with normal vector (pointing into D):

N1(s,t)=(-2 s^2 cos t,-2 s^2 sin t,s);

and

X2(s,t)=(s cos t,s sin t,5), 0<=s<=2, 0<=t<=2π,

with normal vector (pointing away from D):

N2(s,t)=(0,0,s).

```
Clear[X1,X2,F]

X1[s_,t_]:={s Cos[t],s Sin[t],s^2+1}

X2[s_,t_]:={s Cos[t],s Sin[t],5}

F[x_,y_,z_]:={x,-y^2,z}


normal[X1][s,t]//Simplify

normal[X2][s,t]//Simplify


plot[1]=ParametricPlot3D[X1[s,t],{s,0,2},{t,0,2π},Mesh->None,PlotStyle-
>Opacity[0.8],PlotRange->{{-3,3},{-3,3},{0,6}},AxesLabel->
{"x","y","z"},PlotTheme->"Classic"];

plot[2]=ParametricPlot3D[X2[s,t],{s,0,2},{t,0,2π},PlotStyle-
>Opacity[0.3],PlotTheme->"Classic"];
```

Show[Array[plot,2]]

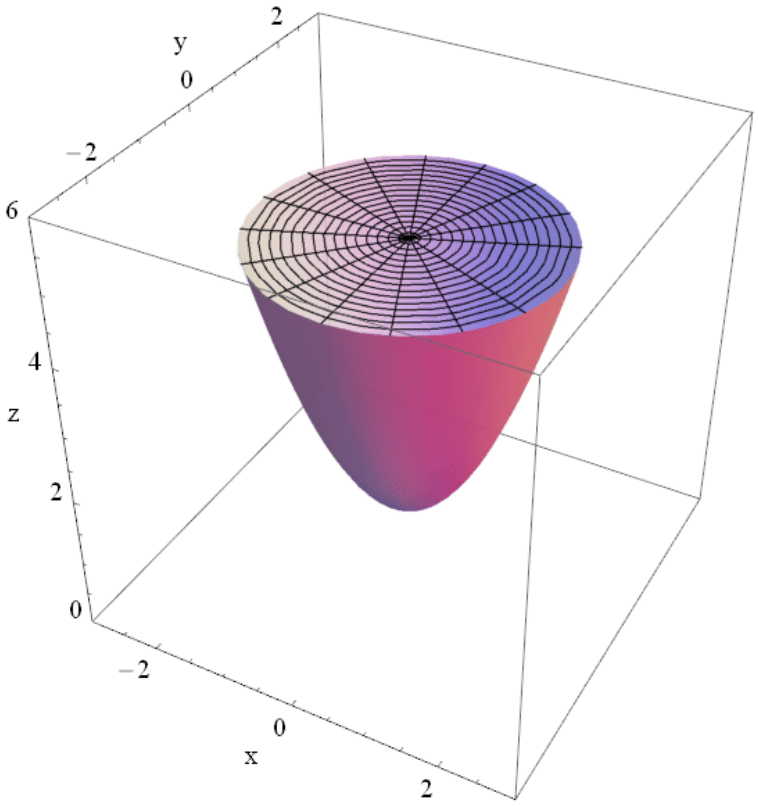


Figure 8.16. The solid region D={ (x,y,z)|x^2+y^2+1<=z<=5 }.

According to Gauss's Theorem, equation (8.5):

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_{D} \nabla \cdot \mathbf{F}\, dV$$

Evaluate the left-hand side of equation (8.5):

$$\oiint_{\partial D}\mathbf{F}\cdot d\mathbf{S}=\int\int_{\partial D_1}\mathbf{F}\cdot d\mathbf{S}+\int\int_{\partial D_2}\mathbf{F}\cdot d\mathbf{S}$$

$$=\int_{s=0}^{2}\int_{t=0}^{2\pi}\mathbf{F}\left(\mathbf{X}_1(s,t)\right)\cdot(-\mathbf{N}_1)\,dt\,ds+\int_{s=0}^{2}\int_{t=0}^{2\pi}\mathbf{F}(\mathbf{X}_2(s,t))\cdot\mathbf{N}_2\,dt\,ds$$

$$=\int_{s=0}^{2}\int_{t=0}^{2\pi}\mathbf{F}\left(s\cos t,\ s\sin t,\ s^2+1\right)\cdot\left(2s^2\cos t,\ 2s^2\sin t,\ -s\right)dt\,ds$$

$$+\int_{s=0}^{2}\int_{t=0}^{2\pi}\mathbf{F}(s\cos t,\ s\sin t,\ 5)\cdot(0,\ 0,\ s)\,dt\,ds$$

$$=\int_{s=0}^{2}\int_{t=0}^{2\pi}\left(s\cos t,\ -s^2\sin^2 t,\ s^2+1\right)\cdot\left(2s^2\cos t,\ 2s^2\sin t,\ -s\right)dt\,ds$$

$$+\int_{s=0}^{2}\int_{t=0}^{2\pi}\left(s\cos t,\ -s^2\sin^2 t,\ 5\right)\cdot(0,\ 0,\ s)\,dt\,ds$$

$$=16\pi$$

Note that a negative sign is assigned to the first integral on the right to orient the normal vector N1 to point away from D, whereas the normal vector N2 is already outward-pointing.

Integrate[{s Cos[t],-s^2 Sin[t]^2,s^2+1}.{2s^2 Cos[t],2s^2 Sin[t],-s},{s,0,2},{t,0,2π}]+Integrate[{s Cos[t],-s^2 Sin[t]^2,5}.{0,0,s},{s,0,2},{t,0,2π}]

We may compute the surface integral using vectorSurfIntegral module:

vectorSurfIntegral[X1,F,{0,2},{0,2π},-1]+vectorSurfIntegral[X2,F,{0,2},{0,2π},1]

Now, we evaluate the right-hand side of equation (8.5):

Since

$$\nabla\cdot\mathbf{F}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(x\,\mathbf{i}-y^2\,\mathbf{j}+z\,\mathbf{k}\right)=2-2y$$

,

and the solid region D is defined by

D={ (x,y,z)|x^2+y^2+1<=z<=5 },

we have

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D (2-2y)\, dV$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=x^2+y^2+1}^{5} (2-2y)\, dx\, dy\, dz$$

$$= 16\pi$$

The two integrals agree:

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = 16\pi$$

.

Div[F[x,y,z],{x,y,z},"Cartesian"]

Integrate[2-2y,{x,-2,2},{y,-Sqrt[4-x^2],Sqrt[4-x^2]},{z,x^2+y^2+1,5}]

Example 8.11. (a) Verify that the inverse square vector field $\mathbf{F}(\mathbf{r}) = k\,\frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ has zero divergence, where k is a constant and r is a position vector: r=x i+y j+z k.

(b) Show that, the flux of F across a piecewise smooth, closed, oriented surface S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \begin{cases} 0 & \text{if } S \text{ does not enclose the origin,} \\ 4\pi k & \text{if } S \text{ encloses the origin.} \end{cases}$$

Solution (a):

We evaluate the divergence of F directly:

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= k \nabla \cdot \left(\frac{\mathbf{r}}{\|\mathbf{r}\|^3} \right) \\
&= k \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial x} \right) \left[\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right] \\
&= k \left[\frac{\partial}{\partial x} \left(\frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right) + \frac{\partial}{\partial y} \left(\frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right) + \frac{\partial}{\partial z} \left(\frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right) \right] \\
&= k \left[\frac{-2x^2 + y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}} + \frac{x^2 - 2y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}} + \frac{x^2 + y^2 - 2z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}} \right] \\
&= k \left[\frac{-2x^2 + y^2 + z^2 + x^2 - 2y^2 + z^2 + x^2 + y^2 - 2z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}} \right] \\
&= k \left[\frac{-2x^2 - 2y^2 - 2z^2 + 2x^2 + 2y^2 + 2z^2}{\left(x^2 + y^2 + z^2\right)^{5/2}} \right] \\
&= 0
\end{aligned}$$

Div[{x,y,z}/(x^2+y^2+z^2)^(3/2),{x,y,z},"Cartesian"]//Simplify

Solution (b):

Case 1. S does not enclose the origin.

Then $\mathbf{F}(\mathbf{r}) = k \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ is defined everywhere and has continuous partial derivatives. From part (a) we obtain $\nabla \cdot \mathbf{F} = 0$. By Gauss's Theorem:

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = 0$$

.

Case 2. S encloses the origin.

Then $\mathbf{F}(\mathbf{r}) = k \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ is defined everywhere except at the origin.

Let Sa denotes a sphere of radius a centered at the origin. Sa has the graph equation x^2+y^2+z^2=a^2. Then the unit normal vector to Sa may be computed using the gradient of the level surface:

$$\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} = \frac{\mathbf{r}}{a}$$

.

Note that n is pointing away from Sa.

Suppose a is so small that Sa is contained inside S. Let D be the solid region between Sa and S. Then ∂D consists of S and Sa, does not contain the origin and Gauss's Theorem is valid in D. As a result from part (a), $\nabla \cdot \mathbf{F} = 0$ throughout D. According to Gauss's Theorem:

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} + \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = 0$$

.

Orienting ∂D with normal vectors that point away from D as stipulated by Gauss's Theorem gives

$$-\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} + \iint_S \mathbf{F} \cdot d\mathbf{S} = 0$$

.

(Sa is now part of the boundary of D, its normal has to be in opposite direction such that it points away from D.)

Thus,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$.

We now compute the flux of F through S indirectly by using instead the flux of F through Sa. Note that Sa is a sphere of radius a so it is easier to compute than a general surface S.

So,

$$\begin{aligned} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_a} \mathbf{F} \cdot \mathbf{n}\, dS \\ &= k \iint_{S_a} \frac{\mathbf{r}}{\|\mathbf{r}\|^3} \cdot \frac{1}{a} \mathbf{r}\, dS \\ &= k \iint_{S_a} \frac{\|\mathbf{r}\|^2}{\|\mathbf{r}\|^3} \cdot \frac{1}{a}\, dS \end{aligned}$$

On Sa, we have ||r||=a, so

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \frac{k}{a^2} \iint_{S_a} dS$$.

Since the surface area of a sphere of radius a is 4π a^2, therefore

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = 4\pi k$$.

Thus,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = 4\pi k$$

.

Hence,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \begin{cases} 0 & \text{if } S \text{ does not enclose the origin,} \\ 4\pi k & \text{if } S \text{ encloses the origin.} \end{cases}$$

Example 8.12. A solid region D is bounded by the paraboloids z=9-x^2-y^2 and z=4x^2+4y^2-16. Use Gauss's Theorem to find the volume of D.

Solution:

The boundary curve C for the two paraboloids can be found by solving the equation:

z=9-x^2-y^2=4x^2+4y^2-16

25=5x^2+5y^2.

x^2+y^2=5.

Substituting back x^2+y^2=5 into z=9-x^2-y^2 gives

z=9-5=4.

Thus, the intersection of the two paraboloids is a circle of radius Sqrt[5] in the xy-plane at z=4.

We may parametrize the circle by

c(t)=(Sqrt[5]cos t,Sqrt[5]sin t,4).

Clear[c]

c[t_]:={Sqrt[5]Cos[t],Sqrt[5]Sin[t],4}

plot[1]=ContourPlot3D[{z==9-x^2-y^2,z==4x^2+4y^2-16},{x,-3,3},
{y,-3,3},{z,-16,9},Mesh->None,ContourStyle->{{Green,Opacity[0.6]},
{Yellow,Opacity[0.4]}},PlotTheme->"Business"];

plot[2]=ParametricPlot3D[c[t],{t,0,2π},PlotStyle->
{Dashed,Red,Thickness[0.007]}];

Show[Array[plot,2]]

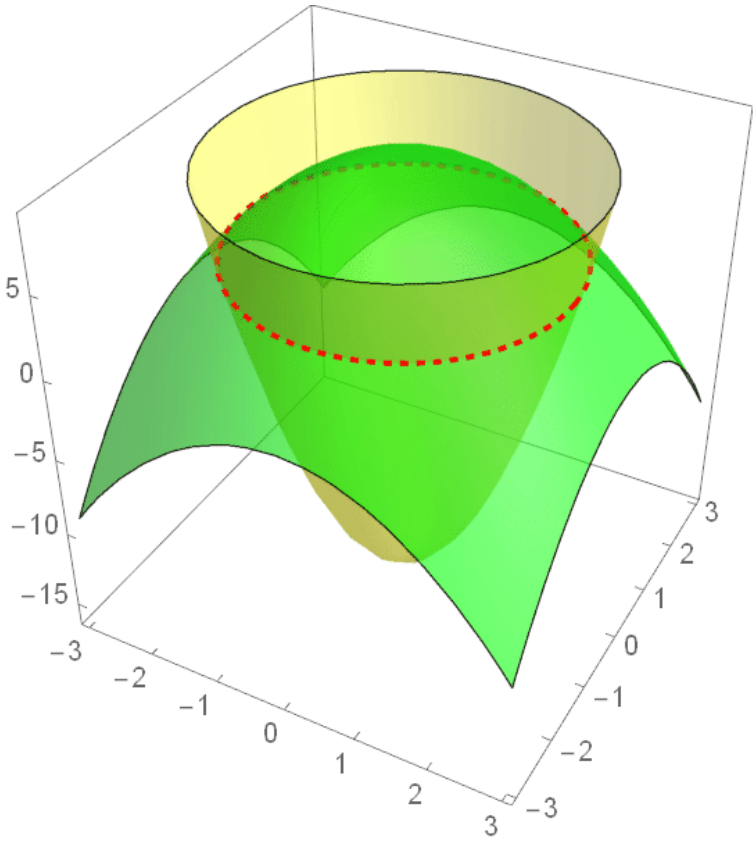


Figure 8.17. The solid region D is bounded by the paraboloids z=9-x^2-y^2 and z=4x^2+4y^2-16.

According to Gauss's Theorem, equation (8.5) of Theorem 8.2:

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

.

Let F=x i+y j+z k. Then $\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$.

Equation (8.5) reduces to

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = 3 \iiint_D dV$$

.

So, the volume of the solid region V is

$$V = \iiint_D dV = \frac{1}{3} \oiint_{\partial D} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot d\mathbf{S}$$

.

The boundary ∂D consists of two surfaces ∂D_1 and ∂D_2 :

1. Paraboloid 1: ∂D_1 ={ (x,y,z)|z=9-x^2-y^2, 4<=z<=9 }

Parametric equation of ∂D_1 :

X1(s,t)=(s cos t,s sin t,9-s^2), 0<=s<=Sqrt[5], 0<=t<=2π,

Normal vector to ∂D_1 :

N1=(2 s^2 cos t,2 s^2 sin t,s).

N1 is outward-pointing normal.

2. Paraboloid 2: ∂D_2 ={ (x,y,z)|z=4x^2+4y^2-16, -16<=z<=4 }.

Parametric equation of ∂D_2 :

X2(s,t)=(s cos t,s sin t,4s^2-16), 0<=s<=Sqrt[5], 0<=t<=2π,

Normal vector to ∂D_2 :

N2=(-8 s^2 cos t,-8 s^2 sin t,s).

N2 is inward-pointing normal, so we need to orient N2 to point away from D when using Gauss's Theorem.

```
Clear[X1,X2,F]

X1[s_,t_]:={s Cos[t],s Sin[t],9-s^2}

X2[s_,t_]:={s Cos[t],s Sin[t],4s^2-16}

F[x_,y_,z_]:={x,y,z}


normal[X1][s,t]//Simplify

normal[X2][s,t]//Simplify
```

```
plot[1]=ParametricPlot3D[X1[s,t],{s,0,Sqrt[5]},{t,0,2π},Mesh-
>None,PlotStyle->{Green,Opacity[0.9]},PlotRange->{{-3,3},{-3,3},
{-16,9}},AxesLabel->{"x","y","z"},PlotTheme->"Business"];

plot[2]=ParametricPlot3D[X2[s,t],{s,0,Sqrt[5]},{t,0,2π},Mesh-
>None,PlotStyle->{Yellow,Opacity[0.4]},PlotTheme->"Business"];

Show[Array[plot,2],BoxRatios->{1, 1, 1}]
```

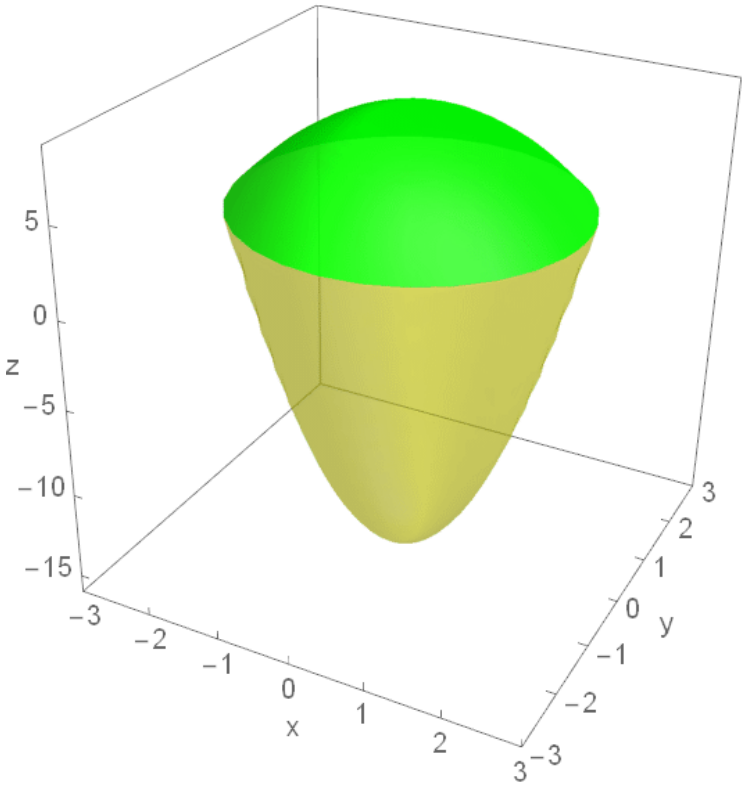


Figure 8.18. The solid region D bounded by two surfaces: ∂D_1 ={ (x,y,z)|z=9-x^2-y^2, 4<=z<=9 } and ∂D_2 ={ (x,y,z)|z=4x^2+4y^2-16, -16<=z<=4 }.

So,

$$
\begin{aligned}
V &= \iiint_D dV \\
&= \frac{1}{3} \oiint_{\partial D} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot d\mathbf{S} \\
&= \frac{1}{3} \left[\iint_{\partial D_1} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot d\mathbf{S} + \iint_{\partial D_2} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot d\mathbf{S} \right] \\
&= \frac{1}{3} \left[\iint_{\partial D_1} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{N}_1 \, dS + \iint_{\partial D_2} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot (-\mathbf{N}_2) \, dS \right] \\
&= \frac{1}{3} \left[\int_{s=0}^{\sqrt{5}} \int_{t=0}^{2\pi} (s\cos t,\, s\sin t,\, 9 - s^2) \cdot (2s^2 \cos t,\, 2s^2 \sin t,\, s)\, dt\, ds \right. \\
&\qquad \left. - \int_{s=0}^{\sqrt{5}} \int_{t=0}^{2\pi} (s\cos t,\, s\sin t,\, 4s^2 - 16) \cdot (-8s^2 \cos t,\, -8s^2 \sin t,\, s)\, dt\, ds \right] \\
&= \frac{125\pi}{2}
\end{aligned}
$$

1/3 (Integrate[{s Cos[t],s Sin[t],9-s^2}.{2s^2 Cos[t],2s^2 Sin[t],s}, {s,0,Sqrt[5]},{t,0,2π}]-Integrate[{s Cos[t],s Sin[t],4s^2-16}.{-8 s^2 Cos[t],-8 s^2 Sin[t],s},{s,0,Sqrt[5]},{t,0,2π}])

We may also use vectorSurfIntegral module to compute this integral:

1/3 (vectorSurfIntegral[X1,F,{0,Sqrt[5]},{0,2π}]+vectorSurfIntegral[X2,F, {0,Sqrt[5]},{0,2π},-1])

Example 8.13. Use Gauss's Theorem to evaluate the flux of the vector field F=y i+x j+(y^2+z)k across the boundary S of the solid region bounded by the cylinder x^2+y^2=4, the plane z=x+1, and the xy-plane (see Figure 8.19).

Solution:

According to Gauss's Theorem, equation (8.5):

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

.

The left-hand side of the equation above is the flux of F across S= ∂D , so

$$\text{flux} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$.

Since $\nabla \cdot \mathbf{F} = 1$, the flux of F across S is equal to the volume of D:

$$\text{flux} = \iiint_D dV$$.

The intersection of the cylinder x^2+y^2=4 with the plane z=x+1 is a curve parametrized by

(2 cos t,2 sin t,2 cos t+1), 0<=t<=2π.

Since the cylinder is bounded by the xy-plane, at z=0 the intersection is the line x=-1. Therefore, D is a solid region lying over the intersection of the disk x^2+y^2<=4 with the line x=-1 in the xy-plane; whereas z ranges from 0 to z=x+1. So,

$$\text{flux} = \iiint_D dV = \int_{x=-1}^{2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=0}^{x+1} dz\, dy\, dx$$

$$= 3\sqrt{3} + \frac{8\pi}{3}$$

Integrate[1,{x,-1,2},{y,-Sqrt[4-x^2],Sqrt[4-x^2]},{z,0,x+1}]

```
Clear[F]

F[x_,y_,z_]:={y,x,y^2+z}


Div[F[x,y,z],{x,y,z},"Cartesian"]


plot[1]=ContourPlot3D[{x^2+y^2==4,z==x+1},{x,-2,2},{y,-2,2},
{z,0,3},Mesh->None,ContourStyle->{{Opacity[0.3],Yellow},
{Opacity[0.5],Green}},AxesLabel->{"x","y","z"},PlotRange->{{-3,3},
{-3,3},{0,4}},ViewPoint->{3,-3,2}];

plot[2]=Graphics3D[{Dashed,Red,Thickness[0.01],Line[{{-1,-Sqrt[3],0},
{-1,Sqrt[3],0}}]}];

plot[3]=ParametricPlot3D[{{2Cos[t],2Sin[t],2Cos[t]+1},
{2Cos[t],2Sin[t],0}},{t,0,(2π)/3},PlotStyle->
{{Dashed,Red,Thickness[0.01]},{Dashed,Red,Thickness[0.01]}}];

plot[4]=ParametricPlot3D[{{2Cos[t],2Sin[t],2Cos[t]+1},
{2Cos[t],2Sin[t],0}},{t,(4π)/3,2π},PlotStyle->
{{Dashed,Red,Thickness[0.007]},{Dashed,Red,Thickness[0.01]}}];

plot[5]=VectorPlot3D[F[x,y,z],{x,-3,3},{y,-3,3},{z,0,4},VectorPoints-
>5,VectorScale->{0.2, 0.15, Automatic},VectorStyle->Gray];

plot[6]=Show[Array[plot,4]];

plot[7]=Show[Array[plot,5]];

Show[GraphicsGrid[{{plot[6],plot[7]}}],ImageSize->600]
```

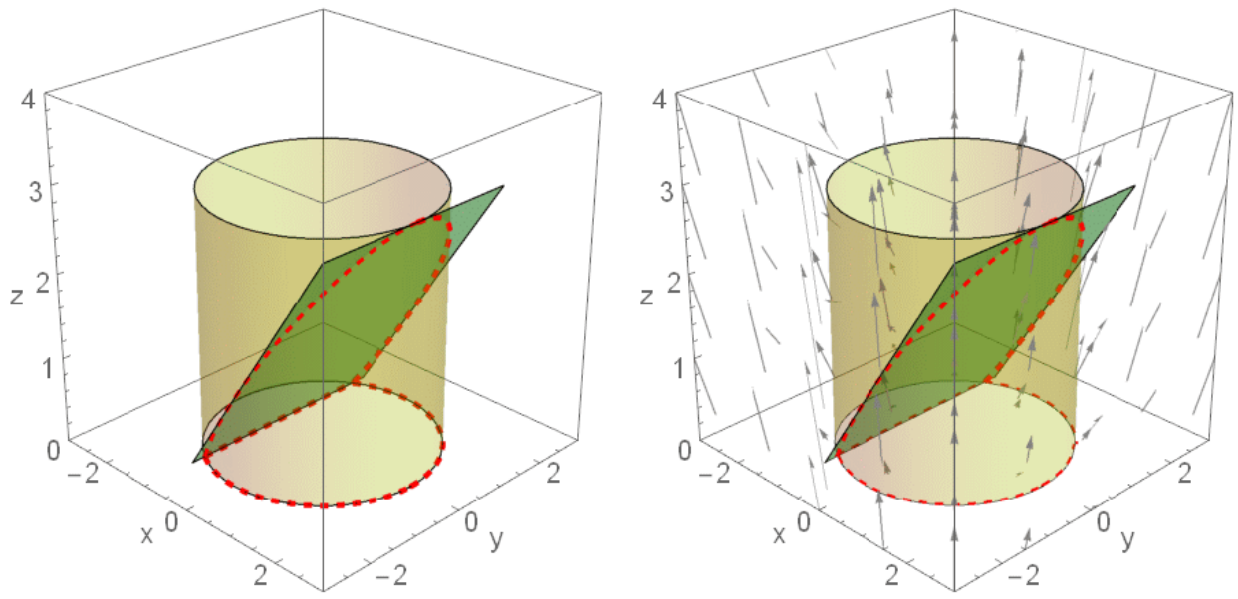


Figure 8.19. The solid region D bounded by the cylinder x^2+y^2=4, the plane z=x+1, and the xy-plane, in the vector field F=y i+x j+(y^2+z)k.

Example 8.14. Let the vector field be F=(x^2-z^2)i+(E^z^2-cos x)j+y^3 k. Use Gauss's Theorem to evaluate the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface x+2y+4z=12 in the first octant.

```
Clear[F]

F[x_,y_,z_]:={x^2-z^2,Exp[z^2]-Cos[x],y^3}


plot[1]=ContourPlot3D[x+2y+4z==12,{x,0,12},{y,0,6},
{z,0,3},ContourStyle->{Opacity[0.3],Blue},AxesLabel->{"x","y","z"}];

plot[2]=Graphics3D[{Opacity[0.2],Red,Triangle[{{0,0,0},{12,0,0},
{0,0,3}}],Green,Triangle[{{0,0,0},{0,6,0},
{0,0,3}}],Yellow,Triangle[{{0,0,0},{12,0,0},{0,6,0}}]}];
```

```
plot[3]=VectorPlot3D[F[x,y,z],{x,0,12},{y,0,6},{z,0,3},VectorPoints->7,VectorScale->{0.1, 0.3, Automatic},VectorStyle->Gray];

plot[4]=Show[Array[plot,2]];

plot[5]=Show[Array[plot,3]];

Show[GraphicsGrid[{{plot[4],plot[5]}}],ImageSize->600]
```

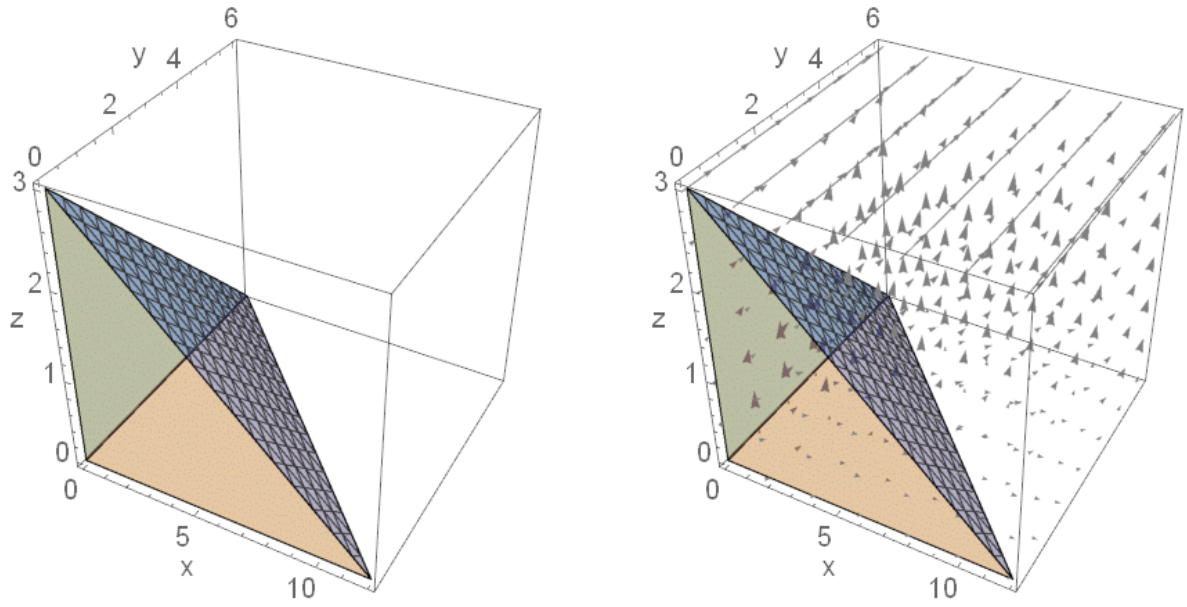


Figure 8.20. The solid region D bounded by the surface x+2y+4z=12 and the coordinate planes in the first octant, in the vector field F=(x^2-z^2)i+ (E^z^2-cos x)j+y^3 k.

According to Gauss's Theorem, equation (8.5):

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_{D} \nabla \cdot \mathbf{F}\, dV$$

.

Evaluate the divergence of F:

$\nabla \cdot \mathbf{F} = 2x$.

Div[F[x,y,z],{x,y,z},"Cartesian"]

The boundary ∂D consists of four surfaces:

1. S1, the surface x+2y+4z=12 in the first octant.

2. Sa, the triangle surface in the xz-plane. The outward-pointing normal is Na=(0,-1,0).

3. Sb, the triangle surface in the yz-plane. The outward-pointing normal is Nb=(-1,0,0).

4. Sc, the triangle surface in the xy-plane. The outward-pointing normal is Nc=(0,0,-1).

Thus,

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_b} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_c} \mathbf{F} \cdot d\mathbf{S}$$
$$= \iiint_D \nabla \cdot \mathbf{F}\, dV$$

So, the flux of F through S1 is

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\, dV - \iint_{S_a} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_b} \mathbf{F}\cdot d\mathbf{S} - \iint_{S_c} \mathbf{F}\cdot d\mathbf{S}$$

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_{x=0}^{12}\int_{y=0}^{\frac{1}{2}(12-x)}\int_{z=0}^{\frac{1}{4}(12-x-2y)} 2x\, dz\, dy\, dx$$

$$-\int_{x=0}^{12}\int_{z=0}^{\frac{1}{4}(12-x)} \left(x^2-z^2, e^{z^2}-\cos x, y^3\right)\cdot(0,-1,0)\, dx\, dz$$

$$-\int_{y=0}^{6}\int_{z=0}^{\frac{1}{4}(12-2y)} \left(x^2-z^2, e^{z^2}-\cos x, y^3\right)\cdot(-1,0,0)\, dz\, dy$$

$$-\int_{x=0}^{12}\int_{y=0}^{\frac{1}{2}(12-x)} \left(x^2-z^2, e^{z^2}-\cos x, y^3\right)\cdot(0,0,-1)\, dy\, dx$$

$$= \int_{x=0}^{12}\int_{y=0}^{\frac{1}{2}(12-x)}\int_{z=0}^{\frac{1}{4}(12-x-2y)} 2x\, dz\, dy\, dx$$

$$+\int_{x=0}^{12}\int_{z=0}^{\frac{1}{4}(12-x)} \left(e^{z^2}-\cos x\right) dz\, dx$$

$$-\int_{y=0}^{6}\int_{z=0}^{\frac{1}{4}(12-2y)} z^2\, dz\, dy$$

$$+\int_{x=0}^{12}\int_{y=0}^{\frac{1}{2}(12-x)} y^3\, dy\, dx$$

$$\simeq 2110.43$$

Integrate[2x,{x,0,12},{y,0,1/2 (12-x)},{z,0,1/4 (12-x-2y)}]+Integrate[Exp[z^2]-Cos[x],{x,0,12},{z,0,1/4 (12-x)}]-Integrate[z^2,{y,0,6},{z,0,1/4 (12-2y)}]+Integrate[y^3,{x,0,12},{y,0,1/2 (12-x)}]//N

Example 8.15. Let S be an oriented surface defined by z=E^(1-x^2-y^2), for z>=1.

(a) Use Gauss's theorem to calculate the flux of F through S where F=-2x i+y j+(2+z)k.

(b) Evaluate the flux of F through S directly.

Solution (a):

Write z=E^(1-x^2-y^2)=E^(1-(x^2 + y^2)). For z>=1, S is the graph of the function f(x,y)=E^(1-(x^2 + y^2)) defined over a planar region R={ (x,y)|x^2+y^2<=1 }.

We may consider D as a solid region where the boundary ∂D consists of S and R, to which Gauss's Theorem applies. Then by equation (8.5):

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

,

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_R \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

.

plot[1]=ContourPlot3D[z==Exp[1-x^2-y^2]//Evaluate,{x,-1,1},{y,-1,1},{z,1,3},Mesh->None,ContourStyle->{Green,Opacity[0.8]},PlotRange->{{-1,1},{-1,1},{0,3}},AxesLabel->{"x","y","z"}];

plot[2]=ParametricPlot3D[{Cos[t],Sin[t],1},{t,0,2π},PlotStyle->{Red,Thickness[0.007]}];

Show[Array[plot,2]]

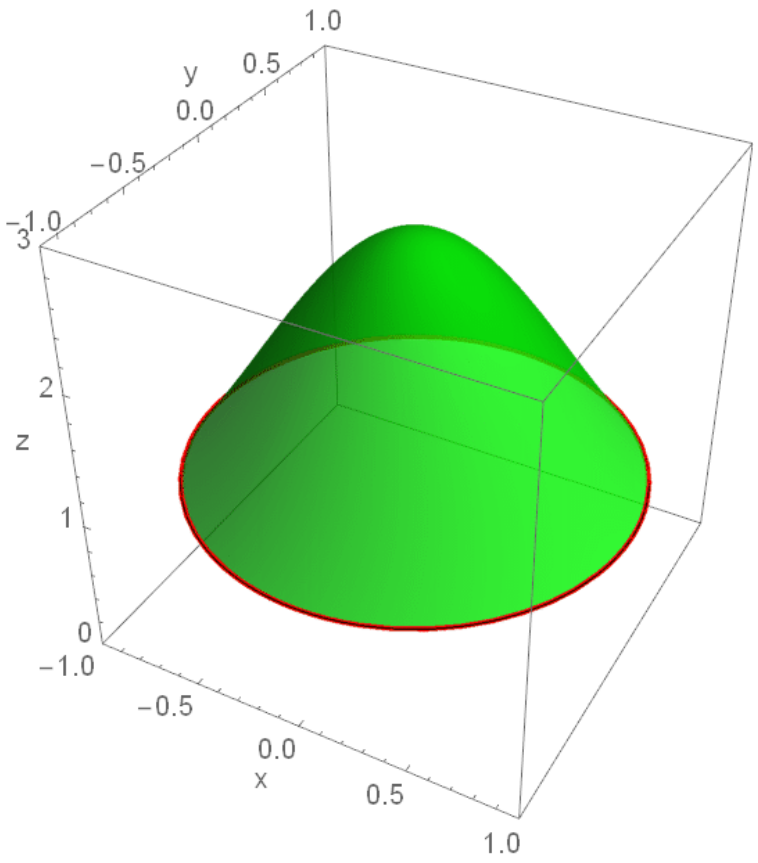


Figure 8.21. S is the surface defined by z=E^(1-x^2-y^2), for z>=1.

We may parametrize S by

X1(s,t)=(s cos t,s sin t,E^(1-s^2)), 0<=s<=1, 0<=t<=2π.

Normal vector: N1=(2 E^(1-s^2) s^2 cos t,2 E^(1-s^2) s^2 sin t,s).

N1 is pointing away from D.

And the disk R parametrized by

X2(s,t)=(s cos t,s sin t,1), 0<=s<=1, 0<=t<=2π.

Normal vector: N2=(0,0,s).

N2 is pointing into D. So, we need to orient N2 such that it points away from D.

```
Clear[X1,X2,F]

X1[s_,t_]:={s Cos[t],s Sin[t],Exp[1-s^2]}

X2[s_,t_]:={s Cos[t],s Sin[t],1}

F[x_,y_,z_]:={-2x,y,2+z}


normal[X1][s,t]//Simplify

normal[X2][s,t]//Simplify
```

The divergence of F is zero.

```
Div[F[x,y,z],{x,y,z},"Cartesian"]


plot[1]=ParametricPlot3D[X1[s,t]//Evaluate,{s,0,1},{t,0,2π},Mesh-
>None,PlotStyle->{Green,Opacity[0.2]},PlotRange->{{-1,1},{-1,1},
{0,3}},AxesLabel->{"x","y","z"},PlotTheme->"Classic"];

plot[2]=ParametricPlot3D[X2[s,t]//Evaluate,{s,0,1},{t,0,2π}];

plot[3]=ParametricPlot3D[{Cos[t],Sin[t],1},{t,0,2π},PlotStyle->
{Red,Thickness[0.004]}];

Show[Array[plot,3]]
```

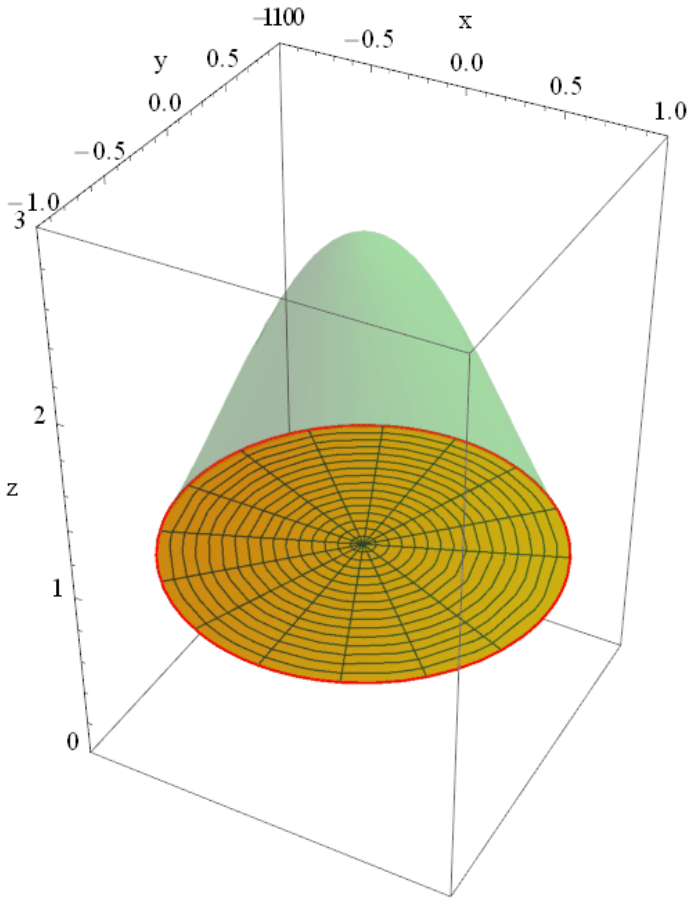


Figure 8.22. The solid region D whose boundary consists of two surfaces.

So,

$$\iint_S \mathbf{F}\cdot\mathbf{N}_1\,dS + \iint_R \mathbf{F}\cdot(-\mathbf{N}_2)\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV$$
$$= 0$$

The flux of F through S is

$$\iint_S \mathbf{F}\cdot\mathbf{N}_1\,dS = \iint_R \mathbf{F}\cdot\mathbf{N}_2\,dS$$
$$= \int_{s=0}^{1}\int_{t=0}^{2\pi}(-2s\cos t,\ s\sin t,\ 3)\cdot(0,\ 0,\ s)\,dt\,ds$$
$$= \int_{s=0}^{1}\int_{t=0}^{2\pi} 3s\,dt\,ds$$
$$= 6\pi\left[\frac{1}{2}s^2\right]_0^1$$
$$= 3\pi$$

Integrate[3s,{s,0,1},{t,0,2π}]

We may verify our result using vectorSurfIntegral module:

vectorSurfIntegral[X2,F,{0,1},{0,2π}]

Solution (b):

Now, we evaluate the flux of F through S directly:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{N}_1\,dS$$
$$= \int_{s=0}^{1}\int_{t=0}^{2\pi}\left(-2s\cos t,\ s\sin t,\ 2+e^{1-s^2}\right)\cdot$$
$$\left(2e^{1-s^2}s^2\cos t,\ 2e^{1-s^2}s^2\sin t,\ s\right)dt\,ds$$
$$= \int_{s=0}^{1}\int_{t=0}^{2\pi}\left(-4e^{1-s^2}s^3\cos^2 t + 2e^{1-s^2}s^3\sin^2 t + s\left(2+e^{1-s^2}\right)\right)dt\,ds$$
$$= 3\pi$$

Integrate[-4Exp[1-s^2]s^3 Cos[t]^2+2Exp[1-s^2]s^3 Sin[t]^2+s(2+Exp[1-s^2]),{s,0,1},{t,0,2π}]

Let us verify our result using vectorSurfIntegral module:

vectorSurfIntegral[X1,F,{0,1},{0,2π}]

Example 8.16. Find the fallacy in the following derivation that concludes "all magnetic fields are not real". This example is taken from the article: "G. Arfken, Magnetic Fields Are Not Real, American Journal of Physics, vol. 27, issue 7 (1959), p.526". See also "B. E. Blank, S. G. Krantz, Calculus, Multivariable 2nd Edition, John Wiley & Sons, 2011, p.1139"

Solution:

One of Maxwell's equations says that if B is a magnetic field, then $\nabla \cdot \mathbf{B} = 0$. Furthermore, B may be expressed as the curl of some vector A:

$\mathbf{B} = \nabla \times \mathbf{A}$,

where A is called the vector potential of B.

According to Gauss's Theorem (equation 8.5) for B:

$$\oiint_{\partial D} \mathbf{B} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{B}\, dV$$

.

Since $\nabla \cdot \mathbf{B} = 0$, we obtain

(8.9) $$\oiint_{\partial D} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = 0$$.

Here lies the common mistakes to apply Stokes's Theorem (equation 8.1) to equation (8.9), by writing it as

(8.10)

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = 0 = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{s}$$

.

Note that equation (8.9) refers to a closed surface integral (hence, the sign $\oiint$), thus there is no boundary curve; whereas Stokes's Theorem refers to open surface integral with boundary curve. Therefore, equation (8.10) is not valid.

If we follow the fallacy by assuming equation (8.10) is valid, we will come to the conclusion that A has path-independent line integrals. For it follows from Theorem 7.7 that there is a scalar function V such that $\mathbf{A} = \nabla V$. So then (the curl of a gradient vanishes)

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\nabla V) = \mathbf{0}$$.

This implies that all magnetic fields B vanish. Of course this is absurd.

Appendix A. Introduction to Mathematica

A.1 Executing Simple Expressions

In Mathematica, any text enclosed in matching (* and *) denotes comments. A comment can be inserted anywhere in Mathematica code.

To execute an expression, place the cursor at the expression, and then press [SHIFT]+[ENTER] on a PC or [COMMAND]+[RETURN] on a Mac.

Or alternatively, from the menu bar on top of Mathematica, click on the Evaluation -> Evaluate Cells. (click on the "Evaluation" menu and then follows by clicking "Evaluate Cells")

```
(* First clear any values that may already have been assigned
  to the names of the various objects to be calculated *)
Clear["Global`*"]
```

```
(* Addition. Type 1+2 and then execute *)
1+2
```

Parentheses, (), are used only for grouping such as the expression below:

```
(* Multiplication *)
```

(1+2)*3

To assign a value to the variable a, we use equal (=) sign. Notice that equal sign in Mathematica is used for assigning value to a variable. It is not the same meaning as equality in standard mathematical expressions.

Since all built-in Mathematica objects begin with capital letters, they are reserved names. Mathematica will complain if we assign value to a reserved objects. Some reserved names are: Cos, Pi, Sin, Exp. Some function names are: Plot, FindRoot, ParametricPlot. The only Greek character reserved in Mathematica is π.

Some common mistakes:

sin[pi]

cos[pi]

The correct expressions are

Sin[Pi]

Cos[π]

Mathematica is case-sensitive. A variable name A is different from a variable name a. So it is good practice to name variables begin with lowercase letters.

Let us compute the expression (1+2)*3 and assign the result to a variable called a:

```
a=(1+2)*3
```

A semicolon (;) at the end of a Mathematica command will suppress output. For example:

```
b=(1+2)*3;
```

To check the value of b, we type the following expression and execute it

```
(* Type b and then execute *)
b
```

Multiplication in Mathematica can be represented by a space between expressions, for example

```
(* (1+2) and multiplied by 3 and the result is assigned to c.
```

A space is used in between to denote multiplication *)

c=(1+2) 3

(* Check if a equals b equals c *)

a==b==c

Notice that Mathematica uses double equal sign (==) to mean equality to check if a equals b equals c.

Mathematica keeps records of all the inputs and outputs in a session. Thus, we can refer to the last result generated by using the percent sign (%) notation. Double percent signs (%%) gives the next-to-last result generated. %%...% (k times) gives the k-th previous result and %k refers to output line numbered k.

A.2 Defining a function

Besides the built-in functions, we can also define a function in Mathematica. For example, a function f which takes an input x, and output x^2-4,

f(x)=x^2-4

we type in

f[x_]:=x^2-4

Notice that the function argument x must be followed by an underscore (_) inside the square bracket [ ]. Here we have used a colon-equal sign (:=) to separate the left side of the function definition from the right.

To compute the output of the function when x=1, we type in the following and execute the expression

f[4]

Similarly, to compute the output of the function when x=-1:

f[-1]

We can also use Mathematica built-in function such as Sin in our function. For instance

g(x)=x-sin (x^2).

g[x_]:=x-Sin[x^2]

g[1]

The result may not be what we wanted. If we wish to have numerically value, we can use the built-in function N. Here is an example:

g[2]//N

The above expression is equivalent to

N[g[2]]

where the N command gives the numerical value of g[2].

There are times when we need to define a different function using the same function name f(x). However, f(x) is still retained in memory as can be checked below

```
?f
```

Thus, we want to clear out the function f(x) from memory using Clear command and then define the new function using the same function name.

```
Clear[f]
```

```
(* Define a new function with function name f *)
f[x_]:=2x^3+1
```

Now, f(x) is a totally different function. We can verify this using the ? command:

```
?f
```

Notice in our examples in defining functions above, we have used the := operator, which is called the SetDelayed operator. Mathematica will evaluate the expression appearing to its right afresh each time the expression appearing to its left is called.

While the = operator, which is called Set operator evaluates the expression on its right only once, at the time the assignment is made.

A.3 Lists

A list is a collection of elements separated by commas and enclosed in curly brackets. For example:

{2, 3, 5, 8}

To make a list above we can use List command:

List[2,3,5,8]

We may enter a list directly without using the List command, such as

{3,6,7,8}

We can use lists to build up vectors, matrices, tensors and other arrays. For instance, we use Table command in the following to generate a one-dimensional array:

myList=Table[2^i,{i,5}]

and assign the list with the name myList.

To extract element of a list, we use double square bracket [[and]]. For instance, to extract the second element we type in

```
myList[[2]]
```

To extract a sequence of elements, from the second to the fourth element of myList

```
myList[[2;;4]]
```

To extract an element 2 from the end of a list:

```
myList[[-2]]
```

As mentioned earlier, we can build a matrix by using list. In fact, a matrix is a list of lists. For instance, we build a 3x3 matrix by typing in

```
myMatrix={{9,3,2},{3,3,1},{2,2,0}}
```

To show the elements of list arranged in a regular array, we use MatrixForm command:

MatrixForm[myMatrix]

or equivalently

myMatrix//MatrixForm

To switch rows and columns, we use Transpose command and then follow by MatrixForm command to show it in regular array.

Transpose[myMatrix]//MatrixForm

The inverse of our matrix myMatrix is

myInverse=Inverse[myMatrix]

myInverse//MatrixForm

Let us verify that multiplication of the matrix with its inverse matrix is an identity matrix:

myMatrix.myInverse//MatrixForm

myInverse.myMatrix//MatrixForm

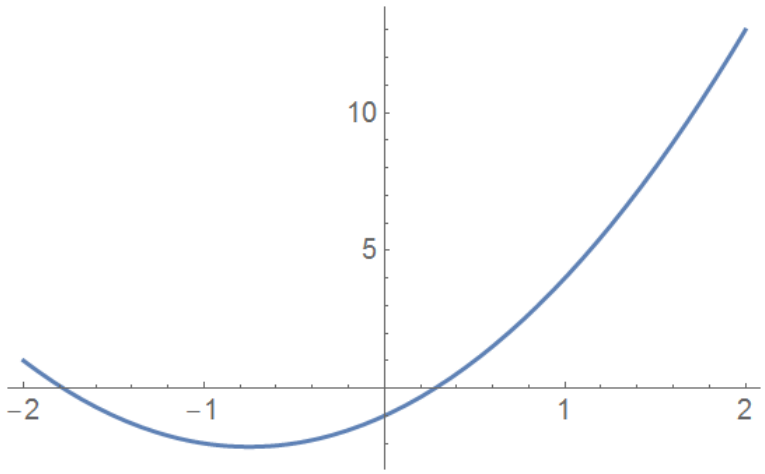
A.4 Plotting Graphs

To generate a graph of a function f, we use the Plot command.

f(x)=2x^2+3x-1.

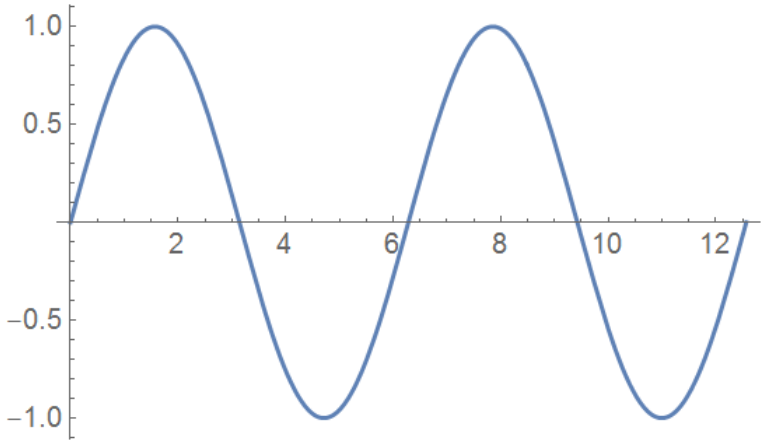
First of all, we clear memory of any previously defined variable f and then define the function:

```
Clear[f]

f[x_]:=2x^2+3x-1

(* Plot the function for -2≤x≤2 *)

Plot[f[x],{x,-2,2},PlotStyle->AbsoluteThickness[2]]
```



We may also write the function explicitly inside the Plot command, such as

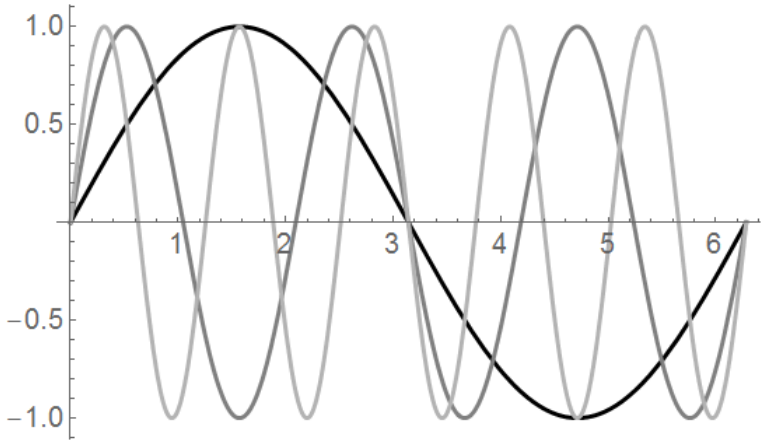
Plot[Sin[x],{x,0,4π},PlotStyle->AbsoluteThickness[2]]



To plot several functions at once, we separate the functions with comma inside a curly bracket. The functions have different gray color using GrayLevel option:

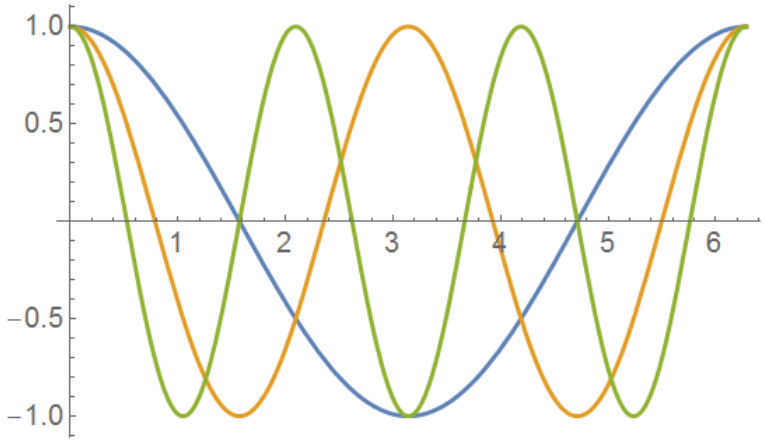
Plot[{Sin[x],Sin[3x],Sin[5x]},{x,0,2π},PlotStyle->
{{AbsoluteThickness[2],GrayLevel[0.0]},

{AbsoluteThickness[2],GrayLevel[0.5]},
{AbsoluteThickness[2],GrayLevel[0.7]}}]



Here is another example in which we use a Table command to generate a list of cosine functions and sent it to Plot command:

Plot[Evaluate[Table[Cos[k x],{k,1,3}]],{x,0,2π},PlotStyle->AbsoluteThickness[2]]

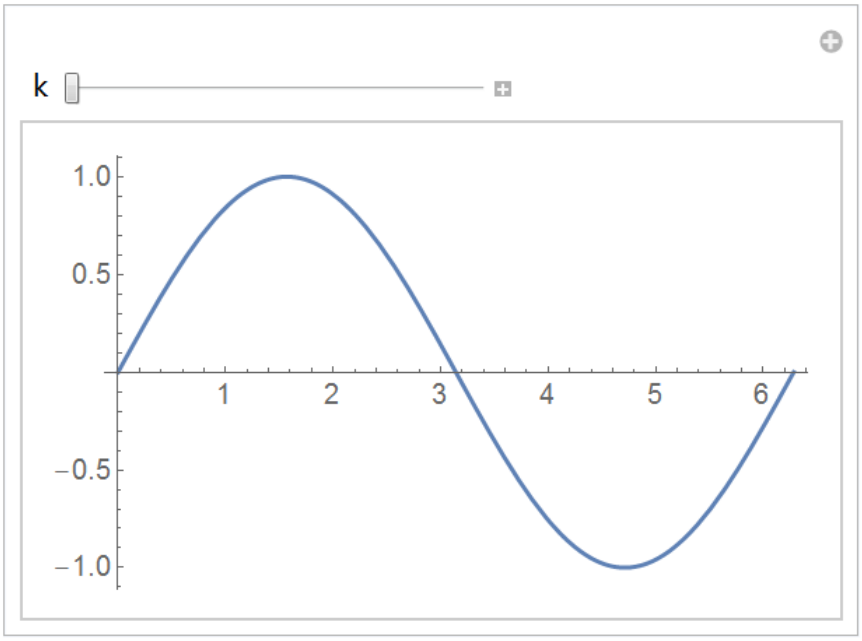


Mathematica has a Manipulate command to allow us interactively manipulate the values of a variable of a function. For instance, we plot the function

sin (k x),

in the range 0<=x<=2π, while changing the value of k from 1 to 3.

Manipulate[Plot[Sin[k x],{x,0,2π},PlotStyle->AbsoluteThickness[2]],
{k,1,3}]



Notice that Manipulate takes two arguments. The first is the expression to be manipulated. In this case:

Plot[Sin[k x],{x,0,2π},PlotStyle->AbsoluteThickness[2]]

The second is an iterator which specifies the range of values that the controller variable k is to assume. In this case k ranges from the value of 1 to 3.

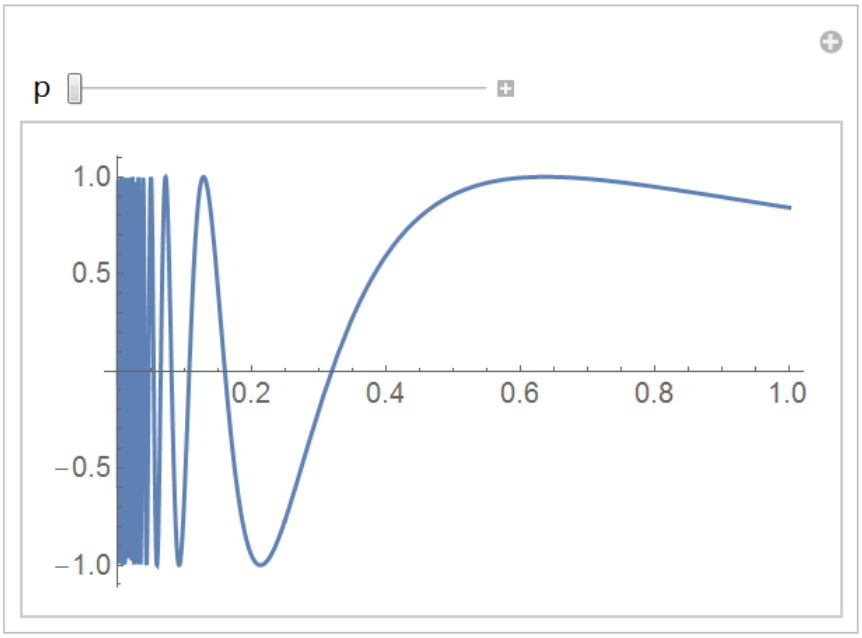
Notice that there is a slider inside the frame. You can move the slider with your mouse to control the value assumed by k. Click on the ▫ button to the right of the slider to show a user control panel. Then, click on the ▶ button to watch as the curve is displayed while k changes in real time.

Here is another example of Manipulate command for function

f(x)=sin (1/x),

in which we allow the upper limit of x to vary from p=1 to p=3, in steps 0.2:

```
Manipulate[Plot[Sin[1/x],{x,0,p},PlotStyle-
>AbsoluteThickness[2],PlotRange->Full],{p,1,3,0.2}]
```



Mathematica can draw curves that are defined parametrically using ParametricPlot command. For instance, the x- and y-coordinates of

points are defined as two independent functions of a parameter t:

x(t)=3cos t,

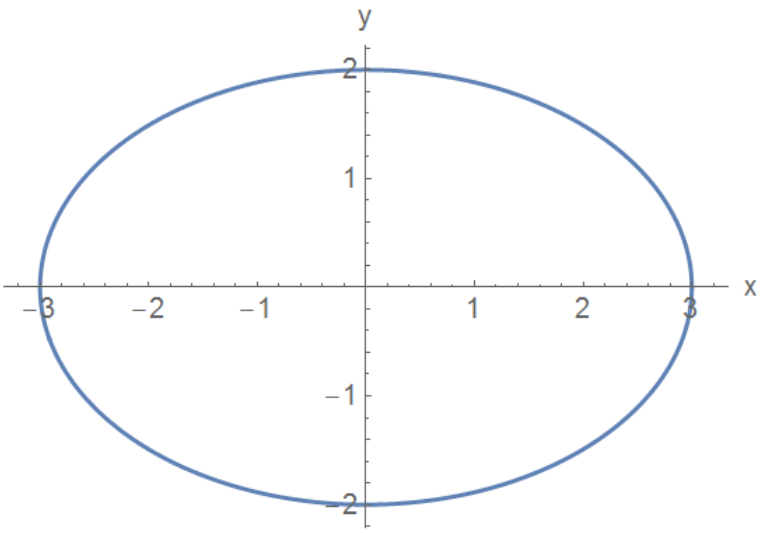
y(t)=2sin t.

Clear[x,y]

x[t_]:=3Cos[t]

y[t_]:=2Sin[t]

ParametricPlot[Evaluate[{x[t],y[t]}],{t,0,2π},PlotStyle->AbsoluteThickness[2],AxesLabel->{"x","y"}]

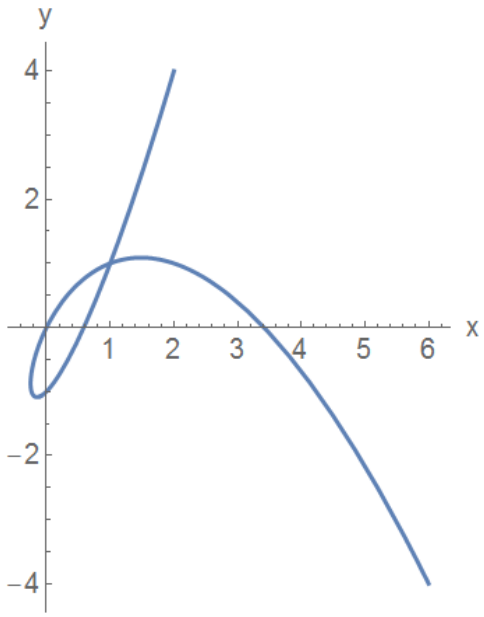


Here is another example

x(t)=t^2-t,

y(t)=t^3-2t.

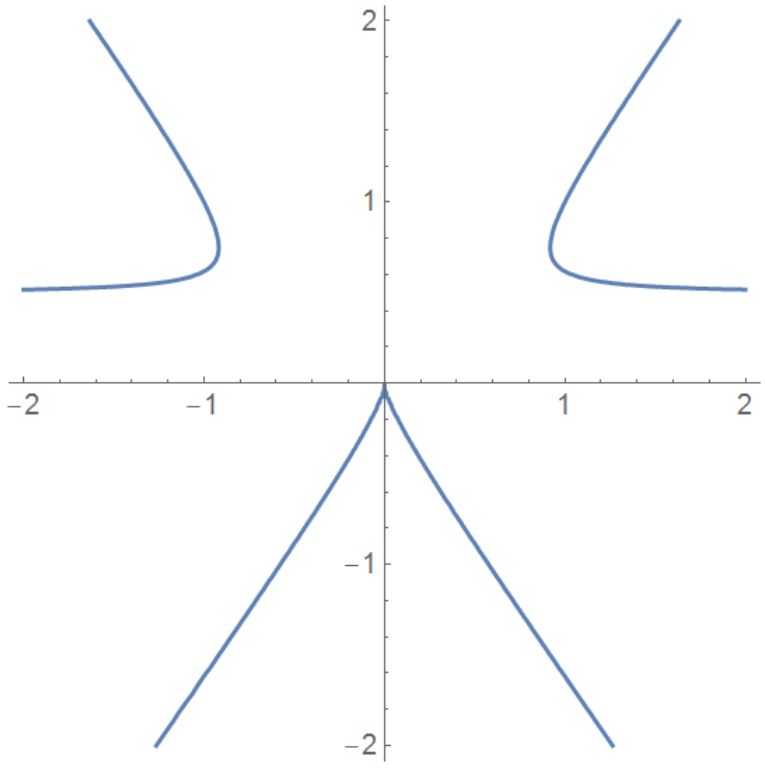
```
ParametricPlot[Evaluate[{t^2-t,t^3-2t}],{t,-2,2},PlotStyle-
>AbsoluteThickness[2],AxesLabel->{"x","y"}]
```



Implicitly defined curves can be graphed with the ContourPlot command. For example curve defined implicitly by the equation:

x^2+y^3=2x^2 y.

```
ContourPlot[x^2+y^3==2x^2 y,{x,-2,2},{y,-2,2},Axes->True,Frame-
>False,ContourStyle->AbsoluteThickness[2]]
```

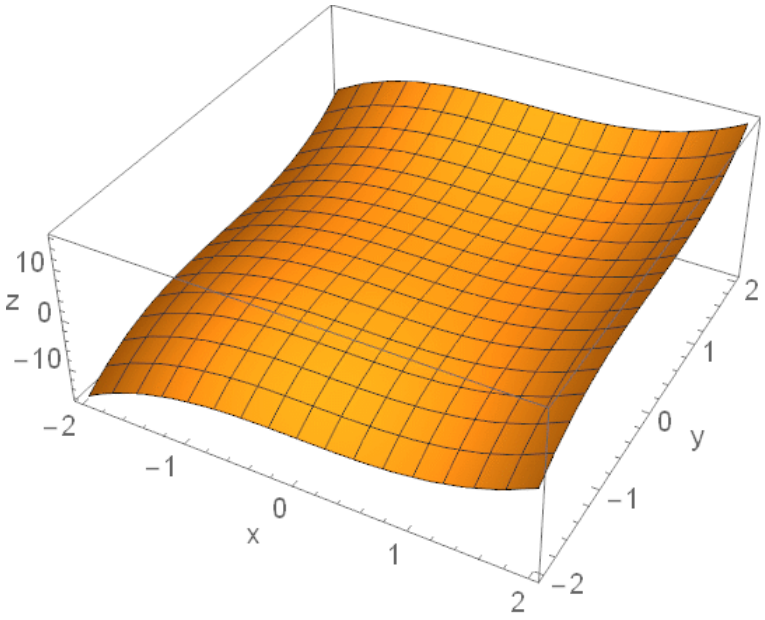


For a function of two independent variables, the graph is a surface in three-dimensional space. Suppose a function f(x,y) is defined as

f(x,y)=x^3+y^3.

To draw the surface in three-dimensional space we can use Plot3D command.

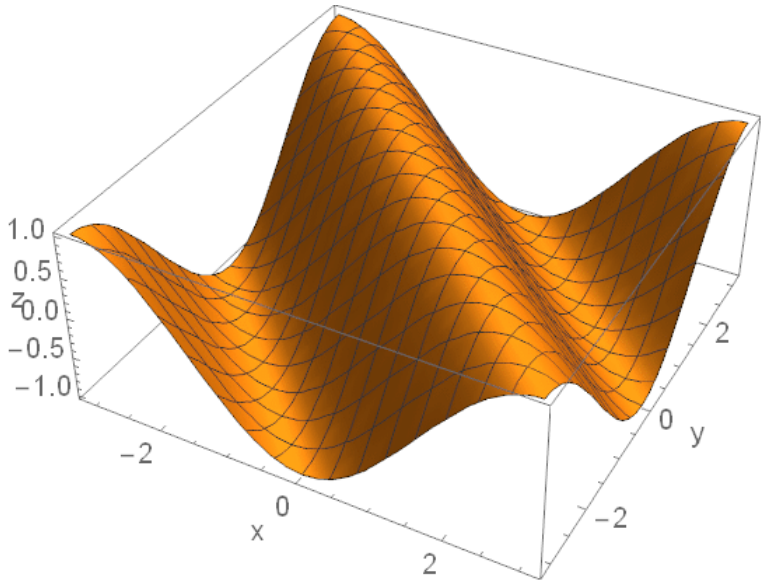
Plot3D[x^3+y^3,{x,-2,2},{y,-2,2},AxesLabel->{"x","y","z"}]



Here is another surface defined as

f(x,y)=cos(x+y).

Plot3D[Cos[x+y],{x,-π,π},{y,-π,π},AxesLabel->{"x","y","z"}]

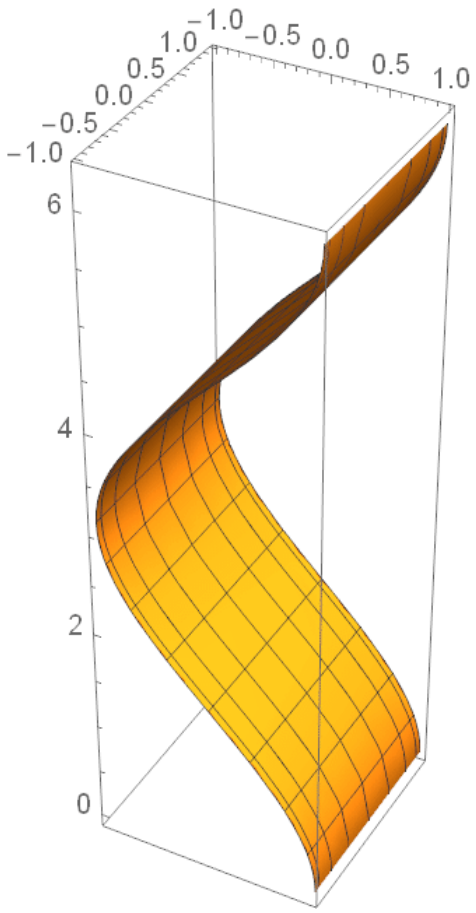


If a three-dimensional space surface is defined parametrically by two parameters, s and t, such as

f(t)=(cos s,sin t,s),

we can use ParametricPlot3D to draw the surface.

ParametricPlot3D[Evaluate[{Cos[s],Sin[t],s}],{s,0,2π},{t,0,2π}]



ParametricPlot3D command can also be used to produce three-dimensional space curve defined parametrically by a variable t only.

Suppose a vector valued function is defined as

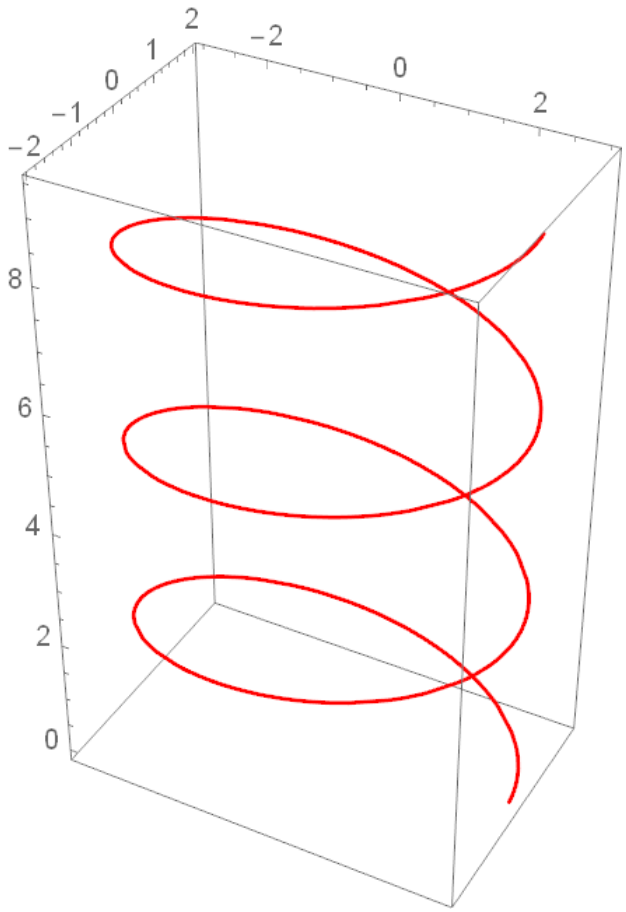
f(t)=(3 cos t, 2 sin t, t/2).

Clear[f]

f[t_]:={3Cos[t],2Sin[t],t/2}

To plot the curve f(t) we type in

ParametricPlot3D[Evaluate[f[t]],{t,0,6π},PlotStyle->
{Red,AbsoluteThickness[2]}]


2
1
0
−1
−2
−2
0
2
8
6
4
2
0

A.5 Calculus

The Limit command can be used to find the limiting value of a function when the independent variable approaches a particular value. For instance, compute the following:

$$\lim_{x \to 3} (x - 3) \csc(\pi x)$$

Limit[(x-3) Csc[π x],x->3]

and

$$\lim_{x \to 0} (1 + \sin x)^{\cot(2x)}$$

Limit[(1+Sin[x])^Cot[2x],x->0]

If f[x] represents a function, its derivative is represented by f'[x]. Higher derivatives are represented by f''[x], f'''[x], and so on.

Clear[f]

f[x_]:=Cosh[x^2-3x+1]

f'[x]

Another method to find the derivative of a function is by using the D command

D[f[x],x]

The second derivative of f(x)=cosh (x^2-3x+1)

f''[x]

Or equivalently by using the D command

D[f[x],{x,2}]

The third derivative of f(x)=cosh (x^2-3x+1)

f'''[x]

D[f[x],{x,3}]

In Mathematica, Integrate command computes antiderivatives. For instance, compute the integral

$$\int (x+2)\sin\left(x^2+4x-6\right) dx$$

.

Integrate[(x+2) Sin[x^2+4x-6],x]

We may use the traditional symbol such as

$$\int (x+2)\ \mathrm{Sin}\left[x^2+4x-6\right]\ dx$$

Note that Mathematica omitted the constant of integration C from the answer.

The Integrate command can also be used to compute definite integrals:

$$\int_0^{\infty} \frac{1}{1+x^2}\, dx$$.

Integrate[1/(1+x^2),{x,0,∞}]

Or using the traditional symbol

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

Here is another example

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x\,\text{ArcSin}\left[x^2\right]}{\sqrt{1-x^4}}\,dx$$

A.6 Differential Equations

```
(* First clear any values that may already have been assigned

to the names of the various objects to be calculated *)

Clear["Global`*"]
```

To solve differential equations for analytical solutions, Mathematica provides DSolve command to find the solutions.

Example 1. To solve the first-order differential equation $\frac{dy}{dx} - y^2 \cos x = 0$, we simply type

```
sol=DSolve[y'[x]-y[x]^2 Cos[x]==0,y[x],x]
```

Note that the result is given as a list and we assign it to sol. The formula for the solution is in the form of replacement. Thus to extract the solution we use ReplaceAll (/.) command.

```
y[x]/.sol
```

Example 2. Find the solution of the second-order differential equation $2\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$, with initial conditions x(0)=1, x'(0)=0.

```
sol=DSolve[{2x''[t]+x'[t]+x[t]==0,x[0]==1,x'[0]==0},x[t],t]
```

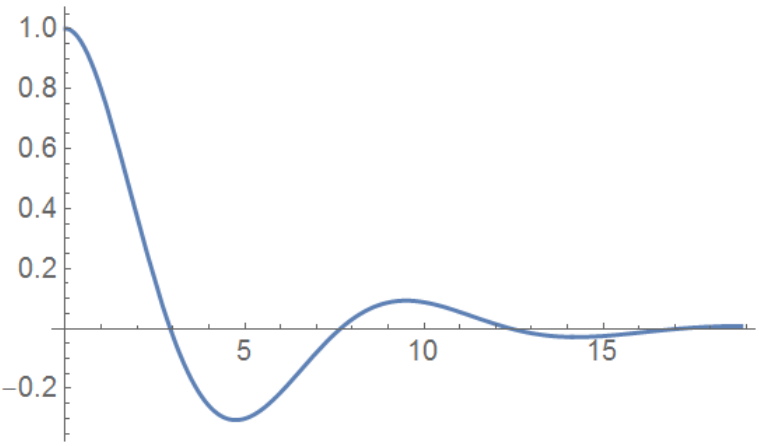
Note that we put the initial conditions together with the differential equation inside curly brackets. We can simplify the list by using the Flatten command:

```
sol=sol//Flatten
```

To extract the solution we use the ReplaceAll (/.) command

```
x[t]/.sol
```

```
(* Plot the analytical solution *)
Plot[x[t]/.sol,{t,0,6π},PlotRange->All,PlotStyle->AbsoluteThickness[2]]
```



If the differential equation cannot be solve analytically, Mathematica provides the command NDSolve to obtain numerical solutions.

Example 3. Find the solution of the differential equation $\frac{dy}{dx} = 1 + y^2$ with initial condition y(0)=1.

First we try with the DSolve command.

```
DSolve[{x''[t]+Cos[x[t]]==0,x[0]==0,x'[0]==1},x[t],t]
```

However, Mathematica complains that DSolve is unable to solve the differential equation.

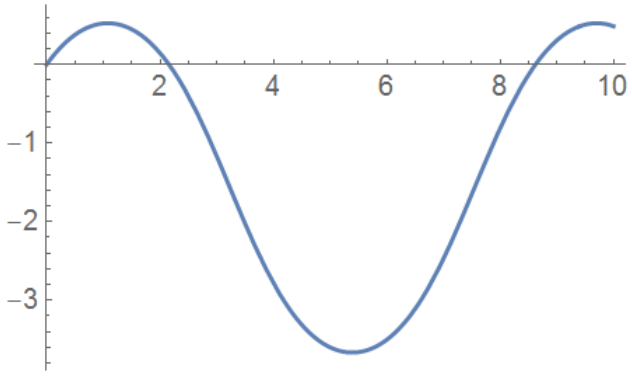
Let us now try using numerical approximation:

```
numsol=NDSolve[{x''[t]+Cos[x[t]]==0,x[0]==0,x'[0]==1},x[t],{t,0,10}]
```

Notice that to perform the integration, NDSolve command needs the initial conditions and the desired range of solution, in this case 0<=t<=10.

(* Plot the numerical solution *)

```
Plot[x[t]/.numsol,{t,0,10},PlotStyle->AbsoluteThickness[2]]
```



A.7 Modules

Thus far, all objects that have been defined are global, meaning that, for example, if we define x to have a value 1, then x will remain 1 until its value is changed. In contrast, a local object has a limited scope valid only within a certain group of instructions.

Modules in Mathematica allow us to define local variables whose values are defined only within the module. Outside of the module, the object may either be defined or undefined. Module is especially useful when we need to group several commands into one unit to be executed as a group.

?Module

For example:

Module[{var1,var2,...},body]

defines a module with local variables var1, var2, ...

Note that the local variables are enclosed inside the curly bracket { } and then followed by the "body" part.

Module[{var1=v1,var2=v2,...},body]

defines a module with local variables var1, var2, ... initialized to v1, v2, ..., respectively.

Note that local variables var1, var2, ... only defined in the module, hence they can only be referred to within the module. Outside of the module, they may or may not have any definition.

The body part of the module can have one or more commands that are separated by semicolons.

Example 4. Define a module with local variables x and y

```
Module[{x,y},
x=1;
y=3;
z=x+y;
Print[x*y]
]
```

Note that x and y are defined locally inside the module because they are included inside the curly brackets. However, z is a global variable. We can refer to z by typing

```
z
```

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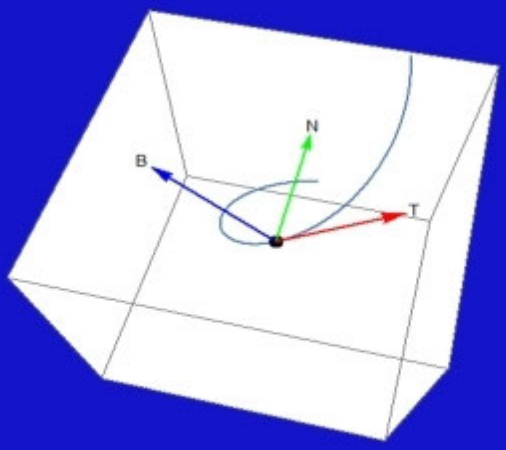
Second Edition